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Two New Contributions to the Three-dimensional Hardy-Hilbert-type Integral Inequalities

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Abstract

The Hardy-Hilbert integral inequality is one of the most celebrated results in mathematical analysis, inspiring numerous variants and extensions. In this paper, we further advance the study of three-dimensional Hardy-Hilbert-type integral inequalities by proving two new theorems. One of these is notable for its inclusion of a maximum function, a feature rarely encountered in this three-dimensional context. The associated constant factors are determined explicitly and detailed proofs are provided, without recourse to special functions.

1 Introduction

Integral inequalities occupy a central place in mathematical analysis. Among the most significant results in this domain is the Hardy-Hilbert integral inequality. It can be stated as follows. Let $p, q > 1$ such that $1/p + 1/q = 1$, and $f, g : [0, \infty) \to [0, \infty)$ be two functions such that

$$\int_{(0,\infty)} f^p(x)dx < \infty, \quad \int_{(0,\infty)} g^q(x)dx < \infty.$$

Then we have

$$\iint_{(0,\infty)^2} \frac{f(x)g(y)}{x+y} dxdy \le \frac{\pi}{\sin(\pi/p)} \left(\int_{(0,\infty)} f^p(x)dx \right)^{1/p} \left(\int_{(0,\infty)} g^q(x)dx \right)^{1/q}. \tag{1}$$

The constant factor $\pi/\sin(\pi/p)$ is the best possible; it cannot be replaced by a smaller constant without violating the inequality. More precisely, there exist functions f and g for which the inequality is almost satisfied. For a detailed exposition, see [7].

Over the years, the Hardy-Hilbert integral inequality has inspired numerous extensions and generalizations, including discrete and multidimensional analogues, parameterized kernel functions, and weighted versions. These developments have found applications in operator theory, function spaces and

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Fourier analysis. A comprehensive survey of these results can be found in [3] and in the monograph [17]. Recent works on various refinements and extensions include [1, 2, 4–6, 8–16].

In particular, in [11], three significant two- and three-dimensional Hardy-Hilbert-type integral inequalities were established. These results are remarkable for their originality, the innovative structure of their kernel functions, and the relative simplicity of their upper bounds. Of these, [11, Theorem 3] stands out for its treatment of the three-dimensional case and the use of primitives. It can be stated as follows.

Theorem 1.1. *Let $p, q, r > 2$ such that $1/p + 1/q + 1/r = 1$ and $f, g, h : [0,\infty) \to [0,\infty)$ be three functions such that*

$$\int_{(0,\infty)} f^{p/2}(x)dx < \infty, \quad \int_{(0,\infty)} g^{q/2}(y)dy < \infty, \quad \int_{(0,\infty)} h^{r/2}(z)dz < \infty.$$

For any $x, y, z \geq 0$, we define

$$F(x) = \int_{(0,x)} f(t)dt, \quad G(y) = \int_{(0,y)} g(t)dt, \quad H(z) = \int_{(0,z)} h(t)dt.$$

Then we have

$$\begin{aligned}
&\iiint_{(0,\infty)^3} \frac{x^{1/p}y^{1/q}z^{1/r}\sqrt{xyzF(x)G(y)H(z)}}{(x+y+z)^8}dxdydz \\
&\leq B^{1/p}\left(\frac{p}{2},\frac{p}{2}\right)B^{1/p}(p,p)B^{1/q}\left(\frac{q}{2},\frac{q}{2}\right)B^{1/q}(q,q)B^{1/r}\left(\frac{r}{2},\frac{r}{2}\right)B^{1/r}(r,r) \\
&\times \sqrt{\frac{pqr}{(p-2)(q-2)(r-2)}}\left(\int_{(0,\infty)} f^{p/2}(x)dx\right)^{1/p}\left(\int_{(0,\infty)} g^{q/2}(y)dy\right)^{1/q}\left(\int_{(0,\infty)} h^{r/2}(z)dz\right)^{1/r},
\end{aligned}$$

where $B(u,v)$ denotes the classical beta function, i.e., $B(u,v) = \int_{(0,1)} t^{u-1}(1-t)^{v-1}dt$, with $u, v > 0$.

We also emphasize a distinctive feature of this result: the originality of the constant factor constructed from the beta function.

Motivated by these observations, we further develop the study of three-dimensional Hardy-Hilbert-type integral inequalities by establishing two new theorems. Using standard notation (to be specified later), the first theorem concerns the following triple integral:

$$\iiint_{(0,\infty)^3} \frac{x^{a/p}y^{b/q}z^{c/r}F(x)G(y)H(z)}{(x+y+z)^{a+b+c+3}}dxdydz,$$

which is closely related in spirit to the integral considered in [11, Theorem 3]. The second theorem is based on the following triple integral:

$$\iiint_{(0,\infty)^3} \frac{F(x)G(y)H(z)}{\max\{x^5, y^5, z^5\}}dxdydz.$$

It represents one of the few known results in the literature on three-dimensional Hardy-Hilbert-type integral inequalities that involve a maximum function. For each of these integrals, explicit and tractable upper bounds are derived, with constant factors that are expressed in closed form and do not involve special functions. Complete proofs are provided and rely primarily on a suitable decomposition of the integrands, the application of the Hölder integral inequality, the Fubini-Tonelli integral theorem, and specific integral evaluations.

The remainder of this paper is organized as follows: The main results are presented in the next section, while Section 3 contains concluding remarks.

2 Main Results

2.1 First theorem with proof

The first theorem is stated below, followed by its proof and a subsequent discussion.

Theorem 2.1. *Let $p, q, r > 1$ such that $1/p + 1/q + 1/r = 1$ and $f, g, h : [0, \infty) \to [0, \infty)$ be three functions such that*

$$\int_{(0,\infty)} f^p(x)dx < \infty, \quad \int_{(0,\infty)} g^q(y)dy < \infty, \quad \int_{(0,\infty)} h^r(z)dz < \infty.$$

For any $x, y, z \geq 0$, we define

$$F(x) = \int_{(0,x)} f(t)dt, \quad G(y) = \int_{(0,y)} g(t)dt, \quad H(z) = \int_{(0,z)} h(t)dt.$$

For the sake of brevity, we set

$$a = \frac{2}{p-1}, \quad b = \frac{2}{q-1}, \quad c = \frac{2}{r-1}.$$

Then we have

$$\begin{aligned}
&\iiint_{(0,\infty)^3} \frac{x^{a/p} y^{b/q} z^{c/r} F(x)G(y)H(z)}{(x+y+z)^{a+b+c+3}} dxdydz \\
&\leq \frac{(p-1)^{2/p}(q-1)^{2/q}(r-1)^{2/r}}{(p^2+1)^{1/p}(p^2-p+2)^{1/p}(q^2+1)^{1/q}(q^2-q+2)^{1/q}(r^2+1)^{1/r}(r^2-r+2)^{1/r}} \\
&\times \frac{pqr}{(p-1)(q-1)(r-1)} \left(\int_{(0,\infty)} f^p(x)dx \right)^{1/p} \left(\int_{(0,\infty)} g^q(y)dy \right)^{1/q} \left(\int_{(0,\infty)} h^r(z)dz \right)^{1/r}.
\end{aligned}$$

Proof. A suitable decomposition of the integrand and the Hölder integral inequality give

$$\begin{aligned}
&\iiint_{(0,\infty)^3} \frac{x^{a/p} y^{b/q} z^{c/r} F(x)G(y)H(z)}{(x+y+z)^{a+b+c+3}} dxdydz \\
&= \iiint_{(0,\infty)^3} \left(\frac{x^{a/p} F(x)}{(x+y+z)^{a+1}} \right) \left(\frac{y^{b/q} G(y)}{(x+y+z)^{b+1}} \right) \left(\frac{z^{c/r} H(z)}{(x+y+z)^{c+1}} \right) dxdydz \\
&\leq J^{1/p} K^{1/q} L^{1/r}, \qquad (2)
\end{aligned}$$

where

$$J = \iiint_{(0,\infty)^3} \frac{x^a F^p(x)}{(x+y+z)^{(a+1)p}} dxdydz,$$

$$K = \iiint_{(0,\infty)^3} \frac{y^b G^q(y)}{(x+y+z)^{(b+1)q}} dxdydz$$

and

$$L = \iiint_{(0,\infty)^3} \frac{z^c H^r(z)}{(x+y+z)^{(c+1)r}} dxdydz.$$

Let us work on the term J, the other terms being similar in form. Using the Fubini-Tonelli integral theorem, we can write

$$J = \int_{(0,\infty)} x^a F^p(x)\Omega(x)dx, \tag{3}$$

where

$$\Omega(x) = \iint_{(0,\infty)^2} \frac{1}{(x+y+z)^{(a+1)p}} dydz.$$

Making the changes of variables $u = y/x$ and $v = z/x$, and applying standard limits that implicitly use the value of a, we get

$$\begin{aligned}
\Omega(x) &= x^2 \iint_{(0,\infty)^2} \frac{1}{(x(1+u+v))^{(a+1)p}} dudv \\
&= x^{2-(a+1)p} \iint_{(0,\infty)^2} \frac{1}{(1+u+v)^{(a+1)p}} dudv \\
&= x^{2-(a+1)p} \int_{(0,\infty)} \left[\frac{1}{(1-(a+1)p)(1+u+v)^{(a+1)p-1}} \right]_{u=0}^{\infty} dv \\
&= x^{2-(a+1)p} \frac{1}{(a+1)p-1} \int_{(0,\infty)} \frac{1}{(1+v)^{(a+1)p-1}} dv \\
&= x^{2-(a+1)p} \frac{1}{(a+1)p-1} \left[\frac{1}{(2-(a+1)p)(1+v)^{(a+1)p-2}} \right]_{v=0}^{\infty} \\
&= x^{2-(a+1)p} \frac{1}{((a+1)p-1)((a+1)p-2)}.
\end{aligned}$$

Putting this in Equation (3) gives

$$J = \frac{1}{((a+1)p-1)((a+1)p-2)} \int_{(0,\infty)} F^p(x) x^{a+2-(a+1)p} dx.$$

Using the exact value of a, i.e., $a = 2/(p-1)$, we get

$$J = \frac{(p-1)^2}{(p^2+1)(p^2-p+2)} \int_{(0,\infty)} \left(\frac{F(x)}{x} \right)^p dx.$$

Proceeding in a similar way, we find that

$$K = \frac{(q-1)^2}{(q^2+1)(q^2-q+2)} \int_{(0,\infty)} \left(\frac{G(y)}{y}\right)^q dy$$

and

$$L = \frac{(r-1)^2}{(r^2+1)(r^2-r+2)} \int_{(0,\infty)} \left(\frac{H(z)}{z}\right)^r dz.$$

The classical Hardy integral inequality gives

$$\int_{(0,\infty)} \left(\frac{F(x)}{x}\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_{(0,\infty)} f^p(x)dx.$$

Therefore, we have

$$J \leq \frac{(p-1)^2}{(p^2+1)(p^2-p+2)} \left(\frac{p}{p-1}\right)^p \int_{(0,\infty)} f^p(x)dx. \tag{4}$$

In a similar way, we obtain

$$K \leq \frac{(q-1)^2}{(q^2+1)(q^2-q+2)} \left(\frac{q}{q-1}\right)^q \int_{(0,\infty)} g^q(y)dy \tag{5}$$

and

$$L \leq \frac{(r-1)^2}{(r^2+1)(r^2-r+2)} \left(\frac{r}{r-1}\right)^r \int_{(0,\infty)} h^r(z)dz. \tag{6}$$

Combining Equations (2), (4), (5) and (6), we derive

$$\begin{aligned}
&\iiint_{(0,\infty)^3} \frac{x^{a/p}y^{b/q}z^{c/r}F(x)G(y)H(z)}{(x+y+z)^{a+b+c+3}} dxdydz \\
&\leq \left(\frac{(p-1)^2}{(p^2+1)(p^2-p+2)} \left(\frac{p}{p-1}\right)^p \int_{(0,\infty)} f^p(x)dx\right)^{1/p} \\
&\times \left(\frac{(q-1)^2}{(q^2+1)(q^2-q+2)} \left(\frac{q}{q-1}\right)^q \int_{(0,\infty)} g^q(y)dy\right)^{1/q} \\
&\times \left(\frac{(r-1)^2}{(r^2+1)(r^2-r+2)} \left(\frac{r}{r-1}\right)^r \int_{(0,\infty)} h^r(z)dz\right)^{1/r} \\
&= \frac{(p-1)^{2/p}(q-1)^{2/q}(r-1)^{2/r}}{(p^2+1)^{1/p}(p^2-p+2)^{1/p}(q^2+1)^{1/q}(q^2-q+2)^{1/q}(r^2+1)^{1/r}(r^2-r+2)^{1/r}} \\
&\times \frac{pqr}{(p-1)(q-1)(r-1)} \left(\int_{(0,\infty)} f^p(x)dx\right)^{1/p} \left(\int_{(0,\infty)} g^q(y)dy\right)^{1/q} \left(\int_{(0,\infty)} h^r(z)dz\right)^{1/r}.
\end{aligned}$$

This completes the proof. □

We emphasize the simplicity of the obtained upper bound, which is expressed in terms of the unweighted integral norms of the main functions and involves a constant factor that is independent of any special functions. The integral norms of the main functions can be adjusted using p, q and r. For these reasons, the result complements [11, Theorem 3].

2.2 Second theorem with proof

The second theorem is stated below, followed by its proof and a subsequent discussion.

Theorem 2.2. *Let $f, g, h : [0,\infty) \to [0,\infty)$ be three functions such that*

$$\int_{(0,\infty)} f^3(x)dx < \infty, \quad \int_{(0,\infty)} g^3(y)dy < \infty, \quad \int_{(0,\infty)} h^3(z)dz < \infty.$$

For any $x, y, z \geq 0$, we define

$$F(x) = \int_{(0,x)} f(t)dt, \quad G(y) = \int_{(0,y)} g(t)dt, \quad H(z) = \int_{(0,z)} h(t)dt.$$

Then we have

$$\begin{aligned}
&\iiint_{(0,\infty)^3} \frac{F(x)G(y)H(z)}{\max\{x^5, y^5, z^5\}} dxdydz \\
&\leq \frac{45}{8} \left(\int_{(0,\infty)} f^3(x)dx \right)^{1/3} \left(\int_{(0,\infty)} g^3(y)dy \right)^{1/3} \left(\int_{(0,\infty)} h^3(z)dz \right)^{1/3}.
\end{aligned}$$

Proof. A suitable decomposition of the integrand and the Hölder integral inequality give

$$\begin{aligned}
&\iiint_{(0,\infty)^3} \frac{F(x)G(y)H(z)}{\max\{x^5, y^5, z^5\}} dxdydz \\
&= \iiint_{(0,\infty)^3} \left(\frac{F(x)}{\max\{x^5, y^5, z^5\}^{1/3}} \right) \left(\frac{G(y)}{\max\{x^5, y^5, z^5\}^{1/3}} \right) \left(\frac{H(z)}{\max\{x^5, y^5, z^5\}^{1/3}} \right) dxdydz \\
&\leq M^{1/3} N^{1/3} O^{1/3}, \qquad (7)
\end{aligned}$$

where

$$M = \iiint_{(0,\infty)^3} \frac{F^3(x)}{\max\{x^5, y^5, z^5\}} dxdydz,$$

$$N = \iiint_{(0,\infty)^3} \frac{G^3(y)}{\max\{x^5, y^5, z^5\}} dxdydz$$

and

$$O = \iiint_{(0,\infty)^3} \frac{H^3(z)}{\max\{x^5, y^5, z^5\}} dxdydz.$$

Let us work on the term M, the other terms being similar in form. Using the Fubini-Tonelli integral theorem, we can write

$$M = \int_{(0,\infty)} F^3(x)\Upsilon(x)dx, \qquad (8)$$

with

$$\Upsilon(x) = \iint_{(0,\infty)^2} \frac{1}{\max\{x^5, y^5, z^5\}} dydz.$$

Let us determine this integral by splitting the (y, z)-plane according to which of x, y, z is the largest. By the Chasles integral relation, the following decomposition holds:

$$\begin{aligned}\Upsilon(x) = & \iint_{(0,\infty)^2 \cap \{y<x,\ z<x\}} \frac{1}{\max\{x^5, y^5, z^5\}} dydz \\ & + \iint_{(0,\infty)^2 \cap \{y\geq x,\ z<x\}} \frac{1}{\max\{x^5, y^5, z^5\}} dydz \\ & + \iint_{(0,\infty)^2 \cap \{y<x,\ z\geq x\}} \frac{1}{\max\{x^5, y^5, z^5\}} dydz \\ & + \iint_{(0,\infty)^2 \cap \{y\geq x,\ z\geq x\}} \frac{1}{\max\{x^5, y^5, z^5\}} dydz.\end{aligned}$$

Let us now determine each of these integrals. First, we have

$$\iint_{(0,\infty)^2 \cap \{y<x,\ z<x\}} \frac{1}{\max\{x^5, y^5, z^5\}} dydz = \iint_{(0,x)^2} \frac{1}{x^5} dydz = \frac{1}{x^5} \iint_{(0,x)^2} dydz = x^{-3}.$$

Taking into account the specific domain of integration, we get

$$\begin{aligned}& \iint_{(0,\infty)^2 \cap \{y\geq x,\ z<x\}} \frac{1}{\max\{x^5, y^5, z^5\}} dydz = \int_{(x,\infty)} \left(\int_{(0,x)} \frac{1}{y^5} dz \right) dy \\ & = x \int_{(x,\infty)} \frac{1}{y^5} dy = x \left[-\frac{1}{4y^4} \right]_{y=x}^{\infty} = \frac{1}{4} x^{-3}.\end{aligned}$$

Proceeding in a similar way, we derive

$$\begin{aligned}& \iint_{(0,\infty)^2 \cap \{y<x,\ z\geq x\}} \frac{1}{\max\{x^5, y^5, z^5\}} dydz = \int_{(x,\infty)} \left(\int_{(0,x)} \frac{1}{z^5} dy \right) dz \\ & = x \int_{(x,\infty)} \frac{1}{z^5} dz = x \left[-\frac{1}{4z^4} \right]_{z=x}^{\infty} = \frac{1}{4} x^{-3}.\end{aligned}$$

Making the changes of variables $u = y/x$ and $v = z/x$, we get

$$\begin{aligned}& \iint_{(0,\infty)^2 \cap \{y\geq x,\ z\geq x\}} \frac{1}{\max\{x^5, y^5, z^5\}} dydz = \iint_{[x,\infty)^2} \frac{1}{\max\{y^5, z^5\}} dydz \\ & = x^{-3} \iint_{[1,\infty)^2} \frac{1}{\max\{u^5, v^5\}} dudv.\end{aligned}$$

Using a symmetry argument, we can express this last integral as

$$\begin{aligned}& \iint_{[1,\infty)^2} \frac{1}{\max\{u^5, v^5\}} dudv = 2 \int_{[1,\infty)} \left(\int_{[1,u)} \frac{1}{u^5} dv \right) du = 2 \int_{[1,\infty)} (u-1) \frac{1}{u^5} du \\ & = 2 \left[-\frac{1}{3u^3} + \frac{1}{4u^4} \right]_{u=1}^{\infty} = \frac{1}{6}.\end{aligned}$$

Hence, we have

$$\iint_{(0,\infty)^2\cap\{y\geq x,\ z\geq x\}} \frac{1}{\max\{x^5,y^5,z^5\}}dydz = \frac{1}{6}x^{-3}.$$

We finally obtain

$$\Upsilon(x) = \left(1+\frac{1}{4}+\frac{1}{4}+\frac{1}{6}\right)x^{-3} = \frac{5}{3}x^{-3}. \tag{9}$$

Combining Equations (8) and (9), we derive

$$M = \frac{5}{3}\int_{(0,\infty)}\left(\frac{F(x)}{x}\right)^3 dx.$$

Proceeding in a similar way, we find that

$$N = \frac{5}{3}\int_{(0,\infty)}\left(\frac{G(y)}{y}\right)^3 dy$$

and

$$O = \frac{5}{3}\int_{(0,\infty)}\left(\frac{H(z)}{z}\right)^3 dz.$$

The classical Hardy integral inequality gives, for any $p > 1$,

$$\int_{(0,\infty)}\left(\frac{F(x)}{x}\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_{(0,\infty)} f^p(x)dx.$$

In particular, for $p = 3$, we get

$$\int_{(0,\infty)}\left(\frac{F(x)}{x}\right)^3 dx \leq \left(\frac{3}{2}\right)^3 \int_{(0,\infty)} f^3(x)dx = \frac{27}{8}\int_{(0,\infty)} f^3(x)dx.$$

Therefore, we have

$$M \leq \frac{5}{3}\times\frac{27}{8}\int_{(0,\infty)} f^3(x)dx = \frac{45}{8}\int_{(0,\infty)} f^3(x)dx. \tag{10}$$

Proceeding in a similar way, we obtain

$$N \leq \frac{45}{8}\int_{(0,\infty)} g^3(y)dy \tag{11}$$

and

$$O \leq \frac{45}{8}\int_{(0,\infty)} h^3(z)dz. \tag{12}$$

Combining Equations (7), (10), (11) and (12), we get

$$\begin{aligned}
&\iiint_{(0,\infty)^3} \frac{F(x)G(y)H(z)}{\max\{x^5, y^5, z^5\}} dxdydz \\
&\leq \left(\frac{45}{8}\right)^{1/3} \left(\int_{(0,\infty)} f^3(x)dx\right)^{1/3} \left(\frac{45}{8}\right)^{1/3} \left(\int_{(0,\infty)} g^3(y)dy\right)^{1/3} \left(\frac{45}{8}\right)^{1/3} \left(\int_{(0,\infty)} h^3(z)dz\right)^{1/3} \\
&= \frac{45}{8} \left(\int_{(0,\infty)} f^3(x)dx\right)^{1/3} \left(\int_{(0,\infty)} g^3(y)dy\right)^{1/3} \left(\int_{(0,\infty)} h^3(z)dz\right)^{1/3}.
\end{aligned}$$

This completes the proof. □

We emphasize the simplicity of the obtained upper bound, which involves the remarkably simple constant factor 45/8. Moreover, this result is among the few theorems on three-dimensional Hardy-Hilbert-type integral inequalities that incorporate a maximum function depending on all three variables. It must therefore be viewed as a complement [11, Theorem 3] and Theorem 2.1. However, the integral norms of the main functions are considered only for the parameter value 3, which restricts the generality of the formulation.

3 Concluding Remarks

In this paper, we contribute to the theory of the Hardy-Hilbert-type integral inequalities by presenting two new results in a three-dimensional context. The obtained inequalities are characterized by explicit constant factors derived in closed form without the use of special functions and complete proofs presented in detail. These findings complement and extend earlier results in the literature, particularly those involving three-dimensional formulations based on kernel functions.

A limitation of this study is that we have not explicitly established the optimality of the constant factors, although the structure of the proofs suggests that they are strong candidates for being optimal.

Future research could proceed in several directions. One obvious extension would be to consider weighted or parameterized versions of the inequalities presented here, or to investigate their corresponding discrete analogues. Another promising area of research involves applying these inequalities to operator theory, functional spaces and harmonic analysis. Finally, developing sharper inequalities or optimality conditions in higher dimensions is a challenging problem that remains to be solved and is worthy of further study.

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