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On the Structures of a Class of Finite Rings of Five Radical Zero Completely Primary of Characteristic p

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Abstract

The classification of finite rings still remains an open problem. For finite rings with identity, attempts have been made to classify them through their units or zero divisors. This study has followed the same trend, where the order structures of the units and zero divisors of a class of five radical zero completely primary finite rings have been precisely discovered.

1 Introduction

Let R be a completely primary finite ring. An automorphism of R is a map that preserves its inherent structures. The automorphism group of R^*, $Aut(R^*)$, is a set whose elements are automorphisms $\Phi : R^* \to R^*$ with group operation given by the composition of automorphisms. It is well known that $Aut(R^*)$ provides treasured information about the structure of R. Alkhamees [2,3] determined the group of automorphisms of finite rings in which the product of any two zero divisors is zero, for both characteristics p and p^2, as well as for both commutative and non- commutative cases in completely primary finite rings. In [10], Hiller and Rhea determined the automorphisms of direct products of finite rings, in which they generated a useful description of the automorphism group of an arbitrary finite group and further obtained the size of the automorphism group. Chikunji [8, 9] determined the structure of the group of automorphisms, $Aut(R)$, of cube radical zero completely primary finite rings of characteristics p, p^2 and p^3. Owino [16] explicitly gave a description of the group of automorphisms of certain class of completely primary finite rings satisfying some special properties. In [14, 15] , Ojiema et al. characterized the automorphisms of the unit groups of both square radical zero as well as power four radical zero completely primary finite rings.

An automorphism h of a graph $\Gamma(R)$ is a bijective mapping $h : \Gamma(R) \to \Gamma(R)$ which preserves both adjacency and non-adjacency. The set of all graph automorphisms of $\Gamma(R)$ forms a group under the usual composition of functions. For a graph $\Gamma(R)$ with p^n vertices, $Aut\,(\Gamma(R)) \cong S_{p^n}$ if and only if $\Gamma(R) = K_{p^n}$.

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Research on zero divisor graphs of completely primary finite rings has attracted much attention, especially those defined by Anderson and Livingstone [6]. Scanty literature has addressed research on the automorphisms of zero divisor graphs. In [17], Owino determined the automorphisms of zero divisor graphs of Galois rings, $R_0 = GR\left(p^{kr}, p^k\right)$, of order p^{kr} and of characteristic p^k for $k \geq 2$. Lao et al. [11,12] characterized the automorphisms of the zero divisor graphs of square radical zero, and power four radical zero completely primary finite rings. Were and Oduor [20] determined graph invariants including the diameter, girth and binding number, as well as graph characteristics including connectedness, completeness, and partiteness of five radical zero commutative completely primary finite rings.

The equivalence class of x, $[x] = \{y : y \sim x, y \, \text{is a set of vertices}\}$. Clearly, $x \in [x]$. The graph of equivalence classes of zero divisors of a ring R, denoted by $\Gamma_E(R)$ is the graph associated to R whose vertices are the classes of the elements in $Z(R)^*$, with each pair of distinct classes $[x]$,$[y]$ joined by an edge if and only if $[x].[y] = 0$. The equivalence classes partition the set $Z\left(R\right)^*$ and the graph of $\Gamma_E(R)$ has vertex set as $Z\left(R\right)^* \big/ \sim$.

Two distinct vertices $x, y \in Z\left(R\right)^*$ are adjacent if and only if $xy = 0$ and are equivalent $(x \sim y)$ if $ann_R(x) = ann_R(y)$. Thus $[x], [y] \in Z\left(R\right)^* \big/ \sim$ are adjacent in $\Gamma_E(R)$ if and only if x is adjacent to y in $\Gamma(R)$.

Compressed zero divisor graphs for commutative ring R with identity $1 \neq 0$ were first introduced by Mulay [13] and they have been explicitly studied in [4–7,18,19]. Some characterizations of the compressed zero divisor graphs of the Galois rings as well as their idealizations have been exposed in [1,17].

Throughout this paper, R denotes a class of five radical zero commutative completely primary finite rings; $Z\left(R\right)$ denotes the set of all zero divisors (including zero); $Z\left(R\right)^*$ denotes the set of nonzero zero divisors; $\Gamma(R)$ denotes the graph of the ring R; $\Gamma_E(R)$ denotes the graph of equivalent vertices in R; $b\left(\Gamma_E(R)\right)$ denotes the binding number of $\Gamma_E(R)$; $gr\left(\Gamma_E(R)\right)$ denotes the girth of $\Gamma_E(R)$; $diam\left(\Gamma_E(R)\right)$ denotes the diameter of $\Gamma_E(R)$; $Aut(R^*)$ denotes the automorphisms of the unit group R^* of the ring R; $\left|Aut(R^*)\right|$ denotes the order of the automorphisms of the unit group R^*; and $CharR$ denotes the characteristic of the ring R.

The rest of this paper is organized as follows. In Section 2, important results that are useful in this work are given. The construction of five radical zero completely primary finite rings whose structures of the unit groups and zero divisor graphs are to be considered is also given. In Section 3, a characterization of the structures and orders of the automorphisms of the described class of rings is given. Finally, Section 4 concludes the research and recommends areas that could be further researched.

2 Preliminaries

The following results are useful in this paper and their proofs can be obtained from the cited references.

This work also refers to the given specific case of a general construction of five radical zero completely primary finite rings of characteristic p. For the details of general background of this construction, the reader is referred to ([21], section 3.1).

For any prime integer p and a positive integer r, let $R_0 = GR(p^r, p)$ be a Galois ring of order p^r and characteristic p. Suppose U, V, W and Y are finitely generated R_0 - modules such that $dim_{R_0}U = e$, $dim_{R_0}V = f$, $dim_{R_0}W = g$ and $dim_{R_0}Y = h$. Let $\{u_1, \cdots, u_e\}$, $\{v_1, \cdots, v_f\}$, $\{w_1, \cdots, w_g\}$, and $\{y_1, \cdots, y_h\}$ be the generators of U, V, W and Y respectively so that $R = R_0 \oplus U \oplus V \oplus W \oplus Y$ is an additive abelian group. On the additive group R , we define multiplication by the following relations:

$u_iu_{i'} = u_{i'}u_i = v_j$, $u_iv_j = v_ju_i = w_k$, $u_iw_k = w_ku_i = y_l$, $u_iy_l = y_lu_i = 0$, $v_jv_{j'} = v_{j'}v_j = y_l$, $v_jw_k = w_kv_j = 0$, $v_jy_l = y_lv_j = 0$, $w_kw_{k'} = w_{k'}w_k = 0$, $w_ky_l = y_lw_k = 0$, $y_ly_{l'} = y_{l'}y_l = 0$.

Further $u_iu_{i'}u_{i''}u_{i'''}u_{i^{iv}} = 0$, $u_ir_0 = r_0u_i$, $v_jr_0 = r_0v_j$, $w_kr_0 = r_0w_k$, $y_lr_0 = r_0y_l$, where $r_0 \in R_0$, $1 \le i, i^{'} \le e$, $1 \le j, j^{'} \le f$, $1 \le k, k^{'} \le g$ and $1 \le l, l^{'} \le h$.

From the given multiplication in R, we see that if

$$\left(r_0, \sum_{i=1}^{e} r_iu_i, \sum_{j=1}^{f} s_jv_j, \sum_{k=1}^{g} t_kw_k, \sum_{l=1}^{h} z_ly_l\right) \text{ and } \left(r_0^{'}, \sum_{i=1}^{e} r_i^{'}u_i, \sum_{j=1}^{f} s_j^{'}v_j, \sum_{k=1}^{g} t_k^{'}w_k, \sum_{l=1}^{h} z_l^{'}y_l\right)$$

are any two elements of R, then

$$\left(r_0, \sum_{i=1}^{e} r_iu_i, \sum_{j=1}^{f} s_jv_j, \sum_{k=1}^{g} t_kw_k, \sum_{l=1}^{h} z_ly_l\right)\left(r_0^{'}, \sum_{i=1}^{e} r_i^{'}u_i, \sum_{j=1}^{f} s_j^{'}v_j, \sum_{k=1}^{g} t_k^{'}w_k, \sum_{l=1}^{h} z_l^{'}y_l\right)$$
$$= \Bigg(r_0r_0^{'},\ \sum_{i=1}^{e}\left[r_0r_i^{'} + r_0^{'}r_i\right]u_i,\ \sum_{j=1}^{f}\left[(r_0)s_j^{'} + s_j(r_0^{'}) + \sum_{\mu,\nu}\left(r_\mu r_\nu^{'}\right)\right]v_j,\ \sum_{k=1}^{g}\Bigg[(r_0)\,t_k^{'} + t_k\left(r_0^{'}\right) +$$
$$\sum_{i,j}(r_i)\,s_j^{'} + s_j\left(r_i^{'}\right)\Bigg]w_k,\ \sum_{l=1}^{h}\left[(r_0)\,z_l^{'} + z_l\left(r_0^{'}\right) + \sum_{i,k}\left(r_it_k^{'} + t_kr_i^{'}\right) + \sum_{\kappa,\gamma}\left(s_\kappa s_\gamma^{'}\right)\right]y_l\Bigg)$$

It is well known that this multiplication turns R into a unital commutative completely primary finite ring.

Theorem 2.1. *([21], Proposition 2.1) Let R be the ring described by the above construction with $pu_i = 0$, $pv_j = 0$, $pw_k = 0$ and $py_l = 0$. Then its group of units is given by*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \left(\mathbb{Z}_{2^3}^r\right)^e \times \left(\mathbb{Z}_2^r\right)^g & ,\ p = 2 \\ \mathbb{Z}_{3^r-1} \times \left(\mathbb{Z}_{3^2}^r\right)^e \times \left(\mathbb{Z}_3^r\right)^f \times \left(\mathbb{Z}_3^r\right)^h & ,\ p = 3 \\ \mathbb{Z}_{p^r-1} \times \left(\mathbb{Z}_p^r\right)^e \times \left(\mathbb{Z}_p^r\right)^f \times \left(\mathbb{Z}_p^r\right)^g \times \left(\mathbb{Z}_p^r\right)^h & ,\ p \ge 5 \end{cases}$$

Theorem 2.2. *([20], Proposition 2.4) The ring R constructed above is completely primary of characteristic p with Jacobson radical $Z(R)$:*

$$Z(R) = \sum_{i=1}^{e} R_0 u_i \oplus \sum_{j=1}^{f} R_0 v_j \oplus \sum_{k=1}^{g} R_0 w_k \oplus \sum_{l=1}^{h} R_0 y_l$$

$$\begin{aligned}
(Z(R))^2 &= \sum_{j=1}^{f} R_0 v_j \oplus (Z(R))^3 \\
(Z(R))^3 &= \sum_{k=1}^{g} R_0 w_k \oplus (Z(R))^4 \\
(Z(R))^4 &= \sum_{l=1}^{h} R_0 y_l \\
(Z(R))^5 &= 0
\end{aligned}$$

Theorem 2.3. *([10], Lemma 2.1) Let H and K be finite groups with relatively prime orders. Then $Aut(H) \times Aut(K) \cong Aut(H \times K)$.*

Theorem 2.4. *([10], Theorem 3.3) The map $\Psi : R_p \to End\,(1 + Z(R))$ given by $\Psi\,(A)\,(x_1, \cdots, x_n)^T = \phi\left(A\,(x_1, \cdots, x_n)^T\right)$ is a surjective ring homomorphism, where the matrix $A = (a_{\mu\nu}) \in R_p$ and $\phi : \mathbb{Z}^n \to (1 + Z(R))$ is a homomorphism given by $\phi\,(x_1, \cdots, x_n)^T = (\phi(x_1), \cdots, \phi(x_n))^T = (x_1, \cdots, x_n)^T$.*

Theorem 2.5. *([10], Theorem 3.6) An endomorphism $M = \Psi\,(A)$ is an automorphism if and only if $A\,(modp) \in GL_n\,(\mathbb{F}_p)$.*

Theorem 2.6. *([15], Proposition 8) Let rank $(1 + Z(R)) = n$ and $1 + Z(R) = \mathbb{Z}_p^n$. Then the number of elements in $GL_n\,(\mathbb{F}_p)$ is $\prod_{k=0}^{n-1}\left(p^n - p^k\right)$.*

Theorem 2.7. *([10], Theorem 4.1) The Abelian group $H_p = \mathbb{Z}/p^{e_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{e_n}\mathbb{Z}$ has*

$$\left|Aut(H_p)\right| = \prod_{k=1}^{n}\left(p^{d_k} - p^{k-1}\right)\prod_{j=1}^{n}(p^{e_j})^{n-d_j}\prod_{i=1}^{n}(p^{e_i-1})^{n-c_i+1}.$$

Definition 2.1. *([10], Definition 3.1) The set of rings of matrices describing the representation of the endomorphisms of $1 + Z(R)$ is defined as $R_p = \{(a_{\mu\nu}) \in \mathbb{Z}_{n\times n} : p^{\tau_\mu - \tau_\nu} | a_{\mu\nu}, 1 \le \nu \le \mu \le n\}$ such that*

$$a_{\mu\nu} = \begin{cases} a_{\mu\nu}, & \mu \le \nu \\ p^{\tau_\mu - \tau_\nu} a_{\mu\nu}, & \mu > \nu \end{cases}$$

Example: *Suppose $n = 3$ and $1 + Z(R) = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$ such that $\tau_1 = 1$, $\tau_2 = 2$ and $\tau_3 = 3$ then,*

$$R_2 = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{21} & a_{22} & a_{23} \\ 4a_{31} & 2a_{32} & a_{33} \end{bmatrix} : a_{\mu\nu} \in \mathbb{Z}_2 \right\}.$$

Definition 2.2. *Let α_b and β_b be defined as follows:*

$$\alpha_b = \max\{m : \tau_m = \tau_b\},$$
$$\text{and} \qquad \beta_b = \min\{m : \tau_m = \tau_b\}.$$

From the above definitions, it is clear that $\tau_m = \tau_b$ for $m = b$ resulting into $\alpha_b \geq b$ and $\beta_b \leq b$. Since $\beta_1 = \beta_2 = \cdots = \beta_{\alpha_1}$, we have

$$\beta_1 = \beta_2 = \cdots = \beta_{\alpha_1} \leq \beta_{\alpha_l+1}.$$

When $\tau_\mu = \tau_\nu = \cdots = \tau_{rk(1+Z(R))}$, $\forall\, \mu, \nu$, it results to $\alpha_b = \beta_b$. For different τ_μ's, we can have distinct numbers $\{\tau'_\mu\}$ such that $\{\tau'_\mu\} = \{\tau_\nu\}$ and

$$\tau'_1 < \tau'_2 < \cdots .$$

Let $\tau'_1 = \tau_1,\ \tau'_2 = \tau_{\alpha_1+1},\ \cdots,\ \tau'_l = \tau_n$ for $l \in \mathbb{N}$ and define

$$d_\mu = \max\{m : \tau_m = \tau'_\mu\},$$
$$\text{and} \qquad c_\mu = \min\{m : \tau_m = \tau'_\mu\},$$

such that $c_1 = 1$. Then $d_l = rk\,(1 + Z(R))$ and $c_{l+1} = (rk\,(1 + Z(R))) + 1$.

The number of matrices $A \in R_p$ that are invertible modulo p is given by

$$\prod_{b=1}^{rank(1+Z(R))} \left(p^{\alpha_b} - p^{b-1}\right),$$

since they must be linearly independent columns. Next, we count the number of extensions of A to $Aut(1+Z(R))$. This is done by extending each entry $m_{\mu\nu} \in \mathbb{Z}/p\mathbb{Z}$ to $a_{\mu\nu} \in p^{\tau_\mu-\tau_\nu}\mathbb{Z}/p^{\tau_\mu}\mathbb{Z}$ if $\tau_\mu > \tau_\nu$ such that $a_{\mu\nu} \equiv m_{\mu\nu} \pmod p$. Clearly, we have p^{τ_ν} ways of doing so for necessary zeros as any element $p^{\tau_\mu-\tau_\nu}\mathbb{Z}/p^{\tau_\nu}\mathbb{Z}$ will do.

3 Main Results

Since the group of units R^* of the ring R is defined by $R^* = \langle b\rangle \times (1 + Z(R))$ where$\langle b\rangle = \big(R/Z(R)\big)^*$ is a cyclic group of order $p^r - 1$ and $1 + Z(R)$ is a normal subgroup of R, then the structure problem of R^* reduces to that of $1 + Z(R)$.

Theorem 3.1. *Let R^* be the unit group of a class of five radical zero commutative completely primary finite ring of characteristic p described by construction in section 2. Then:*

$$a) \quad Aut(R^*) \cong \begin{cases} (\mathbb{Z}_{2^r-1})^* \times GL_{r(g+e)}(\mathbb{F}_2), & p = 2, \\ (\mathbb{Z}_{3^r-1})^* \times GL_{r(f+h+e)}(\mathbb{F}_3), & p = 3, \\ (\mathbb{Z}_{p^r-1})^* \times GL_{r(e+f+g+h)}(\mathbb{F}_p), & p \geq 5. \end{cases}$$

$$b) \left|Aut(R^*)\right| = \begin{cases} \varphi(2^r-1)\cdot \prod\limits_{b=1}^{r(g+e)} (2^{\alpha_b}-2^{b-1}) \prod\limits_{\nu=1}^{r(g+e)} (2^{\tau_\nu})^{r(g+e)-\alpha_\nu} \prod\limits_{\mu=1}^{r(g+e)} (2^{\tau_\mu-1})^{r(g+e)-\beta_{\mu+1}}, & p=2,\\ \varphi(3^r-1)\cdot \prod\limits_{b=1}^{r(f+h+e)} (3^{\alpha_b}-3^{b-1}) \prod\limits_{\nu=1}^{r(f+h+e)} (3^{\tau_\nu})^{r(f+h+e)-\alpha_\nu} \prod\limits_{\mu=1}^{r(f+h+e)} (3^{\tau_\mu-1})^{r(f+h+e)-\beta_{\mu+1}}, & p=3,\\ \varphi(p^r-1)\cdot \prod\limits_{b=1}^{r(e+f+g+h)} (p^{\alpha_b}-p^{b-1}), & p\geq 5. \end{cases}$$

Proof: We describe the matrix R_p and determine the endomorphisms $\Psi(A)$ where $A \in R_p$ such that $\Psi : R_p \to End\,(1+Z(R))$. Then, we find all the elements of $Aut\,(1+Z(R))$ that can be extended to a matrix in $End\,(1+Z(R))$ and calculate the distinct ways of extending such an element to an endomorphism. These are done for the cases when $p=2$, $p=3$ and $p\geq 5$.

Case I. $p=2$: Let $1+Z\,(R) = (\mathbb{Z}_2)^{rg} \times (\mathbb{Z}_{2^3})^{re}$. Then the $rank\,((\mathbb{Z}_2)^{rg} \times (\mathbb{Z}_{2^3})^{re}) = r(g+e)$. It is visualizable that $\tau_1 = \cdots = \tau_{rg} = 1$ and $\tau_{rg+1} = \cdots = \tau_{r(g+e)} = 3$ such that for every $a_{\mu\nu} \in \mathbb{Z}_2$,
$R_2 = \{(a_{\mu\nu}) : 2^{\tau_\mu-\tau_\nu} \big| a_{\mu\nu},\ \forall \mu,\nu, 1\leq \nu\leq\mu\leq r(g+e)\} = M_{r(g+e)}\,(\mathbb{Z}_2)$.

Therefore

$$R_2 = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1(rg)} & a_{1(rg+1)} & \cdots & a_{1(r(g+e))}\\ a_{21} & \cdots & a_{2(rg)} & a_{2(rg+1)} & \cdots & a_{2(r(g+e))}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ a_{(rg)1} & \cdots & a_{(rg)(rg)} & a_{(rg)(rg+1)} & \cdots & a_{(rg)(r(g+e))}\\ 4a_{(rg+1)1} & \cdots & 4a_{(rg+1)(rg)} & a_{(rg+1)(rg+1)} & \cdots & a_{(rg+1)(r(g+e))}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 4a_{(r(g+e))1} & \cdots & 4a_{(r(g+e))(rg)} & a_{(r(g+e))(rg+1)} & \cdots & a_{(r(g+e))(r(g+e))} \end{pmatrix} :\ a_{\mu\nu}\in\mathbb{Z}_2 \right\}$$

From R_2, all the endomorphisms of $(\mathbb{Z}_2)^{rg} \times (\mathbb{Z}_{2^3})^{re}$ are identified by defining a surjective ring homomorphism $\Psi : M_{r(g+e)}\,(\mathbb{Z}_2) \to End\,((\mathbb{Z}_2)^{rg} \times (\mathbb{Z}_{2^3})^{re})$. For $A = (a_{\mu\nu}) \in M_{r(g+e)}\,(\mathbb{Z}_2)$, we have $End\,((\mathbb{Z}_2)^{rg} \times (\mathbb{Z}_{2^3})^{re}) \cong \Psi\,(A)$. To specify endomorphisms that are automorphisms, we note that the structure of $Aut\,((\mathbb{Z}_2)^{rg} \times (\mathbb{Z}_{2^3})^{re})$ is a general linear group whose dimension is the rank of $(\mathbb{Z}_2)^{rg} \times (\mathbb{Z}_{2^3})^{re}$. Thus $Aut\,((\mathbb{Z}_2)^{rg} \times (\mathbb{Z}_{2^3})^{re}) = GL_{r(g+e)}\,(\mathbb{F}_2)$. Since $R^* \cong \mathbb{Z}_{2^r-1} \times ((\mathbb{Z}_2)^{rg} \times (\mathbb{Z}_{2^3})^{re})$, $g.c.d\left(\left|\mathbb{Z}_{2^r-1}\right|, \left|\,(\mathbb{Z}_2)^{rg} \times (\mathbb{Z}_{2^3})^{re}\,\right|\right) = 1$ and that $Aut\,(\mathbb{Z}_{2^r-1}) \cong (\mathbb{Z}_{2^r-1})^*$, then from the fact that $Aut\,((\mathbb{Z}_2)^{rg} \times (\mathbb{Z}_{2^3})^{re}) = GL_{r(g+e)}\,(\mathbb{F}_2)$, we have

$$Aut\,(R^*) \cong (\mathbb{Z}_{2^r-1})^* \times GL_{r(g+e)}\,(\mathbb{F}_2)\,.$$

Using definition 2.1 together with theorem 2.7, we have

$$\left|Aut\,((\mathbb{Z}_2)^{rg} \times (\mathbb{Z}_{2^3})^{re})\,\right| = \prod_{b=1}^{r(g+e)} (2^{\alpha_b}-2^{b-1}) \prod_{\nu=1}^{r(g+e)} (2^{\tau_\nu})^{r(g+e)-\alpha_\nu} \prod_{\mu=1}^{r(g+e)} (2^{\tau_\mu-1})^{r(g+e)-\beta_{\mu+1}}$$

and

$$\begin{aligned}\left|Aut(R^*)\right| =& \left| (\mathbb{Z}_{2^r-1})^* \right| \cdot \left|Aut((\mathbb{Z}_2)^{rg} \times (\mathbb{Z}_{2^3})^{re}) \right| \\ =& \varphi(2^r-1)\cdot \prod_{b=1}^{r(g+e)} (2^{\alpha_b} - 2^{b-1}) \prod_{\nu=1}^{r(g+e)} (2^{\tau_\nu})^{r(g+e)-\alpha_\nu} \prod_{\mu=1}^{r(g+e)} (2^{\tau_\mu - 1})^{r(g+e)-\beta_{\mu+1}}\end{aligned}$$

Case II. $p = 3$: Let $1 + Z(R) = (\mathbb{Z}_3)^{r(f+h)} \times (\mathbb{Z}_{3^2})^{re}$. Then the $rank\left((\mathbb{Z}_3)^{r(f+h)} \times (\mathbb{Z}_{3^2})^{re}\right) = r(f+h+e)$. It is visualizable that $\tau_1 = \cdots = \tau_{r(f+h)} = 1$ and $\tau_{r(f+h)+1} = \cdots = \tau_{r(f+h+e)} = 2$ such that for every $a_{\mu\nu} \in \mathbb{Z}_3$, $R_3 = \{(a_{\mu\nu}) : 3^{\tau_\mu - \tau_\nu} \big| a_{\mu\nu}, \forall \mu, \nu, 1 \leq \nu \leq \mu \leq r(f+h+e)\} = M_{r(f+h+e)}(\mathbb{Z}_3)$.

Therefore

$$R_3 = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1(r(f+h))} & a_{1(r(f+h)+1)} & \cdots & a_{1(r(f+h+e))} \\ a_{21} & \cdots & a_{2(r(f+h))} & a_{2(r(f+h)+1)} & \cdots & a_{2(r(f+h+e))} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(r(f+h))1} & \cdots & a_{(r(f+h))(r(f+h))} & a_{(r(f+h))(r(f+h)+1)} & \cdots & a_{(r(f+h)(r(f+h+e))} \\ 3a_{(r(f+h)+1)1} & \cdots & 3a_{(r(f+h)+1)(r(f+h))} & a_{(r(f+h)+1)(r(f+h)+1)} & \cdots & a_{(r(f+h)+1)(r(f+h+e))} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 3a_{(r(f+h+e))1} & \cdots & 3a_{(r(f+h+e))(r(f+h))} & a_{(r(f+h+e))(r(f+h)+1)} & \cdots & a_{(r(f+h+e))(r(f+h+e))} \end{pmatrix} : a_{\mu\nu} \in \mathbb{Z}_3 \right\}$$

From R_3, all the endomorphisms of $(\mathbb{Z}_3)^{r(f+h)} \times (\mathbb{Z}_{3^2})^{re}$ are identified by defining a surjective ring homomorphism $\Psi : M_{r(f+h+e)}(\mathbb{Z}_3) \to End\left((\mathbb{Z}_3)^{r(f+h)} \times (\mathbb{Z}_{3^2})^{re}\right)$. For $A = (a_{\mu\nu}) \in M_{r(f+h+e)}(\mathbb{Z}_3)$, we have $End\left((\mathbb{Z}_3)^{r(f+h)} \times (\mathbb{Z}_{3^2})^{re}\right) \cong \Psi(A)$. To specify endomorphisms that are automorphisms, we note that the structure of $Aut\left((\mathbb{Z}_3)^{r(f+h)} \times (\mathbb{Z}_{3^2})^{re}\right)$ is $GL_{r(f+h+e)}(\mathbb{F}_3)$. Thus $Aut\left((\mathbb{Z}_3)^{r(f+h)} \times (\mathbb{Z}_{3^2})^{re}\right) = GL_{r(f+h+e)}(\mathbb{F}_3)$.

Since $R^* \cong \mathbb{Z}_{3^r-1} \times \left((\mathbb{Z}_3)^{r(f+h)} \times (\mathbb{Z}_{3^2})^{re}\right)$, $g.c.d\left(\left|\mathbb{Z}_{3^r-1}\right|, \left| (\mathbb{Z}_3)^{r(f+h)} \times (\mathbb{Z}_{3^2})^{re} \right|\right) = 1$ and that $Aut(\mathbb{Z}_{3^r-1}) \cong (\mathbb{Z}_{3^r-1})^*$, then from the fact that $Aut\left((\mathbb{Z}_3)^{r(f+h)} \times (\mathbb{Z}_{3^2})^{re}\right) = GL_{r(f+h+e)}(\mathbb{F}_3)$, we have

$$Aut(R^*) \cong (\mathbb{Z}_{3^r-1})^* \times GL_{r(f+h+e)}(\mathbb{F}_3).$$

Using definition 2.1 together with theorem 2.7, we have

$$\left|Aut\left((\mathbb{Z}_3)^{r(f+h)} \times (\mathbb{Z}_{3^2})^{re}\right)\right| = \prod_{b=1}^{r(f+h+e)} (3^{\alpha_b} - 3^{b-1}) \prod_{\nu=1}^{r(f+h+e)} (3^{\tau_\nu})^{r(f+h+e)-\alpha_\nu} \prod_{\mu=1}^{r(f+h+e)} (3^{\tau_\mu - 1})^{r(f+h+e)-\beta_{\mu+1}}$$

and

$$\begin{aligned}\left|Aut(R^*)\right| =& \left| (\mathbb{Z}_{3^r-1})^* \right| \cdot \left|Aut\left((\mathbb{Z}_3)^{r(f+h)} \times (\mathbb{Z}_{3^2})^{re}\right) \right| \\ =& \varphi(3^r-1)\cdot \prod_{b=1}^{r(f+h+e)} (3^{\alpha_b} - 3^{b-1}) \prod_{\nu=1}^{r(f+h+e)} (3^{\tau_\nu})^{r(f+h+e)-\tau_\nu} \prod_{\mu=1}^{r(f+h+e)\tau_\mu} (3^{\tau_\mu - 1})^{r(f+h+e)-\beta_{\mu+1}}\end{aligned}$$

Case III. $p \geq 5$: Let $1 + Z(R) = (\mathbb{Z}_p)^{r(e+f+g+h)}$. Then the $rank\left((\mathbb{Z}_p)^{r(e+f+g+h)}\right) = r(e+f+g+h)$. It is visualizable that $\tau_1 = \cdots = \tau_{r(e+f+g+h)} = 1$ such that for every $a_{\mu\nu} \in \mathbb{Z}_p$,
$R_p = \{(a_{\mu\nu}) : p^{\tau_\mu - \tau_\nu} \big| a_{\mu\nu}, \forall \mu, \nu, 1 \leq \nu \leq \mu \leq r(e+f+g+h) = M_{r(e+f+g+h)}(\mathbb{Z}_p)$.

Therefore

$$R_p = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1(r)} & a_{1(r+1)} & \cdots & a_{1(r(e+f+g+h))} \\ a_{21} & \cdots & a_{2(r)} & a_{2(r+1)} & \cdots & a_{2(r(e+f+g+h))} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(r)1} & \cdots & a_{(r)(r)} & a_{(r)(r+1)} & \cdots & a_{(r)(r(e+f+g+h))} \\ a_{(r+1)1} & \cdots & a_{(r+1)(r)} & a_{(r+1)(r+1)} & \cdots & a_{(r+1)(r(e+f+g+h))} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(r(e+f+g+h))1} & \cdots & a_{(r(e+f+g+h))(r)} & a_{(r(e+f+g+h))(r+1)} & \cdots & a_{(r(e+f+g+h))(r(e+f+g+h))} \end{pmatrix} : a_{\mu\nu} \in \mathbb{Z}_p \right\}$$

From R_p, all the endomorphisms of $(\mathbb{Z}_p)^{r(e+f+g+h)}$ are identified by defining a surjective ring homomorphism $\Psi : M_{r(e+f+g+h)}(\mathbb{Z}_p) \to End\left((\mathbb{Z}_p)^{r(e+f+g+h)}\right)$. For $A = (a_{\mu\nu}) \in M_{r(e+f+g+h)}(\mathbb{Z}_p)$, we have $End\left((\mathbb{Z}_p)^{r(e+f+g+h)}\right) \cong \Psi(A)$. To specify endomorphisms that are automorphisms, we note that the structure of $Aut\left((\mathbb{Z}_p)^{r(e+f+g+h)}\right)$ is $GL_{r(e+f+g+h)}(\mathbb{F}_p)$. Thus, we have $Aut\left((\mathbb{Z}_p)^{r(e+f+g+h)}\right) = GL_{r(e+f+g+h)}(\mathbb{F}_p)$.

Since $R^* \cong \mathbb{Z}_{p^r-1} \times \left((\mathbb{Z}_p)^{r(e+f+g+h)}\right)$, $g.c.d\left(\left|\mathbb{Z}_{p^r-1}\right|, \left|(\mathbb{Z}_p)^{r(e+f+g+h)}\right|\right) = 1$ and that $Aut(\mathbb{Z}_{p^r-1}) \cong (\mathbb{Z}_{p^r-1})^*$, then from the fact that $Aut\left((\mathbb{Z}_p)^{r(e+f+g+h)}\right) = GL_{r(e+f+g+h)}(\mathbb{F}_p)$, we have

$$Aut(R^*) = (\mathbb{Z}_{p^r-1})^* \times GL_{r(e+f+g+h)}(\mathbb{F}_p).$$

Using definition 2.1 together with theorem 2.7, we have

$$\left|Aut(\mathbb{Z}_p)^{r(e+f+g+h)}\right| = \prod_{b=1}^{r(e+f+g+h)} (p^{\alpha_b} - p^{b-1})$$

and

$$\begin{aligned} \left|Aut(R^*)\right| &= \left|(\mathbb{Z}_{p^r-1})^*\right| \cdot \left|Aut\left((\mathbb{Z}_p)^{r(e+f+g+h)}\right)\right| \\ &= \varphi(p^r-1) \cdot \prod_{b=1}^{r(e+f+g+h)} (p^{\alpha_b} - p^{b-1}) \end{aligned}$$

Theorem 3.2. *Let R be the class of completely primary finite ring constructed in section [2] above whose Jacobson radical $Z(R) = U \oplus V \oplus W \oplus Y$ with $pu_i = 0$, $pv_j = 0$, $pw_k = 0$ and $py_l = 0$. Then the $Aut(\Gamma(R)) \cong S_{(p^{hr}-1)} \times S_{(p^{gr}-1)p^{hr}} \times S_{(p^{fr}-1)p^{(g+h)r}} \times S_{(p^{er}-1)p^{(f+g+h)r}}$.*

Proof: Let $\varepsilon_1, \cdots, \varepsilon_r \in R_0$ with $\varepsilon_1 = 1$ such that $\bar{\varepsilon_1}, \cdots, \bar{\varepsilon_r} \in R_0$ form a basis for R_0 over its prime subfield R_0/pR_0. Then the graph set $V(\Gamma(R)) = Z(R)^* = Z(R) - \{(0,0,0,0,0)\}$ is partitioned in such a way that the vertices having the same degree belong to the same partition. Consider $V_1 = ann(Z(R)^*) = V_{\sum_{l=1}^{h}\sum_{\lambda=1}^{r}\varepsilon_\lambda y_l}$ .Then $|V_1| = p^{hr} - 1$ and each vertex $a \in V_1$ is adjacent to every other vertex in $\Gamma(R)$ with the deg $(a) = p^{(e+f+g+h)r} - 2$ for all $a \in V_1$. Let $V_2 = V_{\sum_{k=1}^{g}\sum_{\lambda=1}^{r}\varepsilon_\lambda w_k + \sum_{l=1}^{h}\sum_{\lambda=1}^{r}\varepsilon_\lambda y_l}$ be another partition such that $V_1 \cap V_2 = \varnothing$. Then $|V_2| = (p^{gr} - 1)p^{hr}$ and each vertex $b \in V_2$ is such that the deg$(b) = p^{(f+g+h)r} - 2$. Also considering $V_3 = V_{\sum_{j=1}^{f}\sum_{\lambda=1}^{r}\varepsilon_\lambda v_j + \sum_{k=1}^{g}\sum_{\lambda=1}^{r}\varepsilon_\lambda w_k + \sum_{l=1}^{h}\sum_{\lambda=1}^{r}\varepsilon_\lambda y_l}$ as another vertex set that is disjoint from V_1 and V_2, then $|V_3| = \left(p^{fr} - 1\right)p^{(g+h)r}$ and each $x \in V_3$ is such that the deg $(x) = p^{(g+h)r} - 2$. Finally let the vertex set $V_4 = V_{\sum_{i=1}^{e}\sum_{\lambda=1}^{r}\varepsilon_\lambda u_i + \sum_{j=1}^{f}\sum_{\lambda=1}^{r}\varepsilon_\lambda v_j + \sum_{k=1}^{g}\sum_{\lambda=1}^{r}\varepsilon_\lambda w_k + \sum_{l=1}^{h}\sum_{\lambda=1}^{r}\varepsilon_\lambda y_l}$, then the $|V_4| = (p^{er} - 1)p^{(f+g+h)r}$ and each $y \in V_4$ is such that the deg$(y) = p^{hr} - 1$.

Clearly V_1, V_2, V_3 and V_4 are mutually disjoint partitions of the vertex set $V(\Gamma(R))$. Since automorphisms permute the vertices V_ω, $\omega = 1, 2, 3, 4$ independently, then

$$Aut(\Gamma(R)) \cong S_{\left(p^{hr}-1\right)} \times S_{(p^{gr}-1)p^{hr}} \times S_{\left(p^{fr}-1\right)p^{(g+h)r}} \times S_{(p^{er}-1)p^{(f+g+h)r}}$$

For a symmetric group on n symbols, S_n has $n!$ permutation operations, and the order of the symmetric group $|S_n| = n!$, therefore

$$|Aut(\Gamma(R))| = \left(p^{hr} - 1\right)!\left((p^{gr} - 1)p^{hr}\right)!\left(\left(p^{fr} - 1\right)p^{(g+h)r}\right)!\left((p^{er} - 1)p^{(f+g+h)r}\right)!.$$

Example 1. Let

$$R = GR(2,2) \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

for $p = 2$, $r = 1$, $e = f = g = h = 1$. Then the set

$$Z(R) = 0 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

and

$$\begin{aligned} Z(R)^* = V(\Gamma(R)) = \{&(0,0,0,0,1), (0,0,0,1,0), (0,0,0,1,1), (0,0,1,0,0), (0,0,1,0,1), \\ &(0,0,1,1,0), (0,0,1,1,1), (0,1,0,0,0), (0,1,0,0,1), (0,1,0,1,0), \\ &(0,1,0,1,1), (0,1,1,0,0), (0,1,1,0,1), (0,1,1,1,0), (0,1,1,1,1)\}. \end{aligned}$$

The graph set $V(\Gamma(R))$ is partitioned into vertices having the same degree as

$$
\begin{aligned}
V_1 &= \{(0,0,0,0,1)\} \text{ of degree 14;}\\
V_2 &= \{(0,0,0,1,0),(0,0,0,1,1)\} \text{ each vertex is of degree 6;}\\
V_3 &= \{(0,0,1,0,0),(0,0,1,1,0),(0,0,1,0,1),(0,0,1,1,1)\}\\
&\quad \text{each vertex is of degree 3;}\\
&\text{and}\\
V_4 &= \{(0,1,0,0,0),(0,1,0,0,1),(0,1,0,1,0),(0,1,0,1,1),\\
&\quad (0,1,1,0,0),(0,1,1,1,0),(0,1,1,1,0),(0,1,1,1,1)\}\\
&\quad \text{each vertex is of degree 1.}
\end{aligned}
$$

Clearly, $|V_1| = 1$, $|V_2| = 2$, $|V_3| = 4$, $|V_4| = 8$.

Thus, $Aut(\Gamma(R)) \cong S_1 \times S_2 \times S_4 \times S_8$ and

$$|Aut(\Gamma(R))| = 1!\,2!\,4!\,8! = 1{,}935{,}360.$$

Theorem 3.3. *Let R be the module idealization constructed in Section [2] above whose Jacobson radical*

$$Z(R) = U \oplus V \oplus W \oplus Y$$

with $pu_i = 0$, $pv_j = 0$, $pw_k = 0$ and $py_l = 0$. Then the graph $\Gamma_E(R)$ satisfies the following:

i) $\Gamma_E(R)$ *is connected.*

ii) $\Gamma_E(R)$ *is incomplete.*

iii) $\operatorname{diam}(\Gamma_E(R)) = 2$.

iv) $gr(\Gamma_E(R)) = 3$.

v) $b(\Gamma_E(R)) = 3$.

vi) $\Gamma(R)$ *is 3-partite.*

Proof: The compressed zero divisor graph $\Gamma_E(R)$ associated with Completely Primary Finite Ring R in which characteristic of R is p contain the following equivalent classes:
$\Gamma_E(R) = \{ann\,(Z(R)^*)\,, (Z(R)^* - ann\,(Z(R)^*))^2\,, (Z(R)^*)^3\,, (Z(R)^*)^5\}$

i) At least $ann\,(Z(R)^*)$ is adjacent to all the equivalent classes in $\Gamma_E(R)$. Thus the $\Gamma_E(R)$ is connected.

ii) Since $\left[(Z(R)^*)^3\right]\left[(Z(R)^*)^5\right] \neq (0)$, it is clear that not all the equivalent classes are adjacent. Hence $\Gamma_E(R)$ is incomplete.

iii) Since the $\Gamma_E(R)$ has two non-adjacent equivalent classes whose product with $ann\left(Z(R)^*\right)$ is zero, we have $diam\left(\Gamma_E(R)\right) = 2$.

iv) The $ann\left(Z(R)^*\right)$ is adjacent to both $\left(Z(R)^* - ann\left(Z(R)^*\right)\right)^2$ and $\left(Z(R)^*\right)^3$. Also $\left(Z(R)^* - ann\left(Z(R)^*\right)\right)^2$ is adjacent to $\left(Z(R)^*\right)^3$. Thus $gr\left(\Gamma_E(R)\right) = 3$.

v) The graph set $\Gamma_E(R)$ is partitioned into mutually disjoint subsets having adjacent equivalent classes as follows: $V_1 = \{ann\left(Z(R)^*\right)\}$, $V_2 = \left\{\left(Z(R)^* - ann\left(Z(R)^*\right)\right)^2, \left(Z(R)^*\right)^3\right\}$ and $V_3 = \left\{\left(Z(R)^*\right)^5\right\}$. Since the neighborhood of V_1, $N(V_1) = V_2 \cup V_3$, we have $\left|N(V_1)\right| = \left|V_2\right| + \left|V_3\right| = 3$ and $\left|V_1\right| = 1$. Thus,we have $b\left(\Gamma(R)\right) = \frac{\left|N(V_1)\right|}{\left|V_1\right|} = 3$.

vi) Since $N(V_1) = V_2 \cup V_3$, it follows that $\Gamma_E(R)$ is 3 partite.

Example 2. Let

$$R = GR(2,2) \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

for $p = 2$, $r = 1$, $e = f = g = h = 1$. Then the set

$$Z(R) = 0 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

and

$$\begin{aligned} Z(R)^* = V(\Gamma(R)) = \{&(0,0,0,0,1),(0,0,0,1,0),(0,0,0,1,1),(0,0,1,0,0),(0,0,1,0,1),\\ &(0,0,1,1,0),(0,0,1,1,1),(0,1,0,0,0),(0,1,0,0,1),(0,1,0,1,0),\\ &(0,1,0,1,1),(0,1,1,0,0),(0,1,1,0,1),(0,1,1,1,0),(0,1,1,1,1)\}. \end{aligned}$$

The compressed zero divisor graphs

$$\Gamma_E(R) = \left\{ann(Z(R)^*),\ \left(Z(R)^* - ann(Z(R)^*)\right)^2,\ \left(Z(R)^*\right)^3,\ \left(Z(R)^*\right)^5\right\}$$

where

$$ann(Z(R)^*) = \{(0,0,0,0,1)\},$$
$$\left(Z(R)^* - ann(Z(R)^*)\right)^2 = \{(0,0,0,1,0),(0,0,0,1,1)\},$$
$$\left(Z(R)^*\right)^3 = \{(0,0,1,0,0),(0,0,1,0,1),(0,0,1,1,0),(0,0,1,1,1)\},$$

$$\begin{aligned} \left(Z(R)^*\right)^5 = \{&(0,1,0,0,0),(0,1,0,0,1),(0,1,0,1,0),(0,1,0,1,1),\\ &(0,1,1,0,0),(0,1,1,0,1),(0,1,1,1,0),(0,1,1,1,1)\}. \end{aligned}$$

The equivalence classes are then partitioned as:

$$V_1 = \{ann(Z(R)^*)\}, \quad V_2 = \left\{\left(Z(R)^* - ann(Z(R)^*)\right)^2, \left(Z(R)^*\right)^3\right\}, \quad V_3 = \{\left(Z(R)^*\right)^5\}.$$

4 Conclusion

In a completely primary finite ring R, an element is either a unit or a zero divisor. This study has explicitly determined the automorphisms of the structures of Completely Primary Finite Ring R of characteristic p whose Jacobson radical $Z(R)$ satisfy the condition $(Z(R))^5 = (0)$ and $(Z(R))^4 \neq (0)$. These opportunities have exposed the symmetries of these rings, facilitating their classification into rings with desirable properties as well as understanding their fundamental structures. It has been realized that $\Gamma_E(R)$ and $\Gamma(R)$ share several graph theoretic properties and $\Gamma_E(R)$ is less noisy than $\Gamma(R)$. Since the class of rings studied in this paper was restricted to characteristic p , the automorphisms of the unit groups and zero divisor graphs of the cases where characteristic of R is p^k where $2 \leq k \leq 5$ are left for future work.

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