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Extension and Correction of an Inequality Similar to the Hardy-Hilbert Integral Inequality

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Abstract

In this article, we present an extension to the Hardy-Hilbert integral inequality. This extension incorporates a multivariate parametric power-ratio function. The original formulation of the inequality is also included, along with a correction.

1 Introduction

The celebrated Hardy-Hilbert integral inequality, popularized in the classical work of Hardy, Littlewood, and Pólya [1], plays a fundamental role in real analysis. It provides a sharp upper bound for a bilinear form involving two nonnegative functions and a singular kernel function of the following type:

$$k(x, y) = \frac{1}{x + y}.$$

A detailed statement of this inequality is given below. Let $p > 1$, q such that $1/p + 1/q = 1$ and $f, g : (0, \infty) \to (0, \infty)$. Then, we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x + y} dxdy \leq \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(x)dx \right)^{1/p} \left(\int_0^\infty g^q(y)dy \right)^{1/q},$$

provided that the integrals of the upper bound converge. Here, the constant

$$\sigma = \frac{\pi}{\sin(\pi/p)}$$

is the best possible, in the sense that it cannot be replaced by a smaller constant independent of f and g. Assuming that the integrals in the upper bound converge ensures that the double integral on the left-hand side is finite, thereby providing a well-defined inequality. The Hardy-Hilbert integral inequality can be seen as a continuous version of the Hardy-Hilbert double series inequality and is closely related to classical integral transforms and the theory of special functions. It has a wide range of applications in harmonic

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analysis, interpolation theory and the study of function spaces. It also plays a significant role in operator theory and the theory of integral transforms. Furthermore, it is involved in investigating sharp constants in functional inequalities and is connected to partial differential equations and mathematical physics.

There are numerous variants and extensions of the Hardy-Hilbert integral inequality. For a comprehensive overview, we refer to the monograph [3] and the survey [2]. In this article, we focus on a significant yet under-studied result proposed in [5]. This result establishes an inequality that is similar in spirit to the Hardy-Hilbert integral inequality, but involving multiple integrals, a more sophisticated functional structure, numerous adjustable parameters and the beta function. This result is stated below.

Theorem 1.1. *[5, Main result] Let $n \in \mathbb{N}\backslash\{0,1\}$, $p > 1$, q such that $1/p+1/q = 1$, for any $i = 1,\ldots,n$, $f_i : (0,\infty) \to (0,\infty)$, $a_i > 0$, and, for any $r = 1,\ldots,n$,*

$$\lambda_{r+1} = (a_{r+1} - 1)(1 - q),$$

$$\Upsilon_{r+1} = \prod_{j=r+1}^{n} B\left(a_j, \lambda - \sum_{i=j}^{n} a_i\right),$$

where $B(a,b)$ denotes the beta function, which is defined by the following two integral representations:

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt = \int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}}dt,$$

and λ such that

$$\lambda > \sum_{i=r+1}^{n} a_i.$$

Then, we have

$$\left(\frac{\int_0^\infty \ldots \int_0^\infty \frac{f_1(x_1)\ldots f_n(x_n)}{(x_1+\ldots+x_n)^\lambda}dx_1\ldots dx_n}{\Upsilon_{r+1}\int_0^\infty \ldots \int_0^\infty f_1^p(x_1)\ldots f_r^p(x_r)dx_1\ldots dx_r}\right)^q$$
$$\leq \frac{\int_0^\infty \ldots \int_0^\infty \frac{(x_1+\ldots+x_r)^{\sum_{i=r+1}^n a_i - \lambda} x_{r+1}^{\lambda_{r+1}} f_{r+1}^q(x_{r+1})\ldots x_n^{\lambda_n} f_n^q(x_n)}{(x_1+\ldots+x_n)^\lambda}dx_1\ldots dx_n}{\Upsilon_{r+1}\int_0^\infty \ldots \int_0^\infty f_1^p(x_1)\ldots f_r^p(x_r)dx_1\ldots dx_r},$$

provided that the integrals of the upper bound converge.

Despite its mathematical interest and originality, this result has not received much attention in the literature. The only related work is found in [4], where it has been extended to different integration domains and adjustable exponent parameters have been included.

In this article, we contribute to this line of research in two directions, as described below.

(i) We propose a functional extension of Theorem 1.1 by incorporating the following multivariate function:

$$\left(\frac{x_{r+1}}{x_1+\cdots+x_r}\right)^{\beta_{r+1}}\left(\frac{x_{r+2}}{x_1+\cdots+x_{r+1}}\right)^{\beta_{r+2}}\cdots\left(\frac{x_n}{x_1+\cdots+x_{n-1}}\right)^{\beta_n},$$

where, for any $i=1,\ldots,n$, $\beta_i\geq 0$. It is clear that the main result of [5] corresponds to the particular case

$$\beta_{r+1}=\beta_{r+2}=\cdots=\beta_n=0.$$

Although the incorporation of this function has a moderate effect on the upper bound, it significantly enriches the structure of the inequality.

(ii) We have identified and corrected a technical omission in [5, Main result]; a crucial exponent term "$q-1$" is missing from both the statement and the proof. In doing so, we are revisiting this published result.

The remainder of the article is organized as follows: Section 2 presents the detailed statement and proof of the new result. Section 3 highlights the correction to be made in Theorem 1.1. Several applications are discussed in Section 4. Finally, Section 5 contains concluding remarks and an outlook on future research directions.

2 Main Result

Our main result is given below, followed by the corresponding proof.

Theorem 2.1. *Let $n\in\mathbb{N}\setminus\{0,1\}$, $p>1$, q such that $1/p+1/q=1$, for any $i=1,\ldots,n$, $f_i:(0,\infty)\to(0,\infty)$, $a_i>0$, $\beta_i\geq 0$ and, for any $r=1,\ldots,n$,*

$$\lambda_{r+1}=(a_{r+1}-1)(1-q),$$

$$\Omega_{r+1}=\prod_{j=r+1}^{n}B\left(a_j+\beta_j p,\lambda-\sum_{i=j}^{n}a_i-\beta_j p\right),$$

and λ such that

$$\lambda>\sum_{i=r+1}^{n}a_i+\beta_{r+1}p.$$

Then, we have

$$\left(\frac{\int_0^\infty\cdots\int_0^\infty\frac{f_1(x_1)\ldots f_n(x_n)}{(x_1+\ldots+x_n)^\lambda}\left(\frac{x_{r+1}}{x_1+\ldots+x_r}\right)^{\beta_{r+1}}\cdots\left(\frac{x_n}{x_1+\ldots+x_{n-1}}\right)^{\beta_n}dx_1\ldots dx_n}{\Omega_{r+1}\int_0^\infty\cdots\int_0^\infty f_1^p(x_1)\ldots f_r^p(x_r)dx_1\ldots dx_r}\right)^q$$

$$\leq\frac{\int_0^\infty\cdots\int_0^\infty\frac{(x_1+\ldots+x_r)^{\left(\sum_{i=r+1}^{n}a_i-\lambda\right)(q-1)}x_{r+1}^{\lambda_{r+1}}f_{r+1}^q(x_{r+1})\ldots x_n^{\lambda_n}f_n^q(x_n)}{(x_1+\ldots+x_n)^\lambda}dx_1\ldots dx_n}{\Omega_{r+1}\int_0^\infty\cdots\int_0^\infty f_1^p(x_1)\ldots f_r^p(x_r)dx_1\ldots dx_r},$$

provided that the integrals of the upper bound converge.

Proof of Theorem 2.1. Using the Fubini-Tonelli integral theorem, we can write

$$\int_0^\infty \dots \int_0^\infty \frac{f_1(x_1)\dots f_n(x_n)}{(x_1+\dots+x_n)^\lambda}\left(\frac{x_{r+1}}{x_1+\dots+x_r}\right)^{\beta_{r+1}}\dots\left(\frac{x_n}{x_1+\dots+x_{n-1}}\right)^{\beta_n} dx_1\dots dx_n$$
$$= \int_0^\infty \dots \int_0^\infty f_1(x_1)\dots f_r(x_r) I(x_1,\dots,x_r) dx_1\dots dx_r, \quad (1)$$

where

$$I(x_1,\dots,x_r) = \int_0^\infty \dots \int_0^\infty \frac{f_{r+1}(x_{r+1})\dots f_n(x_n)}{(x_1+\dots+x_n)^\lambda}\left(\frac{x_{r+1}}{x_1+\dots+x_r}\right)^{\beta_{r+1}}\dots\left(\frac{x_n}{x_1+\dots+x_{n-1}}\right)^{\beta_n} dx_{r+1}\dots dx_n.$$

Applying the Hölder integral inequality, we get

$$\int_0^\infty \dots \int_0^\infty f_1(x_1)\dots f_r(x_r) I(x_1,\dots,x_r) dx_1\dots dx_r$$
$$\leq \left(\int_0^\infty \dots \int_0^\infty f_1^p(x_1)\dots f_r^p(x_r) dx_1\dots dx_r\right)^{1/p}$$
$$\times \left(\int_0^\infty \dots \int_0^\infty I^q(x_1,\dots,x_r) dx_1\dots dx_r\right)^{1/q}. \quad (2)$$

Using a suitable decomposition of the integrand and applying the Hölder integral inequality, we obtain

$$I(x_1,\dots,x_r)$$
$$= \int_0^\infty \dots \int_0^\infty \frac{x_{r+1}^{\lambda_{r+1}/q} f_{r+1}(x_{r+1})\dots x_n^{\lambda_n/q} f_n(x_n)}{(x_1+\dots+x_n)^{\lambda/q}} \frac{x_{r+1}^{-\lambda_{r+1}/q}\dots x_n^{-\lambda_n/q}}{(x_1+\dots+x_n)^{\lambda/p}}$$
$$\times \left(\frac{x_{r+1}}{x_1+\dots+x_r}\right)^{\beta_{r+1}}\dots\left(\frac{x_n}{x_1+\dots+x_{n-1}}\right)^{\beta_n} dx_{r+1}\dots dx_n$$
$$\leq \left(\int_0^\infty \dots \int_0^\infty \frac{x_{r+1}^{\lambda_{r+1}} f_{r+1}^q(x_{r+1})\dots x_n^{\lambda_n} f_n^q(x_n)}{(x_1+\dots+x_n)^\lambda} dx_{r+1}\dots dx_n\right)^{1/q}$$
$$\times \left(\int_0^\infty \dots \int_0^\infty \frac{x_{r+1}^{-\lambda_{r+1}/(q-1)}\dots x_n^{-\lambda_n/(q-1)}}{(x_1+\dots+x_n)^\lambda}\left(\frac{x_{r+1}}{x_1+\dots+x_r}\right)^{\beta_{r+1}p}\dots\right.$$
$$\left.\times \left(\frac{x_n}{x_1+\dots+x_{n-1}}\right)^{\beta_n p} dx_{r+1}\dots dx_n\right)^{1/p}$$
$$= \left(\int_0^\infty \dots \int_0^\infty \frac{x_{r+1}^{\lambda_{r+1}} f_{r+1}^q(x_{r+1})\dots x_n^{\lambda_n} f_n^q(x_n)}{(x_1+\dots+x_n)^\lambda} dx_{r+1}\dots dx_n\right)^{1/q}$$
$$\times J^{1/p}(x_1,\dots,x_r), \quad (3)$$

where

$$J(x_1,\dots,x_r) = \int_0^\infty \dots \int_0^\infty \frac{x_{r+1}^{a_{r+1}-1}\dots x_n^{a_n-1}}{(x_1+\dots+x_n)^\lambda}\left(\frac{x_{r+1}}{x_1+\dots+x_r}\right)^{\beta_{r+1}p}\dots\left(\frac{x_n}{x_1+\dots+x_{n-1}}\right)^{\beta_n p} dx_{r+1}\dots dx_n.$$

By the Fubini-Tonelli integral theorem, we now isolate the last variable x_n as follows:

$$\begin{aligned}
&J(x_1,\ldots,x_r)\\
&=\int_0^\infty\ldots\int_0^\infty \frac{x_{r+1}^{a_{r+1}-1}\ldots x_{n-1}^{a_{n-1}-1}}{(x_1+\ldots+x_{n-1})^{\lambda-a_n}}\left(\frac{x_{r+1}}{x_1+\ldots+x_r}\right)^{\beta_{r+1}p}\ldots\\
&\times\left(\frac{x_{n-1}}{x_1+\ldots+x_{n-2}}\right)^{\beta_{n-1}p}K(x_1,\ldots,x_{n-1})dx_{r+1}\ldots dx_{n-1},
\end{aligned}\tag{4}$$

where

$$K(x_1,\ldots,x_{n-1})=\int_0^\infty \frac{x_n^{a_n-1}(x_1+\ldots+x_{n-1})^{\lambda-a_n}}{(x_1+\ldots+x_n)^\lambda}\left(\frac{x_n}{x_1+\ldots+x_{n-1}}\right)^{\beta_n p}dx_n.$$

Performing the change of variables

$$t=\frac{x_n}{x_1+\ldots+x_{n-1}}$$

yields

$$\begin{aligned}
&K(x_1,\ldots,x_{n-1})\\
&=\int_0^\infty \frac{\left(\dfrac{x_n}{x_1+\ldots+x_{n-1}}\right)^{a_n+\beta_n p-1}}{\left(1+\dfrac{x_n}{x_1+\ldots+x_{n-1}}\right)^{\lambda}}\times\frac{1}{x_1+\ldots x_{n-1}}dx_n\\
&=\int_0^\infty \frac{t^{a_n+\beta_n p-1}}{(1+t)^{a_n+\beta_n p+(\lambda-(a_n+\beta_n p))}}dt=B\left(a_n+\beta_n p,\lambda-a_n-\beta_n p\right).
\end{aligned}\tag{5}$$

Combining Equations (5) and (4), we get

$$\begin{aligned}
&J(x_1,\ldots,x_r)=B\left(a_n+\beta_n p,\lambda-a_n-\beta_n p\right)\\
&\times\int_0^\infty\ldots\int_0^\infty \frac{x_{r+1}^{a_{r+1}-1}\ldots x_{n-1}^{a_{n-1}-1}}{(x_1+\ldots+x_{n-1})^{\lambda-a_n}}\left(\frac{x_{r+1}}{x_1+\ldots+x_r}\right)^{\beta_{r+1}p}\ldots\\
&\times\left(\frac{x_{n-1}}{x_1+\ldots+x_{n-2}}\right)^{\beta_{n-1}p}dx_{r+1}\ldots dx_{n-1}.
\end{aligned}$$

Based on this new expression of $J(x_1,\ldots,x_r)$ and the previous one, we can proceed inductively to obtain

$$\begin{aligned}
J(x_1,\ldots,x_r)&=\prod_{j=r+1}^{n-1}B\left(a_j+\beta_j p,\lambda-\sum_{i=j}^{n}a_i-\beta_j p\right)\\
&\times(x_1+\ldots+x_r)^{\sum_{i=r+1}^n a_i-\lambda}\\
&=\Omega_{r+1}(x_1+\ldots+x_r)^{\sum_{i=r+1}^n a_i-\lambda}.
\end{aligned}\tag{6}$$

Combining Equations (6) and (3), we get

$$
\begin{aligned}
&I(x_1,\ldots,x_r)\\
&\leq \left(\int_0^{\infty}\ldots\int_0^{\infty}\frac{x_{r+1}^{\lambda_{r+1}}f_{r+1}^q(x_{r+1})\ldots x_n^{\lambda_n}f_n^q(x_n)}{(x_1+\ldots+x_n)^{\lambda}}dx_{r+1}\ldots dx_n\right)^{1/q}\\
&\times \Omega_{r+1}^{1/p}(x_1+\ldots+x_r)^{\left(\sum_{i=r+1}^n a_i-\lambda\right)/p}. \qquad (7)
\end{aligned}
$$

Combining Equations (7), (1) and (2), we derive

$$
\begin{aligned}
&\int_0^{\infty}\ldots\int_0^{\infty}\frac{f_1(x_1)\ldots f_n(x_n)}{(x_1+\ldots+x_n)^{\lambda}}\left(\frac{x_{r+1}}{x_1+\ldots+x_r}\right)^{\beta_{r+1}}\ldots\left(\frac{x_n}{x_1+\ldots+x_{n-1}}\right)^{\beta_n}dx_1\ldots dx_n\\
&\leq \Omega_{r+1}^{1/p}\left(\int_0^{\infty}\ldots\int_0^{\infty}f_1^p(x_1)\ldots f_r^p(x_r)dx_1\ldots dx_r\right)^{1/p}\\
&\times\left(\int_0^{\infty}\ldots\int_0^{\infty}\frac{(x_1+\ldots+x_r)^{\left(\sum_{i=r+1}^n a_i-\lambda\right)(q-1)}x_{r+1}^{\lambda_{r+1}}f_{r+1}^q(x_{r+1})\ldots x_n^{\lambda_n}f_n^q(x_n)}{(x_1+\ldots+x_n)^{\lambda}}dx_1\ldots dx_n\right)^{1/q}.
\end{aligned}
$$

Raising at the exponent q, this can be rearranged as follows:

$$
\begin{aligned}
&\left(\frac{\int_0^{\infty}\ldots\int_0^{\infty}\frac{f_1(x_1)\ldots f_n(x_n)}{(x_1+\ldots+x_n)^{\lambda}}\left(\frac{x_{r+1}}{x_1+\ldots+x_r}\right)^{\beta_{r+1}}\ldots\left(\frac{x_n}{x_1+\ldots+x_{n-1}}\right)^{\beta_n}dx_1\ldots dx_n}{\Omega_{r+1}\int_0^{\infty}\ldots\int_0^{\infty}f_1^p(x_1)\ldots f_r^p(x_r)dx_1\ldots dx_r}\right)^q\\
&\leq\frac{\int_0^{\infty}\ldots\int_0^{\infty}\frac{(x_1+\ldots+x_r)^{\left(\sum_{i=r+1}^n a_i-\lambda\right)(q-1)}x_{r+1}^{\lambda_{r+1}}f_{r+1}^q(x_{r+1})\ldots x_n^{\lambda_n}f_n^q(x_n)}{(x_1+\ldots+x_n)^{\lambda}}dx_1\ldots dx_n}{\Omega_{r+1}\int_0^{\infty}\ldots\int_0^{\infty}f_1^p(x_1)\ldots f_r^p(x_r)dx_1\ldots dx_r}.
\end{aligned}
$$

This concludes the proof of Theorem 2.1. □

Note that the adjustable parameters $\beta_1,\ldots,\beta_n$ appear only in the constant factor Ω_{r+1}.

It is also worth emphasizing that the upper bounds given in Theorems 2.1 and 1.1 are not identical, a distinction that will be clarified in the subsequent section.

3 A Correction

As outlined in the introduction, it appears that an exponent term "$q-1$" was inadvertently omitted in both the statement and the proof of [5, Main Theorem]. To be more precise, based on the formulation in Theorem 1.1, the following term:

$$
\frac{\int_0^{\infty}\ldots\int_0^{\infty}\frac{(x_1+\ldots+x_r)^{\sum_{i=r+1}^n a_i-\lambda}x_{r+1}^{\lambda_{r+1}}f_{r+1}^q(x_{r+1})\ldots x_n^{\lambda_n}f_n^q(x_n)}{(x_1+\ldots+x_n)^{\lambda}}dx_1\ldots dx_n}{\Upsilon_{r+1}\int_0^{\infty}\ldots\int_0^{\infty}f_1^p(x_1)\ldots f_r^p(x_r)dx_1\ldots dx_r},
$$

must be corrected as follows, with the additional term put in color:

$$\frac{\int_0^\infty \cdots \int_0^\infty \frac{(x_1+\ldots+x_r)^{\left(\sum_{i=r+1}^n a_i-\lambda\right)(q-1)} x_{r+1}^{\lambda_{r+1}} f_{r+1}^q(x_{r+1}) \ldots x_n^{\lambda_n} f_n^q(x_n)}{(x_1+\ldots+x_n)^\lambda} dx_1 \ldots dx_n}{\Upsilon_{r+1} \int_0^\infty \cdots \int_0^\infty f_1^p(x_1) \ldots f_r^p(x_r) dx_1 \ldots dx_r}.$$

Therefore, Theorem 2.1, besides extending the result, also provides a valuable correction.

4 Applications

Several applications of Theorem 2.1, corresponding to specific choices of the parameters n and r, are presented below.

If we take $n = 2$ and $r = 1$, then we obtain

$$\left(\frac{\int_0^\infty \int_0^\infty \frac{f_1(x_1) f_2(x_2)}{(x_1+x_2)^\lambda} \left(\frac{x_2}{x_1}\right)^{\beta_2} dx_1 dx_2}{\Omega_2 \int_0^\infty f_1^p(x_1) dx_1}\right)^q \le \frac{\int_0^\infty \int_0^\infty \frac{x_1^{(a_2-\lambda)(q-1)} x_2^{\lambda_2} f_2^q(x_2)}{(x_1+x_2)^\lambda} dx_1 dx_2}{\Omega_2 \int_0^\infty f_1^p(x_1) dx_1},$$

where

$$\Omega_2 = B\left(a_2 + \beta_2 p, \lambda - a_2 - \beta_2 p\right).$$

For the particular case $\beta_2 = 0$, this inequality reduces to [5, Equation (7)], but with the mentioned correction (and the presence of $f_2^q(x_2)$ which is also omitted in [5, Equation (7)]).

If we take $n = 3$ and $r = 1$, then we get

$$\left(\frac{\int_0^\infty \int_0^\infty \int_0^\infty \frac{f_1(x_1) f_2(x_2) f_3(x_3)}{(x_1+x_2+x_3)^\lambda} \left(\frac{x_2}{x_1}\right)^{\beta_2} \left(\frac{x_3}{x_1+x_2}\right)^{\beta_3} dx_1 dx_2 dx_3}{\Omega_2 \int_0^\infty f_1^p(x_1) dx_1}\right)^q$$
$$\le \frac{\int_0^\infty \int_0^\infty \int_0^\infty \frac{x_1^{(a_2+a_3-\lambda)(q-1)} x_2^{\lambda_2} f_2^q(x_2) x_3^{\lambda_3} f_3^q(x_3)}{(x_1+x_2+x_3)^\lambda} dx_1 dx_2 dx_3}{\Omega_2 \int_0^\infty f_1^p(x_1) dx_1}.$$

For the particular case $\beta_2 = \beta_3 = 0$, this inequality reduces to [5, Equation (8)], but with the mentioned correction.

If we take $n = 3$ and $r = 2$, then we have

$$\left(\frac{\int_0^\infty \int_0^\infty \int_0^\infty \frac{f_1(x_1)f_2(x_2)f_3(x_3)}{(x_1+x_2+x_3)^\lambda} \left(\frac{x_3}{x_1+x_2}\right)^{\beta_3} dx_1dx_2dx_3}{\Omega_3 \int_0^\infty \int_0^\infty f_1^p(x_1)f_2^p(x_2)dx_1dx_2} \right)^q$$
$$\leq \frac{\int_0^\infty \int_0^\infty \int_0^\infty \frac{(x_1+x_2)^{(a_3-\lambda)(q-1)} x_2^{\lambda_2} f_2^q(x_2) x_3^{\lambda_3} f_3^q(x_3)}{(x_1+x_2+x_3)^\lambda} dx_1dx_2dx_3}{\Omega_3 \int_0^\infty \int_0^\infty f_1^p(x_1)f_2^p(x_2)dx_1dx_2},$$

where

$$\Omega_3 = B\left(a_3 + \beta_3 p, \lambda - a_3 - \beta_3 p\right).$$

For the particular case $\beta_3 = 0$, this inequality reduces to [5, Equation (9)], but with the mentioned correction.

5 Conclusion

Building on [5, Main result], we establish a new general inequality in the spirit of the Hardy–Hilbert integral inequality. This extension incorporates a multivariate parametric power–ratio function, thereby broadening the scope of the classical formulation. At the same time, our analysis reveals the necessity of correcting the original statement by adding a missing exponent term. The contributions of this work are therefore twofold: an extension of the inequality to a more general framework and a rigorous correction to the previously stated result.

Potential future research directions include exploring weighted versions of the inequality, studying analogous results on bounded or discrete domains, and investigating possible applications to harmonic analysis, operator theory and inequalities in higher dimensions.

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