Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 15, Number 3, 2025, Pages 289-311
https://doi.org/10.34198/ejms.15325.289311

General Dynamical Systems and Variational Inequalities

Muhammad Aslam Noor[1,2,*] and Khalida Inayat Noor[3,4]

[1] Mathematics Department, COMSATS University Islamabad, Islamabad, Pakistan

[2] Mathematics, University of Wah, Wah, Pakistan
e-mail: nooraslam@gmail.com

[3] Mathematics Department, COMSATS University Islamabad, Islamabad, Pakistan

[4] Mathematics, University of Wah, Wah, Pakistan
e-mail: khalidan@gmail.com

Abstract

In this paper, we introduce and consider a new second order dynamical system for solving general variational inequalities. Using the forward backward finite difference schemes, we suggest some new multi-step iterative methods for solving the variational inequalities and their variants forms. Convergence analysis is investigated under certain mild conditions. We also use the change of variable method to establish the equivalence between the complementarity problems and the fixed point problems. The alternate formulation can exploited to consider the dynamical systems and study the stability properties of the solution. Since the variational inequalities are equivalent to the complementarity problems, our results can be used to develop new techniques for them. It is an interesting problem to compare these methods with other technique for solving variational inequalities and related optimizations for further research activities.

1 Introduction

Stampacchia [50] proved that the minimum of the energy functions of the obstacle problems in potential theory can be characterized by an inequality, which is called the variational inequality. Lions et al. [18] introduced and investigated the variational inequalities using essentially the auxiliary principle technique. Noor [23] introduced and studied some new classes of variational inequalities involving the two monotone operators with applications in Bingham fluid, elasticity and optimizations. Finite element technique [23] was used to derive the error estimate for the solution of the variational inequalities. System of the absolute value equations and hemivariational inequalities are special cases of the nonlinear variational inequalities involving two operators. It is well known facts that variational theory provides us with a simple, natural,

Received: January 4, 2025; Accepted: February 5, 2025; Published: February 12, 2025
2020 Mathematics Subject Classification: 47A12, 49J40, 65K15, 90C33.
Keywords and phrases: variational inequalities, dynamical system, boundary value problems, finite difference interpolation, iterative methods, convergence.

*Corresponding author

unified, novel and general framework to study an extensive range of unilateral, obstacle, free, moving and equilibrium problems arising in fluid flow through porous media, elasticity, circuit analysis, transportation, oceanography, operations research, finance, economics, and optimization. It is worth mentioning that the variational inequalities can be viewed as a significant and novel generalization of the variational principles, the origin can be traced back to Euler, Lagrange, Bernoulli's brother and Fermat. It is very simple fact that the minimum of a differentiable convex functions on the convex sets can be characterized by variational inequality. It is amazing that variational inequalities have influenced various areas of pure and applied sciences and are still continue to influence the recent research, see [5, 8, 11–15, 18, 19, 23–50, 52, 54–57] and the references therein.

One of the most difficult and important problems in variational inequalities is the development of efficient numerical methods. Several numerical methods have been developed for solving the variational inequalities and their variant forms. These methods have been extended and modified in numerous ways exploring novel and innovative to study complicated and complex problems. It is well known established fact that the variational inequalities are equivalent to the fixed point problem. This alternative formulation has been used to consider the existence of a solution, iterative schemes, sensitivity analysis, merit functions dynamical systems and other aspects of the variational inequalities. It is very important to develop some efficient iterative methods for solving the variational inequalities. In this direction, Noor [29, 31] suggested and analyzed several three-step forward-backward splitting algorithms for solving variational by using the updating technique. These three-step methods are also known as Noor's iterations. It is noted that these forward-backward splitting algorithms are similar to those of Glowinski et al. [13], which they suggested by using the Lagrangian technique. It is known that three-step schemes are versatile and efficient. These three-step schemes are a natural generalization of the splitting methods for solving partial differential equations. It has been established [1–6, 10, 13, 20, 21, 29, 31, 32, 35, 41, 45, 47–49, 51–53, 57] that Noor iterations and their modified forms perform better than two-step Ishikawa iteration and one step method Mann iteration In recent years, considerable interest has been shown in developing various extensions and generalizations of Noor iterations, both for their own sake and for their applications. For novel applications, modifications and generalizations of the Noor iterations. These methods include Mann iteration, Ishikawa iteration, modified forward-backward splitting methods of Tseng [48], Noor [29, 31] and Noor et al. [41, 42, 45] as special cases. Noor iterations have been modified and generalized in different directions to explore their applications in fractal, chaos, images, signal recovery, polynomiography, fixed point theory, compress programming, nonlinear equations, compressive sensing and image in painting, see [2–4, 6, 7, 10, 17, 20, 21, 41, 45, 47, 48, 51–53, 57] and the references therein.

Dupuis and Nagurney [11] introduced and studied the projected dynamical systems associated with variational inequalities using the equivalent fixed point formulation. The novel feature of the projected dynamical system is that the its set of stationary points corresponds to the set of the corresponding set of the solutions of the variational inequality problem. It has been shown [11, 14, 19, 30, 31, 36–38, 42, 55, 56] that these dynamical systems are useful in developing efficient and powerful numerical techniques for solving variational inequalities.

Motivated and inspired by ongoing research in these fascinations areas, we consider a dynamical system coupled with second order boundary value problems associated with general variational inequalities. In this paper, we show that the second boundary value problems can be exploited to suggest and analyzed multi step methods for finding the approximate solutions of variational inequalities and related optimization problem. This is a new approach. Using the finite difference schemes, we suggest and analyzed some new multi step iterative methods for solving variational inequalities. Some special cases are also pointed as potential applications of the obtained results. These multi step methods include Mann iteration, Ishikawa iterations and Noor iterations as special cases. Convergence criteria of these methods is discussed under suitable weaker conditions. In section 4, we consider the change of variables method, which have been developed in [1,27,31,35] to show that the complementarity problems are equivalent to the fixed point problems. This alternative formulation can be used to study the existence of the solution as well as to suggest some approximation schemes for solving the variational inequalities and their variant forms. This is another alternative technique. We have only considered theoretical aspects of the suggested methods. It is an interesting problem to implement these methods and to illustrate the their efficiency. Comparison with other methods need further research efforts. The ideas and techniques of this paper may be extended for other classes of general variational inequalities and related optimization problems.

2 Basic Definitions and Results

Let Ω be a closed convex set in a real Hilbert space $\mathcal{H}$ with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, respectively. Let $\mathcal{T}, g : \mathcal{H} \longrightarrow \mathcal{H}$ be nonlinear operators.

We consider the problem of finding $\mu \in \Omega$, such that

$$\langle\, \mathcal{T}\mu, g(\nu) - \mu\rangle \geq 0, \quad \forall\, \nu \in \Omega, \tag{2.1}$$

which is called the general variational inequality, introduced and studied by Noor and Noor [34].

We note that, for $\mu = g(\mu)$, the problem (2.1) is equivalent to finding $\mu \in \mathcal{H}$ such that

$$\langle\, \mathcal{T}\mu + \mu - g(\mu), g(\nu) - \mu\rangle \geq 0, \quad \forall\, \nu \in \Omega, \tag{2.2}$$

It has been shown [34] that the optimality conditions of the differentiable nonconvex functions can be characterized via the general variational inequalities of the type (2.1).

Special cases

We now discuss some special cases of general variational inequalities (2.1)

1. If $g = I$, then the problem (2.1) reduces to finding $\mu \in \Omega$, such that

$$\langle \mathcal{T}\mu, \nu - \mu\rangle \geq 0, \quad \forall\, \nu \in \Omega, \tag{2.3}$$

is called the variational inequality, introduced by Lions and Stampacchia [18]. It has been shown a wide class of obstacle boundary value and initial value problems can be studied in the general framework of variational inequalities. For the applications, motivation, numerical methods, sensitivity analysis, dynamical system, merit functions and other aspects of variational inequalities, see [5, 8, 11–15, 18, 19, 23–25, 27–38, 41–50, 52, 54–56] and the references therein.

2. If $\mu = g(\mu)$, then problem (2.1) is equivalent to finding $u \in \Omega$

$$\langle \mathcal{T}(g(\mu)), g(\nu) - g(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega, \tag{2.4}$$

which is called the general variational inequality, introduced and studied by Noor [29]. Variational inequality of the type (2.4) arises as a minimum of the differentiable nonconvex functions.

3. If $\Omega^* = \{\mu \in \mathcal{H} : \langle \mu, \nu\rangle \geq 0, \forall \nu \in \Omega,\}$ is a polar(dual) cone, then problem (2.1) is equivalent to finding $\mu \in \Omega$ such that

$$g(\mu) \in \Omega, \quad \mathcal{T}\mu \in \Omega^*, \quad \langle \mathcal{T}\mu, g(\mu)\rangle = 0, \tag{2.5}$$

which is called the general complementarity problem, introduced and studied by Noor [25] in 1988. If $g = I$, then the problem 2.5 is called the nonlinear complementarity problem.

For the applications, motivations, generalization, numerical methods and other aspects of the complementarity problems in engineering and applied sciences, see [8, 24, 27, 29, 31, 35, 42, 44] and the references therein.

4. If $\Omega = \mathcal{H}$, then problem (2.1) collapses to finding $\mu \in \mathcal{H}$ such that

$$\langle \rho\mathcal{T}\mu + \mu - g(\mu), g(\nu) - \mu\rangle = 0, \quad \forall \nu \in \mathcal{H}.$$

Consequently, it follows that $\mu \in \mathcal{H}$ satisfies

$$\mu = g(\mu) - \rho\mathcal{T}\mu, \tag{2.6}$$

which is called the general equation and appears to be a new one.

5. If $\mathcal{T} = I$, then problem (2.1) reduces to finding $\mu \in \Omega$ such that

$$\langle \rho\mu, g(\nu) - \mu \rangle \geq 0, \quad \forall \nu \in \Omega. \tag{2.7}$$

is called the inverse general variational inequality and is quite different form the ones which have been considered in the literature.

For a different and appropriate choice of the operators and spaces, one can obtain several known and new classes of variational inequalities and related problems. This clearly shows that the problem (2.1) considered in this paper is more general and unifying one.

We need the following well-known definitions and results in obtaining our results.

Definition 2.1. *Let $\mathcal{T} : \mathcal{H} \longrightarrow \mathcal{H}$ be a given mapping.*

i. *The mapping $\mathcal{T}$ is called strongly monotone, if there exists a constant $\alpha \geq 0$ such that*

$$\big\langle \mathcal{T}\mu - \mathcal{T}\nu\,,\, \mu - \nu \big\rangle \geq \alpha\, \|\mu - \nu\|^2, \qquad \forall\, \mu,\, \nu \,\in\, \mathcal{H}.$$

ii. *The mapping $\mathcal{T}$ is called monotone, if*

$$\big\langle \mathcal{T}\mu - \mathcal{T}\nu\,,\, \mu - \nu \big\rangle \geq 0, \quad \mu,\, \nu \,\in\, \mathcal{H}.$$

iii. *The mapping $\mathcal{T}$ is called Lipschitz continuous, if there exists a constant $\eta > 0$ such that*

$$\|\, \mathcal{T}\mu - \mathcal{T}\nu\, \| \leq\, \eta\, \|\mu - \nu\|, \qquad \forall\, \mu,\, \nu \,\in\, \mathcal{H}.$$

The following projection result plays an indispensable role in achieving our results.

Lemma 2.1. *[12, 15] For a given $z \in \mathcal{H}$, find $\mu \in \Omega$, such that*

$$\langle \mu - z, \nu - \mu \rangle \geq 0, \quad \forall \nu \in \Omega \tag{2.8}$$

if and only if

$$\mu = P_\Omega(z),$$

where P_Ω is the projection of $\mathcal{H}$ onto the closed convex set Ω.

It is known [12, 15] that the projection operator is nonexpansive, that is,

$$\|P_\Omega(u) - P_\Omega(v)\| \leq \|u - v\|, \quad \forall u, v \in \mathcal{H}.$$

3 Dynamic Systems and Iterative Methods

In this section, we use the fixed point formulation to suggest and consider a second order projection dynamical system associated with general variational inequalities (2.1). We use this dynamical system to suggest and investigate some multi step inertial proximal methods for solving the general variational inequalities (2.1). These inertial implicit methods are constructed using the central finite difference schemes and its variant forms, which is a new novel approach. All the results in this section can be viewed as the refinement and significant improvement of the results of Noor et al. [37, 41].

We show that the general variational inequality (2.1) is equivalent to the fixed point problem by using Lemma 2.1.

Lemma 3.1. *The function $\mu \in \Omega$ is solution of variational inequality (2.1), if and only if, $\mu \in \Omega$ satisfies the relation*

$$\mu = P_{\Omega}[g(\mu) - \rho \mathcal{T}\mu], \tag{3.1}$$

where $\rho > 0$ is a constant and P_{Ω} is the projection of $\mathcal{H}$ onto the convex set Ω.

Lemma 3.1 implies that the problem (2.1) is equivalent to a fixed point problem (3.1). This alternate formulation is very useful from both numerical and theoretical point of views.

We define the residue vector $R(\mu)$ as

$$R(\mu) = P_{\Omega}[g(\mu) - \rho \mathcal{T}\mu] - \mu. \tag{3.2}$$

One can easily show that $\mu \in \Omega$ is a solution of the problem (2.1), if and only if, $\mu \in \Omega$ is the solution of the equation.

$$R(\mu) = 0. \tag{3.3}$$

We now introduce the second order dynamical system associated with the variational inequality (2.1), which is the main aim of this paper. To be more precise, we consider the problem of finding $\mu \in \mathcal{H}$ such that

$$\gamma \frac{d^2\mu}{dx^2} + \frac{d\mu}{dx} = \lambda \{ P_{\Omega}[g(\mu) - \rho \mathcal{T}\mu] - \mu \}, \quad \mu(a) = \alpha, \quad \mu(b) = \beta, \tag{3.4}$$

where $\gamma > 0, \lambda > 0$ and $\rho > 0$ are constants. We would like to emphasize that the problem (3.4) is indeed a second order boundary vale problem.

If $\gamma = 0$, then the second order dynamical systems reduces to finding $\mu \in \mathcal{H}$ such that

$$\frac{d\mu}{dt} = \lambda\big\{P_\Omega[g(\mu) - \rho\mathcal{T}\mu] - \mu\big\}, \quad \mu(t_0) = \alpha_1, \tag{3.5}$$

is an initial value problem associated with general variational inequality (2.1), where $\lambda > 0, \alpha_1$ and $\rho > 0$ are constants.

Here the right hand side is related to the projection operator and is discontinuous on the boundary. It is clear from the definition that the solution to (3.5) always stays in the constraint set. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution on the given data can be studied. All the basic concepts and results are mainly due to Noor at al. [29].

The equilibrium points of the dynamical system (3.5) are defined as follows.

Definition 3.1. *An element $\mu \in \mathcal{H}$ is an equilibrium point of the dynamical system (3.5), if, $\frac{d\mu}{dt} = 0$, that is,*

$$P_\Omega[g(\mu) - \rho\mathcal{T}\mu] - \mu = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the general variational inequality (2.1), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point. Also, we can rewrite the dymanical system (3.5) as

$$\mu = P_\Omega[g(\mu) - \rho\mathcal{T}\mu + \frac{d\mu}{dt}].$$

Definition 3.2. *The dynamical system is said to converge to the solution set S^* of (3.5), if , irrespective of the initial point, the trajectory of the dynamical system satisfies*

$$\lim_{t\to\infty} dist(\mu(t), S^*) = 0, \tag{3.6}$$

where

$$dist(\mu, S^*) = inf_{\nu\in S^*}\|\mu - \nu\|.$$

It is easy to see, if the set S^* has a unique point μ^*, then (3.6) implies that

$$\lim_{t\to\infty} \mu(t) = \mu^*.$$

If the dynamical system is still stable at μ^* in the Lyapunov sense, then the dynamical system is globally asymptotically stable at μ^*.

Definition 3.3. *The dynamical system is said to be globally exponentially stable with degree η at μ^*, if, irrespective of the initial point, the trajectory of the system satisfies*

$$\|\mu(t) - \mu^*\| \leq \eta_1\|\mu(t_0) - \mu^*\|exp(-\eta(t - t_0)), \quad \forall t \geq t_0,$$

where μ_1 and η are positive constants independent of the initial point.

It is clear that the globally exponentially stability is necessarily globally asymptotically stable and the dynamical system converges arbitrarily fast.

Lemma 3.2. *(Gronwall Lemma) [11, 20] Let $\hat{\mu}$ and $\hat{\nu}$ be real-valued nonnegative continuous functions with domain $\{t : t \leq t_0\}$ and let $\alpha(t) = \alpha_0(|t - t_0|)$, where α_0 is a monotone increasing function. If for $t \geq t_0$,*

$$\hat{\mu} \leq \alpha(t) + \int_{t_0}^{t} \hat{\mu}(s)\hat{\nu}(s)ds,$$

then

$$\hat{\mu}(s) \leq \alpha(t) exp\{\int_{t_0}^{t} \hat{\nu}(s)ds\}.$$

We now show that the trajectory of the solution of the general dynamical system (3.5) converges to the unique solution of the general variational inequality (2.1). The analysis is in the spirit of Noor [31] and Xia and Wang [55, 56].

Theorem 3.1. *Let the operators $\mathcal{T}, g : \mathcal{H} \longrightarrow \mathcal{H}$ be both Lipschitz continuous with constants $\beta > 0$ and $\mu > 0$ respectively. Then, for each $\mu_0 \in \mathcal{H}$, there exists a unique continuous solution $\mu(t)$ of the dynamical system (3.5) with $\mu(t_0) = \mu_0$ over $[t_0, \infty)$.*

Proof. Let

$$G(\mu) = \lambda P_\Omega[g(\mu) - \rho\mathcal{T}\mu] - \mu\},$$

where $\lambda > 0$ is a constant and $G(\mu) = \frac{d\mu}{dt}$. $\forall \mu, \nu \in \mathcal{H}$, we have

$$\begin{aligned} \|G(\mu) - G(\nu)\| &\leq \lambda\{\|P_\Omega[g(\mu) - \rho\mathcal{T}\mu] - P_\Omega[g(\nu) - \rho\mathcal{T}\nu]\| + \|\mu - \nu\|\} \\ &\leq \lambda\|\mu - \nu\| + \lambda\|g(\mu) - g(\nu)\| + \lambda\rho\|\mathcal{T}\mu - \mathcal{T}\nu\| \\ &\leq \lambda\{1 + \mu + \beta\rho\}\|\mu - \nu\|. \end{aligned}$$

This implies that the operator $G(\mu)$ is a Lipschitz continuous in $\mathcal{H}$, and for each $\mu_0 \in \mathcal{H}$, there exists a unique and continuous solution $\mu(t)$ of the dynamical system (3.5), defined on an interval $t_0 \leq t < \mathcal{T}_1$ with the initial condition $\mu(t_0) = \mu_0$. Let $[t_0, \mathcal{T}_1)$ be its maximal interval of existence. Then we have to show that $\mathcal{T}_1 = \infty$. Consider, for any $\mu \in \mathcal{H}$,

$$\begin{aligned} \|G(\mu)\| = \|\frac{d\mu}{dt}\| &= \lambda\|P_\Omega[g(\mu) - \rho\mathcal{T}\mu] - \mu\| \\ &\leq \lambda\{\|P_\Omega[g(\mu) - \rho\mathcal{T}\mu] - P_\Omega[0]\| + \|P_\Omega[0] - \mu\|\} \\ &\leq \lambda\{\rho\|\mathcal{T}\mu\| + \|J_\phi[u] - P_\Omega[0]\| + \|P_\Omega[0] - \mu\|\} \\ &\leq \lambda\{(\rho\beta + 1 + \eta)\|\mu\| + \|P_\Omega[0]\|\}. \end{aligned}$$

Then

$$\begin{aligned} \|\mu(t)\| &\leq \|\mu_0\| + \int_{t_0}^{t} \|\mathcal{T}\mu(s)\| ds \\ &\leq (\|\mu_0\| + k_1(t-t_0)) + k_2 \int_{t_0}^{t} \|\mu(s)\| ds, \end{aligned}$$

where $k_1 = \lambda\|J_\phi[0]\|$ and $k_2 = \lambda(\rho\beta + 1 + \mu)$. Hence by the Gronwall Lemma 3.2, we have

$$\|u(t)\| \leq \{\|\mu_0\| + k_1(t-t_0)\}e^{k_2(t-t_0)}, \quad t \in [t_0, \mathcal{T}_1).$$

This shows that the solution is bounded on $[t_0, \mathcal{T}_1)$. So $\mathcal{T}_1 = \infty$. □

Theorem 3.2. *Let the operators $\mathcal{T}, g : \mathcal{H} \longrightarrow \mathcal{H}$ be Lipschitz continuous with constants $\beta > 0$ and $\mu > 0$ respectively. If the operator $g : \mathcal{H} \longrightarrow \mathcal{H}$ is strongly monotone with constant $\gamma > 0$ and $\lambda > 0$, then the dynamical system (3.5) converges globally exponentially to the unique solution of the general variational inequality (2.1).*

Proof. Since the operators $\mathcal{T}, g$ are both Lipschitz continuous, it follows from Theorem 3.1 that the dynamical system (3.5) has unique solution $\mu(t)$ over $[t_0, \mathcal{T}_1)$ for any fixed $\mu_0 \in \mathcal{H}$. Let $\mu(t)$ be a solution of the initial value problem (3.5). For a given $\mu^* \in \mathcal{H}$ satisfying (2.1), consider the Lyapunov function

$$L(\mu) = \lambda\|\mu(t) - \mu^*\|^2, \quad \mu(t) \in \mathcal{H}. \tag{3.7}$$

From (3.5) and (3.7), we have

$$\begin{aligned} \frac{dL}{dt} &= 2\lambda\langle \mu(t) - \mu^*, P_\Omega[g(\mu(t)) - \rho\mathcal{T}\mu(t)] - \mu(t)\rangle \\ &= -2\lambda\langle \mu(t) - \mu^*, \mu(t) - \mu^*\rangle \\ &\quad +2\lambda\langle \mu(t) - \mu^*, P_\Omega[g(\mu(t)) - \rho\mathcal{T}\mu(t)] - \mu^*\rangle \\ &\leq -2\lambda\|\mu(t) - \mu^*\|^2 \\ &\quad +2\lambda\langle \mu(t) - \mu^*, P_\Omega[g(\mu(t)) - \rho\mathcal{T}\mu(t)] - \mu^*\rangle, \end{aligned} \tag{3.8}$$

where $u^* \in \mathcal{H}$ is a solution of (2.1). Thus

$$\mu^* = P_\Omega[g(\mu^*) - \rho\mathcal{T}\mu^*].$$

Using the Lipschitz continuity of the operators $\mathcal{T}, g$, we have

$$\begin{aligned} \|P_\Omega[g(\mu) - \rho\mathcal{T}\mu] - P_\Omega[g(\mu^*) - \rho\mathcal{T}\mu^*]\| &\leq \|g(\mu) - g(\mu^*) - \rho(\mathcal{T}\mu - \mathcal{T}\mu^*)\| \\ &\leq (\mu + \rho\beta)\|\mu - \mu^*\|. \end{aligned} \tag{3.9}$$

From (3.8) and (3.9), we have

$$\frac{d}{dt}\|\mu(t)-\mu^*\| \leq 2\alpha\lambda\|\mu(t)-\mu^*\|,$$

where

$$\alpha = \mu + \rho\beta\lambda.$$

Thus, for $\lambda = -\lambda_1$, where λ_1 is a positive constant, we have

$$\|\mu(t)-\mu^*\| \leq \|\mu(t_0)-\mu^*\|e^{-\alpha\lambda_1(t-t_0)},$$

which shows that the trajectory of the solution of the dynamical system (3.5) converges globally exponentially to the unique solution of the general variational inequality (2.1). □

The equilibrium point of the dynamical system (3.4) is defined as follows.

Definition 3.4. *An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (3.4), if $\gamma\frac{d^2\mu}{dx^2}+\frac{d\mu}{dx} = 0$, that is,*

$$\mu = P_\Omega[g(\mu) - \rho\mathcal{T}\mu].$$

This implies that

$$\mu = P_\Omega\big[g(\mu) - \rho\mathcal{T}\mu + \gamma\frac{d^2\mu}{dx^2} + \frac{d\mu}{dx}\big]. \tag{3.10}$$

Thus it is clear that $\mu \in \Omega$ is a solution of the variational inequality (2.1), if and only if, $\mu \in \Omega$ is an equilibrium point.

For simplicity, we take $\lambda = 1$. Thus the problem (3.4) is equivalent to finding $\mu \in \Omega$ such that

$$\gamma\ddot{\mu} + \dot{\mu} + \mu = P_\Omega\big[g(\mu) - \rho\mathcal{T}\mu\big], \quad \mu(a) = \alpha, \quad \mu(b) = \beta. \tag{3.11}$$

The problem (3.11) is called the second order projection dynamical system, which is in fact a second order boundary value problem. This interlink among various areas is fruitful from numerical analysis in developing implementable numerical methods for finding the approximate solutions of the general variational inequalities. Consequently, we can explore the ideas and techniques of the differential equations to suggest and propose hybrid proximal point methods for solving the variational inequalities and related optimization problems.

We discretize the second-order dynamical systems (3.11) using central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1}-2\mu_n+\mu_{n-1}}{h^2}+\frac{\mu_n-\mu_{n-1}}{h}+\mu_n=P_\Omega[g(\mu_n)-\rho(\mathcal{T}\mu_{n+1}], \tag{3.12}$$

where h is the step size.

If $\gamma=1, h=1$, then, from equation(3.12) we have

Algorithm 3.1. *For a given $\mu_0\in\Omega$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1}=P_\Omega[g(\mu_n)-\rho\mathcal{T}\mu_{n+1}].$$

Algorithm 3.1 is an implicit method. To implement the implicit method, we use the predictor-corrector technique to suggest the method.

Algorithm 3.2. *For given $\mu_0,\mu_1\in\Omega$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= P_\Omega[g(\mu_n)-\rho\mathcal{T}\mu_n] \\ \mu_{n+1} &= P_\Omega[g(\mu_n)-\rho\mathcal{T}y_n], \end{aligned}$$

is called the extragradient method of Korpelevich [16] for solving the general variational inequality.

Problem (3.11) can be rewritten as

$$\gamma\ddot{\mu}+\dot{\mu}+\mu=P_\Omega[g((1-\theta_n)\mu+\theta_n\mu)-\rho\mathcal{T}((1-\theta_n)\mu+\theta_n\mu)],\quad \mu(a)=\alpha,\mu(b)=\beta, \tag{3.13}$$

where $\gamma>0,\theta_n$ and $\rho>0$ are constants.

Discretising the system (3.13), we have

$$\begin{aligned} &\gamma\frac{\mu_{n+1}-2\mu_n+\mu_{n-1}}{h^2}+\frac{\mu_{n+1}-\mu_n}{h}+\mu_n \\ &=P_\Omega[g((1-\theta_n)\mu_n+\theta_n\mu_{n-1})-\rho\mathcal{T}((1-\theta_n)\mu_n+\theta_n\mu_{n-1})] \end{aligned}$$

from which, for $\gamma=0,\quad h=1$, we have

Algorithm 3.3. *For a given $\mu_0,\mu_1\in\Omega$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1}=P_\Omega[g((1-\theta_n)u_n+\theta_n\mu_{n-1})-\rho\mathcal{T}((1-\theta_n)\mu_n+\theta_n\mu_{n-1})].$$

Using the predictor corrector technique, Algorithm 3.3 can be written as

Algorithm 3.4. *For a given $\mu_0, \mu_1 \in \Omega$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n + \theta_n \mu_{n-1} \\ \mu_{n+1} &= P_\Omega[g(y_n) - \rho \mathcal{T} y_n], \end{aligned}$$

which is called the new two step inertial iterative method for solving the general variational inequality.

We discretize the second-order dynamical systems (3.11) using central finite difference and backward difference schemes to have

$$\gamma \frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h^2} + \frac{\mu_n - \mu_{n-1}}{h} + \mu_{n+1} = P_\Omega[g(\mu_n) - \rho \mathcal{T} \mu_{n+1}],$$

where h is the step size.

Using this discrete form, we can suggest the following an iterative method for solving the variational inequalities (2.1).

Algorithm 3.5. *For given $\mu_0, \mu_1 \in \Omega$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = P_\Omega[g(\mu_n) - \rho \mathcal{T} \mu_{n+1} - \frac{\gamma \mu_{n+1} - (2\gamma - h)\mu_n + (\gamma - h)\mu_{n-1}}{h^2}].$$

Algorithm 3.5 is called the inertial proximal method for solving the general variational inequalities and related optimization problems. This is a new proposed method.

We can rewrite the Algorithm 3.5 in the equivalent form as follows:

Algorithm 3.6. *For a given $\mu_0, \mu_1 \in \Omega$, compute μ_{n+1} by the iterative scheme*

$$\langle \rho \mathcal{T} \mu_{n+1} + \frac{(\gamma + h^2)\mu_{n+1} - (2\gamma - h)\mu_n + (\gamma - h)\mu_{n-1}}{h^2} - g(\mu_n), g(v) - \nu_{n+1} \rangle \geq 0, \forall v \in \Omega. \quad (3.14)$$

We note that, for $\gamma = 0, \quad h = 1$, Algorithm 3.6 reduces to the following iterative method for solving general variational inequalities (2.1).

Algorithm 3.7. *For given $\mu_0, \mu_1 \in \Omega$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = P_\Omega[g(\mu_n) - (\mu_n - \mu_{n-1}) - \rho \mathcal{T} \mu_{n+1)}].$$

We again discretize the second-order dynamical systems (3.11) using central difference scheme and forward difference scheme to suggest the following inertial proximal method for solving (2.1).

Algorithm 3.8. *For a given $\mu_0, \mu_1 \in \Omega$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = P_\Omega[g(\mu_{n+1}) - \rho \mathcal{T} \mu_{n+1} - \frac{(\gamma + h)\mu_{n+1} - (2\gamma + h)\mu_n + \gamma \mu_{n-1}}{h^2}].$$

Algorithm 3.8 is quite different from other inertial proximal methods for solving the variational inequalities.

If $\gamma = 0$, then Algorithm 3.8 collapses to:

Algorithm 3.9. *For a given $\mu_0 \in \Omega$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = P_\Omega[g(\mu_{n+1}) - \rho\mathcal{T}\mu_{n+1} - \frac{\mu_{n+1} - \mu_n}{h}].$$

Algorithm 3.8 is an proximal method for solving the variational inequalities. Such type of proximal methods were suggested by Noor [35] using the fixed point problems.

In brief, by suitable descritization of the second-order dynamical systems (3.11), one can construct a wide class of explicit and implicit method for solving inequalities.

Rewriting the problem (3.11) in the following form

$$\gamma\ddot{\mu} + \dot{\mu} + \mu = P_\Omega\big[g(\frac{\mu + \mu}{2}) - \rho\mathcal{T}(\frac{\mu + \mu}{2})\big], \tag{3.15}$$

and descretizing, taking $\lambda = 1, h = 1$, we obtain

Algorithm 3.10. *For given $\mu_0 \in \Omega$, compute the approximate solution u_{n+1} by the iterative scheme*

$$u_{n+1} = P_\Omega\big[g(\frac{\mu_n + \mu_{n+1}}{2}) - \rho\mathcal{T}(\frac{\mu_n + \mu_{n+1}}{2})\big],$$

which is an implicit iterative method. Using the predictor and corrector technique, we suggest the following two-step iterative method for solving the variational inequalities.

Algorithm 3.11. *For given $\mu_0 \in \Omega$, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= P_\Omega\big[g(\mu_n) - \rho\mathcal{T}\mu_n\big] \\ u_{n+1} &= P_\Omega\big[g(\frac{\mu_n + y_n}{2}) - \rho\mathcal{T}(\frac{\mu_n + y_n}{2})\big]. \end{aligned}$$

Algorithm 3.11 is the two step iterative method.

Clearly Algorithm 3.10 and Algorithm 3.11 are equivalent. It is enough to prove the convergence of Algorithm 3.10, which is the main motivation of our next result.

Theorem 3.3. *Let the operator T, g be Lipschitz continuous with constant $\beta > 0, \sigma > 0$, respectively. Let $u \in \Omega$ be solution of (2.1) and μ_{n+1} be an approximate solution obtained from Algorithm 3.10. If there exists a constant $\rho > 0$, such that*

$$\rho < \frac{1 - \sigma}{\beta}, \quad \sigma < 1, \tag{3.16}$$

then the approximate solution μ_{n+1} converge to the exact solution $\mu \in \Omega$.

Proof. Let $\mu \in \Omega$ be a solution of (2.1) and μ_{n+1} be the approximate solution obtained from Algorithm 3.10. Then, using the Lipschitz continuity of the operators $\mathcal{T}$ and g with constants $\beta > 0, \sigma > 0$, respectively, we obtain

$$\begin{aligned}
\|\mu_{n+1} - \mu\| &= \|P_\Omega[g(\frac{\mu_n + \mu_{n+1}}{2}) - \rho\mathcal{T}(\frac{\mu_n + \mu_{n+1}}{2})] - P_\Omega[g(\frac{\mu + \mu}{2}) - \rho\mathcal{T}(\frac{\mu + \mu}{2})]\| \\
&\leq \|g(\frac{\mu_n + \mu_{n+1}}{2}) - g(\frac{\mu + \mu}{2}) - \rho(\mathcal{T}(\frac{\mu_{n+1} + \mu_n}{2}) - \mathcal{T}(\frac{\mu + \mu}{2}))\| \\
&\leq \|g(\frac{\mu_n + \mu_{n+1}}{2}) - g(\frac{\mu + \mu}{2})\| + \rho\|\mathcal{T}(\frac{\mu_{n+1} + \mu_n}{2}) - \mathcal{T}(\frac{\mu + \mu}{2})\| \\
&\leq (\sigma + \rho\beta)\|(\frac{\mu_n + \mu_{n+1}}{2}) - (\frac{\mu + \mu}{2})\| \\
&\leq \frac{(\sigma + \rho\beta)}{2}\|\mu_{n+1} - \mu\| + \|\mu_n - \mu\|\},
\end{aligned}$$

which implies that

$$\begin{aligned}
\|\mu_{n+1} - \mu\| &\leq \frac{\sigma + \rho\beta}{2 - \sigma - \rho\beta}\|\mu_n - \mu\| \\
&= \theta\|\mu_n - \mu\|,
\end{aligned}$$

where

$$\theta = \frac{\sigma + \rho\beta}{2 - \sigma - \rho\beta}.$$

From (3.16), it implies that $\theta < 1$. This shows that the approximate solution μ_{n+1} obtained from Algorithm 3.10 converges to the exact solution $\mu \in \Omega$ satisfying the general variational inequality (2.1). $\square$

To implement the implicit Algorithm 3.10, one uses the predictor-corrector technique. Thus, we obtain new multi step step methods for solving variational inequalities.

Algorithm 3.12. *For given $\mu_0 \in \Omega$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned}
y_n &= (1 - \alpha_n)\mu_n + \alpha_n P_\Omega\big[\mu_n - \rho\mathcal{T}\mu_n\big] \\
w_n &= (1 - \eta_n)y_n + \eta_n P_\Omega\big[g(\frac{\mu_n + y_n}{2}) - \rho\mathcal{T}(\frac{\mu_n + y_n}{2})\big] \\
\mu_{n+1} &= (1 - \beta_n)w_n + \beta_n P_\Omega\big[g(\frac{\mu_n + w_n + y_n}{3}) - \rho\mathcal{T}(\frac{\mu_n + w_n + y_n}{3})\big],
\end{aligned}$$

which is a three step method, where $\alpha_n, \eta_n, \beta_n$ are constants.

Algorithm 3.13. *For given $\mu_0, \mu_1 \in \Omega$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned}
t_n &= (1-\theta_n)\mu_n + \theta_n \mu_{n-1} \\
y_n &= (1-\alpha_n)(\frac{\mu_n + t_n}{2}) + \alpha_n P_\Omega\big[g(\frac{\mu_n + t_n}{2}) - \rho \mathcal{T}(\frac{\mu_n + t_n}{2})\big] \\
w_n &= (1-\beta_n)(\frac{\mu_n + t_n + y_n}{3}) + \beta_n P_\Omega\big[g(\frac{\mu_n + t_n + y_n}{3}) - \rho \mathcal{T}(\frac{\mu_n + t_n + y_n}{3})\big] \\
\mu_{n+1} &= (1-\zeta_n)(\frac{\mu_n + t_n + y_n + w_n}{4}) \\
&\quad + \zeta_n P_\Omega\big[g(\frac{\mu_n + t_n + w_n + y_n}{4}) - \rho \mathcal{T}(\frac{\mu_n + t_n + w_n + y_n}{4})\big],
\end{aligned}$$

which is a four step inertial iterative method, where $\theta_n, \alpha_n, \beta_n, \zeta_n$ are constants.

For $g = I$, Algorithm 3.14 reduces to the following inertial multi-step iterative method for solving variational inequalities.

Algorithm 3.14. *For given $\mu_0, \mu_1 \in \Omega$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned}
t_n &= (1-\theta_n)\mu_n + \theta_n \mu_{n-1} \\
y_n &= (1-\alpha_n)(\frac{\mu_n + t_n}{2}) + \alpha_n P_\Omega\big[(\frac{\mu_n + t_n}{2}) - \rho \mathcal{T}(\frac{\mu_n + t_n}{2})\big] \\
w_n &= (1-\beta_n)(\frac{\mu_n + t_n + y_n}{3}) + \beta_n P_\Omega\big[(\frac{\mu_n + t_n + y_n}{3}) - \rho \mathcal{T}(\frac{\mu_n + t_n + y_n}{3})\big] \\
\mu_{n+1} &= (1-\zeta_n)(\frac{\mu_n + t_n + y_n + w_n}{4}) \\
&\quad + \zeta_n P_\Omega\big[(\frac{\mu_n + t_n + w_n + y_n}{4}) - \rho \mathcal{T}(\frac{\mu_n + t_n + w_n + y_n}{4})\big],
\end{aligned}$$

which is called the four step inertial iterative method for solving the variational inequalities and appears to be a new one.

Remark 3.1. *These multi-step methods contain Mann iteration, Ishikawa two-step iterations and Noor three-step iterations as special cases. Noor [27,29] has proposed and suggested three step forward-backward iterative methods for finding the approximate solution of general variational inequalities using the technique of updating the solution and auxiliary principle. These three-step methods are known as Noor iterations. Suantai et al. [52] have also considered some novel forward-backward algorithms for optimization and their applications to compressive sensing and image inpainting. For the recent applications and generalizations of the Noor iterations, see [2–5,10,21,26,39,51–53]. We have shown that these multi step iterative methods can be proposed and suggested using the dynamical systems coupled with boundary value problems, which is can be considered entirely new approach.*

4 Change of Variable Method

In this section, we consider the change of variable method for solving general variational inequalities (2.1). This technique is mainly due to Noor [27] and Noor et al. [35]. For the sake of completeness and to convey the main idea, we include some details.

We note that the complementarity problem (2.5) can be rewritten in the following form:

$$w = g(\mu) \in \Omega, \quad \nu = \mathcal{T}\mu \in \Omega^*, \quad \langle \mathcal{T}\mu, g(\mu)\rangle = 0. \tag{4.1}$$

which is useful in developing a fixed point formulation.

It is well known that, for $z \in \mathcal{H}$, we have

$$z = P_\Omega z + P_{-\Omega^*} z = P_\Omega z + P_{\Omega^*}(-z). \tag{4.2}$$

Following the idea of Noor and Al-Said [35], we consider the following change of variables

$$g(\mu) = \frac{|z| + z}{2} = z^+ = P_\Omega(z) \tag{4.3}$$

and

$$\nu = \frac{|z| - z}{2\rho} = \rho^{-1} z^-. \tag{4.4}$$

From (4.2), (4.3) and (4.4), we have

$$\mu = g^{-1} P_\Omega z \tag{4.5}$$

$$\begin{aligned} z &= z^+ - z^- = P_\Omega z + P_\omega(-z) \\ &= g(\mu) - \rho\mathcal{T} g^{-1} P_\Omega z = g(\mu) - \rho\mathcal{T}\mu. \end{aligned} \tag{4.6}$$

Combining (4.5) and (4.6), we obtain

$$\mu = P_\Omega[g(\mu) - \rho\mathcal{T}\mu]. \tag{4.7}$$

Thus, we have shown that the complementarity problem (2.5) is equivalent to the fixed point problem (4.7). This implies that $\mu \in \Omega$ is the solution of the general variational inequality (2.1). That is, $\mu \in \Omega$ satisfies the inequality

$$\langle \mathcal{T}\mu, g(\nu) - \mu\rangle \geq 0, \quad \forall \nu \in \Omega,$$

which can be rewritten equivalently as finding $\mu \in \Omega$ such that

$$\langle \mathcal{T}\mu + \mu - g(\mu), g(\nu) - \mu\rangle \geq 0, \quad \forall \nu \in \Omega,$$

which is the general variational inequality (2.4).

In recent years, this technique have been used to develop modulus based methods for solving the system of absolute value equations, which is another area in numerical analysis and optimization. This approach can be extended for solving the mixed variational inequalities, which needs further research efforts.

5 Generalizations and Future Research

We would like to mention that some of the results obtained and presented in this paper can be extended for multivalued variational inequalities. To be more precise, let $C(H)$ be a family of nonempty compact subsets of H. Let $T, V : H \longrightarrow C(H)$ be the multivalued operators. For a given nonlinear bifunction $N(.,.) : H \times H \longrightarrow H$, consider the problem of finding $u \in \Omega, w \in T(\mu), y \in V(\mu)$ such that

$$\langle N(w, y) + \mu - g(\mu), \nu - \mu\rangle \geq 0, \quad \forall \nu \in \Omega, \tag{5.1}$$

which is called the multi-valued variational inequality. We would like to mention that one can obtain various classes of variational inequalities for appropriate and suitable choices of the bifunction $N(.,.)$, and the operators.

1. For $g = I$, the identity operator, the problem (5.1) reduces to finding $u \in \Omega, w \in T(\mu), y \in V(\mu)$ such that

   $$\langle N(w, y), \nu - \mu\rangle \geq 0, \quad \forall \quad \nu \in \Omega, \tag{5.2}$$

   is called the multi-valued variational inequality, which appears to be a new one.

2. If $\Omega^* = \{\mu \in \mathcal{H} : \langle \mu, \nu\rangle \geq 0, \forall \nu \in \Omega\}$ is a polar(dual) cone, then problem (5.2) is equivalent to finding $\mu \in \Omega$ such that

   $$g(\mu) \in \Omega, \quad N(w, y) \in \Omega^*, \quad \langle N(w, y), g(\mu)\rangle = 0, \tag{5.3}$$

   which is called the multi-vlaued complementarity problem and appears to be a new one

3. If $N(w, y) = T\mu + V\mu$, then the problem (5.1) is equivalent to find $\mu \in \Omega$, such that

   $$\langle T\mu + V\mu + \mu - g(\mu), \nu - \mu\rangle \geq 0 \quad \forall \nu \in \Omega,$$

which is the general variational inequality involving the sum of two operators. Such type of variational inequalities were introduced and investigated by Noor [25] in 1975, which contain the system of absolute equations, hemivariational inequalities and their variant forms as special cases. The interested readers and researchers can explore the applications of the multivalued variational inequalities in machine learning, date analysis, information technology and finance mathematics.

Using Lemma 3.1, one can prove that the problem (5.1) is equivalent to finding $u \in \Omega$ such that

$$\mu = P_\Omega[g(\mu) - \rho N(w, y)]. \tag{5.4}$$

This show that the problem (5.1) is equivalent to the fixed point problem (5.4). This equivalent formulation is applied to consider the third order dynamical system associated with the problem (5.1) as

$$\gamma\dddot{\mu} + \zeta\ddot{\mu} + \xi\dot{\mu} + \mu = P_\Omega[g(\mu) - \rho N(w, y)], \quad u(a) = \alpha, \quad \dot{u}(a) = 0, \quad \dot{u}(b) = 0 \tag{5.5}$$

where $\gamma, \zeta, \xi, \alpha$, are constants. The problem (5.5) is the third order boundary value problem and may be applied to suggest and investigate proximal point methods for solving the multivalued variational inequality (5.1) applying the techniques developed in this paper. Consequently, all the results obtained for the problem (2.1) continue to hold for the problem (5.1) with suitable modifications and adjustments. Since the problem (5.1) and problem (5.2) are equivalent, if the convex set is a convex cone. This implies that the dynamical system approach may be exploited to solve the complementarity problems. The development of efficient implementable numerical methods for solving the multi-valued variational inequalities, random elastic traffic equilibrium problem and optimization problems requires further efforts. Despite the current research activates, very few results are available. The development of efficient implementable numerical methods for solving the general quasi variational inequalities and non optimizations problems requires further efforts.

Conclusion

In this paper, we have used the technique of the dynamical systems coupled with the second order boundary value problem to suggest some multi step inertial proximal methods for solving variational inequalities. The convergence analysis of these methods have beeb considered under some weaker conditions. Our method of convergence criteria is very simple as compared with other techniques. Comparison and implementation of these new methods need further efforts. We have only discussed the theoretical aspects of the proposed iterative methods. It is an interesting problem to discuss the implementation and performance of these new methods with other methods. Applications of the fuzzy set theory, stochastic, quantum calculus, fractal, fractional and random can be found in many branches of mathematical and engineering sciences including artificial intelligence, computer science, control engineering, management science, operations research and variational inequalities. Similar methods can be suggested for stochastic, fuzzy, quantum, random and fractional variational inequalities, which is an interesting and challenging problem. Despite the recent research activates, very few results are available. The development of efficient numerical methods requires further efforts. The ideas and techniques presented in this paper may be starting point for further developments.

Contributions of the authors:

All the authors contributed equally in writing, editing, reviewing and agreed for the final version for publication.

Data availability:

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study

Conflict of Interest:

All authors have no conflict of interest.

Acknowledgement:

The authors wish to express their deepest gratitude to their respected teachers, students, colleagues, collaborators and friends, who have directly or indirectly contributed.

References

[1] Al-Said, E. A., & Noor, M. A. (1999). An iterative scheme for generalized mildly nonlinear complementarity problems. *Applied Mathematics Letters*, *12*, 7-11. https://doi.org/10.1016/S0893-9659(99)00071-3

[2] Ashish, Rani, M., & Chugh, R. (2014). Julia sets and Mandelbrot sets in Noor orbit. *Applied Mathematics and Computation*, *228*(1), 615-631. https://doi.org/10.1016/j.amc.2013.11.077

[3] Ashish, Chugh, R., & Rani, M. (2021). *Fractals and chaos in Noor orbit: a four-step feedback approach.* Lap Lambert Academic Publishing, Saarbrücken, Germany.

[4] Ashish, Cao, J., & Noor, M. A. (2023). Stabilization of fixed points in chaotic maps using Noor orbit with applications in cardiac arrhythmia. *Journal of Applied Analysis and Computation*, *13*(5), 2452-2470. https://doi.org/10.11948/20220350

[5] Bnouhachem, A., Noor, M. A., & Rassias, T. M. (2006). Three-step iterative algorithms for mixed variational inequalities. *Applied Mathematics and Computation*, *183*, 436-446. https://doi.org/10.1016/j.amc.2006.05.086

[6] Chairatsiripong, C., & Thianwan, T. (2022). Novel Noor iterations technique for solving nonlinear equations. *AIMS Mathematics*, *7*(6), 10958-10976. https://doi.org/10.3934/math.2022612

[7] Cho, S. Y., Shahid, A. A., Nazeer, W., & Kang, S. M. (2006). Fixed point results for fractal generation in Noor orbit and s-convexity. *SpringerPlus*, *5*, 1843. https://doi.org/10.1186/s40064-016-3530-5

[8] Cottle, R. W., Pang, J.-S., & Stone, R. E. (2009). *The linear complementarity problem*. SIAM Publications. https://doi.org/10.1137/1.9780898719000

[9] Cristescu, G., & Lupsa, L. (2002). *Non-connected convexities and applications*. Kluwer Academic Publishers, Dordrecht, Holland. https://doi.org/10.1007/978-1-4615-0003-2

[10] Cristescu, G., & Miauna, G. (2009). Shape properties of Noor's g-convex sets. *Proceedings of the Twelfth Symposium of Mathematical Applications, Timisoara, Romania*, 1-13.

[11] Dupuis, P., & Nagurney, A. (1993). Dynamical systems and variational inequalities. *Annals of Operations Research*, *44*, 7-42. https://doi.org/10.1007/BF02073589

[12] Glowinski, R., Lions, J. L., & Tremolieres, R. (1981). *Numerical analysis of variational inequalities*. North-Holland, Amsterdam.

[13] Glowinski, R., & Le Tallec, P. (1989). *Augmented Lagrangian and operator splitting methods in nonlinear mechanics*. SIAM, Philadelphia, Pennsylvania, USA. https://doi.org/10.1137/1.9781611970838

[14] Khan, A. G., Noor, M. A., & Noor, K. I. (2015). Dynamical systems for general quasi variational inequalities. *Annals of Functional Analysis*, *6*(1), 193-209. https://doi.org/10.15352/afa/06-1-14

[15] Kinderlehrer, D., & Stampacchia, G. (2000). *An introduction to variational inequalities and their applications*. SIAM, Philadelphia, PA, USA. https://doi.org/10.1137/1.9780898719451

[16] Korpelevich, G. M. (1976). The extragradient method for finding saddle points and other problems. *Ekonomika i Matematicheskie Metody*, *12*, 747-756.

[17] Kwuni, Y. C., Shahid, A. A., Nazeer, W., Butt, S. I., Abbas, M., & Kang, S. M. (2019). Tricorns and multicorns in Noor orbit with s-convexity. *IEEE Access*, *7*. https://doi.org/10.1109/ACCESS.2019.2928796

[18] Lions, J., & Stampacchia, G. (1967). Variational inequalities. *Communications on Pure and Applied Mathematics*, *20*, 493-519. https://doi.org/10.1002/cpa.3160200302

[19] Nagurney, A., & Zhang, D. (1996). *Projected dynamical systems and variational inequalities with applications*. Kluwer Academic Publishers, Boston, Dordrecht, London. https://doi.org/10.1007/978-1-4615-2301-7_2

[20] Nammanee, K., Noor, M. A., & Suantai, S. (2006). Convergence criteria of modified Noor iterations with errors for asymptotically nonexpansive mappings. *Journal of Mathematical Analysis and Applications*, *314*, 320-334. https://doi.org/10.1016/j.jmaa.2005.03.094

[21] Natarajan, S. K., & Negi, D. (2024). Green innovations utilizing fractal and power for solar panel optimization. In R. Sharma, G. Rana, & S. Agarwal (Eds.), *Green Innovations for Industrial Development and Business Sustainability* (pp. 146-152). CRC Press, Florida, Boca Raton, USA. https://doi.org/10.1201/9781003458944-10

[22] Niculescu, C. P., & Persson, L. E. (2018). *Convex functions and their applications*. Springer-Verlag, New York. https://doi.org/10.1007/978-3-319-78337-6_1

[23] Noor, M. A. (1975). *On variational inequalities* (PhD thesis). Brunel University, London, U.K.

[24] Noor, M. A. (1998). The quasi-complementarity problem. *Journal of Mathematical Analysis and Applications*, *130*, 344-353. https://doi.org/10.1016/0022-247X(88)90310-1

[25] Noor, M. A. (1988). General variational inequalities. *Applied Mathematics Letters*, *1*, 119-121. https://doi.org/10.1016/0893-9659(88)90054-7

[26] Noor, M. A. (1988). Quasi variational inequalities. *Applied Mathematics Letters*, *1*(4), 367-370. https://doi.org/10.1016/0893-9659(88)90152-8

[27] Noor, M. A. (1988). Fixed point approach for complementarity problems. *Journal of Mathematical Analysis and Applications*, *33*, 437-448. https://doi.org/10.1016/0022-247X(88)90413-1

[28] Noor, M. A. (2000). Variational inequalities for fuzzy mappings (III). *Fuzzy Sets and Systems*, *110*(1), 101-108. https://doi.org/10.1016/S0165-0114(98)00131-6

[29] Noor, M. A. (2000). New approximation schemes for general variational inequalities. *Journal of Mathematical Analysis and Applications*, *251*, 217-230. https://doi.org/10.1006/jmaa.2000.7042

[30] Noor, M. A. (2002). A Wiener-Hopf dynamical system and variational inequalities. *New Zealand Journal of Mathematics*, *31*, 173-182.

[31] Noor, M. A. (2004). Some developments in general variational inequalities. *Applied Mathematics and Computation*, *152*, 199-277. https://doi.org/10.1016/S0096-3003(03)00558-7

[32] Noor, M. A. (2001). Three-step iterative algorithms for multivalued quasi variational inclusions. *Journal of Mathematical Analysis and Applications*, *255*(2), 589-604. https://doi.org/10.1006/jmaa.2000.7298

[33] Noor, M. A. (2009). Extended general variational inequalities. *Applied Mathematics Letters*, *22*, 182-185. https://doi.org/10.1016/j.aml.2008.03.007

[34] Noor, M. A. (2008). Differentiable non-convex functions and general variational inequalities. *Applied Mathematics and Computation*, *199*(2), 623-630. https://doi.org/10.1016/j.amc.2007.10.023

[35] Noor, M. A., & Al-Said, E. A. (1999). Change of variable method for generalized complementarity problems. *Journal of Optimization Theory and Applications*, *100*(2), 389-395. https://doi.org/10.1023/A:1021790404792

[36] Noor, M. A., & Noor, K. I. (2022). Dynamical system technique for solving quasi variational inequalities. *U.P.B. Scientific Bulletin, Series A*, *84*(4), 55-66. https://doi.org/10.34198/ejms.10122.166

[37] Noor, M. A., & Noot, K. I. (2023). Iterative schemes for solving general variational inequalities. *Differential Equations and Applications*, *15*(2), 113-134. https://doi.org/10.7153/dea-2023-15-07

[38] Noor, M. A., Noor, K. I., & Khan, A. G. (2015). Dynamical systems for quasi variational inequalities. *Annals of Functional Analysis*, *6*(1), 193-209. https://doi.org/10.15352/afa/06-1-14

[39] Noor, M. A., & Noor, K. I. (2024). Some novel aspects and applications of Noor iterations and Noor orbits. *Journal of Advanced Mathematical Studies*, *17*(3), 276-284.

[40] Noor, M. A., & Noor, K. I. (2025). General harmonic biconvex directional variational inequalities. *Journal of Advanced Mathematical Studies*, *18*(2), (in press).

[41] Noor, M. A., & Noor, K. I. (2025). Second-order dynamical systems technique for general variational inequalities. *Journal of Advanced Mathematical Studies*, *18*(1), 1-17.

[42] Noor, M. A., Noor, K. I., & Rassias, M. T. (2020). New trends in general variational inequalities. *Acta Applicandae Mathematicae*, *170*(1), 981-1046. https://doi.org/10.1007/s10440-020-00366-2

[43] Noor, M. A., Noor, K. I., & Rassias, Th. M. (1993). Some aspects of variational inequalities. *Journal of Computational and Applied Mathematics*, *47*, 285-312. https://doi.org/10.1016/0377-0427(93)90058-J

[44] Noor, M. A., & Oettlie, W. (1994). On general nonlinear complementarity problems. *Le Matematiche*, *49*, 313-331.

[45] Paimsang, S., Yambangwai, D., & Thianwan, T. (2024). A novel Noor iterative method of operators with property (E) as concerns convex programming applicable in signal recovery and polynomiography. *Mathematical Methods in the Applied Sciences*, 1-18. https://doi.org/10.1002/mma.10083

[46] Patriksson, M. (1998). *Nonlinear programming and variational inequalities: a unified approach*. Kluwer Academic Publishers, Dordrecht.

[47] Phuengrattana, W., & Suantai, S. (2011). On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval. *Journal of Computational and Applied Mathematics*, *235*(9), 3006-3014. https://doi.org/10.1016/j.cam.2010.12.022

[48] Rattanaseeha, K., Imnang, S., Inkrong, P., & Thianwan, T. (2023). Novel Noor iterative methods for mixed-type asymptotically nonexpansive mappings from the perspective of convex programming in hyperbolic spaces. *International Journal of Innovative Computing Information and Control*, *19*(6), 1717-1734.

[49] Sanaullah, K., Ullah, S., & Aloraini, N. M. (2024). A self-adaptive three-step numerical scheme for variational inequalities. *Axioms*, *13*, 57. https://doi.org/10.3390/axioms13010057

[50] Stampacchia, G. (1964). Formes bilinéaires coercitives sur les ensembles convexes. *Comptes Rendus de l'Académie des Sciences de Paris*, *258*, 4413-4416.

[51] Suantai, S. (2005). Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings. *Journal of Mathematical Analysis and Applications*, *331*, 506-517. https://doi.org/10.1016/j.jmaa.2005.03.002

[52] Suantai, S., Noor, M. A., Kankam, K., & Cholamjiak, P. (2021). Novel forward-backward algorithms for optimization and applications to compressive sensing and image inpainting. *Advances in Difference Equations*, *2021*, ID-265. https://doi.org/10.1186/s13662-021-03422-9

[53] Tassaddiq, A. (2022). General escape criteria for the generation of fractals in extended Jungck-Noor orbit. *Mathematics and Computers in Simulation*, *196*, 1-14. https://doi.org/10.1016/j.matcom.2022.01.003

[54] Tseng, P. (2000). A modified forward-backward splitting method for maximal monotone mappings. *SIAM Journal on Control and Optimization*, *38*, 431-446. https://doi.org/10.1137/S0363012998338806

[55] Xia, Y. S., & Wang, J. (2000). A recurrent neural network for solving linear projection equations. *Neural Networks*, *13*, 337-350. https://doi.org/10.1016/S0893-6080(00)00019-8

[56] Xia, Y. S., & Wang, J. (2000). On the stability of globally projected dynamical systems. *Journal of Optimization Theory and Applications*, *106*, 129-150. https://doi.org/10.1023/A:1004611224835

[57] Yadav, A., & Jha, K. (2016). Parrondo's paradox in the Noor logistic map. *International Journal of Advanced Research in Engineering Technology*, *7*(5), 1-6.