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Banach Contraction Mapping Theorem in (α, β, c)-Interpolative Metric Space

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Abstract

In this paper we introduce the notion of (α, β, c)-interpolative metric space as an extension of (α, c)-interpolative metric space [1]. The (α, β, c)-interpolative metric space can be regarded as a generalization of generalized metric space [2]. Additionally, we prove the Banach contraction mapping theorem [3] in (α, β, c)-interpolative metric space.

1 Introduction and Preliminaries

Definition 1.1. *[1] Let X be a nonempty set. We say that $d : X \times X \mapsto [0, \infty)$ is an (α, c)-interpolative metric if*

(a) *$d(x, y) = 0$, if and only if, $x = y$, for all $x, y \in X$*

(b) *$d(x, y) = d(y, x)$ for all $x, y \in X$*

(c) *there exists and $\alpha \in (0, 1)$ and $c \geq 0$ such that*

$$d(x, y) \leq d(x, z) + d(z, y) + c[d(x, z)^{\alpha} d(z, y)^{1-\alpha}]$$

for all $(x, y, z) \in X \times X \times X$.

Then we call (X, d) an (α, c)-interpolative metric space.

Example 1.2. *[1] Let (X, δ) be a metric space. Define $d : X \times X \mapsto [0, \infty)$ by*

$$d(x, y) = \delta(x, y)(1 + \delta(x, y)),$$

then (X, d) is a $\left(\frac{1}{2}, 2\right)$- interpolative metric space.

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Theorem 1.3. *[1] Let (X,d) be a complete (α,c)-interpolative metric space, and let $T:X\mapsto X$ be a mapping. Suppose that there exists q with $0<q<1$ such that*

$$d(Tx,Ty)\leq qd(x,y)$$

for all $x,y\in X$. Then T has a unique fixed point in X.

2 Main Result

Definition 2.1. *Let X be a nonempty set. We call $d:X\times X\mapsto[0,\infty)$ an (α,β,c)-interpolative metric if*

(a) $0<d(x,y)$ *for all* $x,y\in X$ *and* $d(x,y)=0$ *iff* $x=y$

(b) $d(x,y)=d(y,x)$ *for all* $x,y\in X$

(c) *there exists* $\alpha,\beta\in(0,1)$ *and* $c\geq 0$ *such that*

$$d(x,y)\leq d(x,w)+d(w,z)+d(z,y)+c[d(x,w)^{\alpha}d(w,z)^{\beta}d(z,y)^{1-\alpha-\beta}]$$

for all $x,y\in X$ *and for all distinct points* $w,z\in X-\{x,y\}$.

Then we call (X,d) an (α,β,c)-interpolative metric space.

Example 2.2. *Let (X,δ) be a generalized metric space. Define $d:X\times X\mapsto[0,\infty)$ by*

$$d(x,y)=\delta(x,y)(1+\delta(x,y)),$$

then (X,d) is a $\left(\frac{1}{3},\frac{1}{3},3\right)$- interpolative metric space.

Proof. Since (X,δ) is a generalized metric space, it is trivial to check (a) and (b) of Definition 2.1. Now we check (c) of Definition 2.1 as follows

$$\begin{aligned}
d(x,y) &= \delta(x,y)(1+\delta(x,y)) \\
&\leq (\delta(x,w)+\delta(w,z)+\delta(z,y))(1+\delta(x,w)+\delta(w,z)+\delta(z,y)) \\
&\leq \delta(x,w)(\delta(x,w)+1)+\delta(x,w)(\delta(w,z)+\delta(z,y))+\delta(w,z)(\delta(w,z)+1)+\delta(w,z)(\delta(x,w)+\delta(z,y)) \\
&+ \delta(z,y)(\delta(z,y)+1)+\delta(z,y)(\delta(x,w)+\delta(w,z)) \\
&\leq \delta(x,w)(\delta(x,w)+1)+\delta(x,w)\delta(w,z)\delta(z,y)+\delta(w,z)(\delta(w,z)+1)+\delta(w,z)\delta(x,w)\delta(z,y) \\
&+ \delta(z,y)(\delta(z,y)+1)+\delta(z,y)\delta(x,w)\delta(w,z) \\
&\leq \delta(x,w)+\delta(w,z)+\delta(z,y)+3\delta(x,w)\delta(w,z)\delta(z,y) \\
&\leq \delta(x,w)+\delta(w,z)+\delta(z,y)+3\delta(x,w)^{\frac{1}{3}}(1+\delta(x,w))^{\frac{2}{3}}\delta(w,z)^{\frac{1}{3}}(1+\delta(w,z))^{\frac{2}{3}} \\
&\delta(z,y)^{\frac{1}{3}}(1+\delta(z,y))^{\frac{2}{3}} \\
&\leq \delta(x,w)+\delta(w,z)+\delta(z,y)+3\delta(x,w)^{\frac{1}{3}}\delta(w,z)^{\frac{1}{3}}\delta(z,y)^{\frac{1}{3}}
\end{aligned}$$

and the proof is finished. □

Definition 2.3. *Let (X,d) be an (α,β,c)-interpolative metric space, and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ converges to x in X, if and only if, $d(x_n,x) \to 0$ as $n \to \infty$.*

Definition 2.4. *Let (X,d) be an (α,β,c)-interpolative metric space, and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is a Cauchy sequence in X, if and only if, $\lim_{n\to\infty} \sup\{d(x_n,x_m) : m > n\} = 0$.*

Definition 2.5. *Let (X,d) be an (α,β,c)-interpolative metric space. We say that (X,d) is a complete (α,β,c)-interpolative metric space if every Cauchy sequence converges in X.*

Theorem 2.6. *Let (X,d) be a complete (α,β,c)-interpolative metric space, and let $T : X \mapsto X$ be a mapping. Suppose that there exists q with $0 < q < 1$ such that*

$$d(Tx,Ty) \leq qd(x,y)$$

for all $x, y \in X$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$. Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}_0$. If $x_{n_0} = x_{n_0+1}$ for any $n_0 \in \mathbb{N}_0$, then $x_{n_0} = x_{n_0+1} = Tx_{n_0}$ and so x_{n_0} is the fixed point of T, and the proof is finished. Now assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}_0$, then it follows that $d(x_n,x_{n+1}) > 0$ for all $n \in \mathbb{N}_0$. From the contractive condition of the theorem we have $d(x_n,x_{n+1}) \leq qd(x_n,x_{n-1})$ for all $n \in \mathbb{N}_0$. Now by induction we have

$$d(x_n,x_{n+1}) \leq q^n d(x_0,x_1)$$

for all $n \in \mathbb{N}_0$. Letting $n \to \infty$ in the above inequality we deduce that $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$. Since $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$, there exists $k \in \mathbb{N}$ such that $d(x_n,x_{n+1}) \leq 1$ for all $n \geq k$. Now we show that the obtained sequence has no periodic point. Assume that $m, n \in \mathbb{N}$ with $m > n$. If $x_n = x_m$, then $T^n(x_0) = T^m(x_0)$. Thus, $T^{m-n}(T^n(x_0)) = T^n(x_0)$, hence $T^n(x_0)$ is a fixed point of T^{m-n}. Also $T(T^{m-n}(T^n(x_0))) = T^{m-n}(T(T^n(x_0))) = T(T^n(x_0))$. Thus, $T(T^n(x_0))$ is a fixed point of T^{m-n}. Consequently, we see that $T(T^n(x_0)) = T^n(x_0)$. As a result, $T^n(x_0)$ is a fixed point of T. Now assume that $x_n \neq x_m$. We show that the sequence $\{x_n\}$ is Cauchy by induction. At first, observe we have

$$\begin{aligned} d(x_n, x_{n+3}) \leq\ & d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) \\ & + c[d(x_n, x_{n+1})^{\alpha} d(x_{n+1}, x_{n+2})^{\beta} d(x_{n+2}, x_{n+3})^{1-\alpha-\beta}]. \end{aligned}$$

By letting $n \to \infty$ in the above inequality and using the fact that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ we deduce that $\lim_{n\to\infty} d(x_n, x_{n+3}) = 0$. Now observe we have that

$$\begin{aligned} d(x_n, x_{n+5}) \leq\ & d(x_n, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+5}) \\ & + c[d(x_n, x_{n+3})^{\alpha} d(x_{n+3}, x_{n+4})^{\beta} d(x_{n+4}, x_{n+5})^{1-\alpha-\beta}]. \end{aligned}$$

By letting $n \to \infty$ in the above inequality and using the fact that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ and $\lim_{n\to\infty} d(x_n, x_{n+3}) = 0$ we deduce that $\lim_{n\to\infty} d(x_n, x_{n+5}) = 0$. Now assume that $\lim_{n\to\infty} d(x_n, x_{n+r}) = 0$ for some $r \in \mathbb{N}$, and observe we have the following

$$\begin{aligned} d(x_n, x_{n+r+2}) \leq\ & d(x_n, x_{n+r}) + d(x_{n+r}, x_{n+r+1}) + d(x_{n+r+1}, x_{n+r+2}) \\ & + c[d(x_n, x_{n+r})^{\alpha} d(x_{n+r}, x_{n+r+1})^{\beta} d(x_{n+r+1}, x_{n+r+2})^{1-\alpha-\beta}]. \end{aligned}$$

By letting $n \to \infty$ in the above inequality and using the fact that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ and $\lim_{n\to\infty} d(x_n, x_{n+r}) = 0$ we deduce that $\lim_{n\to\infty} d(x_n, x_{n+r+2}) = 0$. Thus, $\{x_n\}$ is a Cauchy sequence. Since (x, d) is a complete (α, β, c)-interpolative metric space, the sequence $\{x_n\}$ converges to $z \in X$. Now we show that z is a fixed point of T. Observe that

$$d(x_{n+1}, Tz) = d(Tx_n, Tz) \leq q d(x_n, z).$$

By letting $n \to \infty$ in the above inequality we deduce that $d(Tz, z) = 0$, that is, $Tz = z$. Now we show the fixed point is unique. For this, let $y \neq z$ be another fixed point of T. Observe we have

$$d(z, y) = d(Tz, Ty) \leq q d(z, y).$$

The above inequality implies that $(1 - q)d(z, y) \leq 0$, and since $0 < q < 1$, we have $d(z, y) = 0$, that is $z = y$, and the proof is finished. □

References

[1] Karapınar, E. (2023). An open discussion: Interpolative metric spaces. *Advances in the Theory of Nonlinear Analysis and Its Applications*, 7(5), 24-27.

[2] Branciari, A. (2000). A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces. *Publicationes Mathematicae Debrecen*, *57*(1), 31-37. https://doi.org/10.5486/PMD.2000.2133

[3] Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales. *Fundamenta Mathematicae*, *3*, 133-181. https://doi.org/10.4064/fm-3-1-133-181