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New Aspects of Extended General Equilibrium Inclusions

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Abstract

Some new classes of extended general equilibrium inclusions are introduced and investigated. We have established the equivalence between the general equilibrium inclusions and the fixed point problems, which is used to discuss the unique existence of the solution. Using various techniques such as resolvent methods, dynamical systems coupled with finite difference approach, we suggest and analyze a number of new multi step methods for solving equilibrium inclusions. Convergence analysis of these methods is investigated under suitable conditions. Sensitivity analysis is also investigated. Various special cases are discussed as applications of the main results. Several open problems are suggested for future research.

1 Introduction

Equilibrium problems were introduced by Blum et al. [7] and Noor et al. [52] provide us with a unified, natural, novel, innovative and general technique to study a wide class of problems arising in different branches of mathematical and engineering sciences. Variational inequality theory can be viewed as a novel and important generalization of the variational principles. By variational principles, we mean maximum and minimum problems arising in game theory, mechanics, geometrical optics, general relativity theory, field theory, economics, transportation, differential geometry and related areas. These are fascinating interesting fields that a wide class of unrelated problems can be studied in the general and unified framework of variational inequalities and equilibrium problems. For more details of the applications and generalizations of the variational inequalities, see [5–7, 9, 12–14, 16–18, 20, 22, 24–32, 32–60, 62] and the references therein.

One of the most difficult and important problem is the development of efficient numerical methods. Lions and Stampacchia [20] and Noor [25] proved that the quasi variational inequalities are equivalent to the fixed point problem. This alternative formulation was used to suggest and investigate three-step

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iterations for solving the variational inequities. These three-step iterations contain Noor (three step) iterations [29–31], Picard method, Mann(one step)iteration and Ishikawa (two-step) iterations as special cases. Suantai et al. [56] have also considered some novel forward-backward algorithms for optimization and their applications to compressive sensing and image inpainting. Noor iterations have influenced the research in the fixed point theory and will continue to inspire further research in fractal geometry, chaos theory, coding, number theory, spectral geometry, dynamical systems, complex analysis, nonlinear programming, graphics and computer aided design. These three-step schemes are a natural generalization of the splitting methods for solving partial differential equations.

The projected dynamical systems associated with variational inequalities were considered by Dupuis and Nagurney [13]. The novel feature of the projected dynamical system is that the its set of stationary points corresponds to the set of the corresponding set of the solutions of the variational inequality problem. This dynamical system is a first order initial value problem. Consequently, equilibrium and nonlinear problems arising in various branches in pure and applied sciences can now be studied in the setting of dynamical systems. It has been shown [13, 22, 31, 36, 39, 48, 49, 60] that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems.

The sensitivity analysis provides useful information for designing or planning various equilibrium systems. Sensitivity analysis can provide new insight and stimulate new ideas and techniques for problem solving. Dafermos [12] studied the sensitivity analysis of the variational inequalities using the fixed point technique. This approach has strong geometrical flavour and has been investigated for various classes of quasi variational inequalities. Also see, [2, 12, 28, 31, 40, 44, 48, 49, 51] and the references therein. We would like to point out that it is not possible to establish the equivalence between the equilibrium problems and the fixed point problems. Due to these drawback, one can not suggest the multistep iterative methods for solving the equilibrium problems.

In this paper, we introduce some new classes of extended general equilibrium inclusions involving the maximal monotone operator. We establish the equivalence between the quasi extended general equilibrium inclusions and fixed point problem exploring the resolvent operator approach. This alternative equivalent formulation is used to consider the existence of the solution as well as to consider some multi step an iterative method for solving equilibrium inclusions. Several special cases are discussed as applications of the equilibrium l inclusions in Section 2. These multi step methods can be viewed as a novel generalization of the Noor (three step) iterations [29], which have applications in fixed point, fractal geometry, information technology, machine learning and medical sciences and signal processing. In section 3, we discuss the unique existence of the solution as well as to suggest several inertial iterative method along with the convergence analysis. In Section 4, dynamical system approach is applied to study the stability of the solution as well as to suggest some iterative methods for solving the extended general equilibrium problems exploring the finite difference idea. Our results in this section can be viewed as significant refinement of the known results. Sensitivity analysis for variational inequalities has been studied by many authors using quite different techniques. In Section 5, we obtain some new results for the sensitivity analysis of the extended general equilibrium inclusions.

One of the main purposes of this paper is to demonstrate the close connection among various classes of algorithms for the solution of the extended general equilibrium inclusions and to point out that researchers in different field of equilibrium inclusions and optimization. These results may motivate and bring a large number of novel, innovate potential applications, extensions and interesting topics in these areas. We have given only a brief introduction of this new field of equilibrium inclusions. The interested readers may explore this field further and discover novel and fascinating applications of the extended general equilibrium inclusions in other areas of sciences such as fractal geometry, chaos theory, coding, number theory, spectral geometry, dynamical systems, complex analysis, nonlinear programming, graphics, computer aided design and related other optimization problems. It is expected the techniques and ideas of this paper may be starting point for further research.

2 Formulations and Basic Facts

Let Ω be a nonempty closed convex set in a real Hilbert space $\mathcal{H}$. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the inner product and norm, respectively. First of all, we recall some concepts from convex analysis [1, 10, 24, 33] which are needed in the derivation of the main results.

We consider the extended general equilibrium inclusion problem. For given nonlinear operators $g, h = \mathcal{H} \longrightarrow \mathcal{H}$, a bifunction $E(.,.) : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ and maximal monotone operator $A(.)$, we consider the problem of finding $\mu \in \mathcal{H}$, such that

$$0 \in \rho E(\mu, \nu) + g(\mu) - h(\mu) + \rho A(g(\mu)), \quad \forall \nu \in \mathcal{H}, \tag{2.1}$$

which is called the extended general equilibrium inclusion.

Special Cases.

1. For $g = h$, the problem (2.1) reduces to finding $\mu \in \mathcal{H}$ such that

$$0 \in \rho E(\mu, \nu) + g(\mu) - g(\mu) + \rho A(g(\mu)), \quad \forall \nu \in \mathcal{H}, \tag{2.2}$$

is also known as the general equilibrium inclusion.

2. If $A(\cdot) = \partial \varphi(\cdot) : \mathcal{H} \longrightarrow R \cup \{+\infty\}$, the subdifferential of a convex, proper and lower semi-continuous function $\varphi(\cdot)$, then problem (2.1) is equivalent to finding $\mu \in \mathcal{H}$ such that

$$\langle \rho E(\mu, \nu) + g(\mu) - h(\mu), h(\nu) - g(\mu) \rangle + \rho(\varphi(h(\nu)) - \varphi(g(\mu)) \geq 0, \quad \forall \nu \in \mathcal{H}, \tag{2.3}$$

which is called the mixed general equilibrium variational inequality.

3. If $g = I$, the identity operator, then problem (2.1) reduces to finding $\mu \in H$ such that

$$0 \in \rho E(\mu, \nu) + \mu - h(\mu) + \rho A(\mu), \quad \forall \nu \in \mathcal{H}, \tag{2.4}$$

which is called the equilibrium inclusion.

4. If the function $\varphi(\cdot)$ is the indicator function of a closed convex set Ω in H, that is,

$$\varphi(\mu) = \begin{cases} 0, & \text{if} \quad \mu \in \Omega \\ +\infty, & \text{otherwise}, \end{cases}$$

then problem (2.3) is equivalent to finding $\mu \in \Omega$, such that

$$\langle E(\mu,\nu) + g(\mu) - h(\mu), h(\nu) - g(\mu)\rangle \geq 0, \quad \forall \quad \nu \in \Omega, \tag{2.5}$$

is called the mixed general equilibrium variational inequality.

5. For $g = h$, and $E(\mu,\nu) = \langle \mathcal{T}\mu), g(\mu) - g(\nu)\rangle$, where $\mathcal{T} : \mathcal{H} \longrightarrow \mathcal{H}$ is an arbitrary operator, the problem (2.5) reduces to finding $\mu \in \Omega$ such that

$$\langle \mathcal{T}(\mu), g(\mu) - g(\nu)\rangle \geq 0, \quad \forall \nu \in \mathcal{H}, \tag{2.6}$$

is called the general variational inequality, introduced and studied by Noor [26] in 1988. For the applications, modification and numerical aspects of the general variational inequalities, see [31,50,51].

6. If $\Omega^* = \{\mu \in H, \langle \mu, \nu\rangle \geq 0, \quad \forall \nu \in \Omega\}$ is a polar cone of the convex cone Ω in H and $h = g$, then the problem (2.5) is equivalent to finding $\mu \in H$, such that

$$g(\mu) \in \Omega, \quad E(\mu,\nu) \in \Omega^*, \quad \langle E(\mu,\nu), g(\mu)\rangle = 0, \tag{2.7}$$

is called the equilibrium complementarity problem, which appears to be new one. For $E(\mu,\nu) = \langle \mathcal{T}(\mu)$, the equilibrium problem (2.7) reduces to finding $\mu \in \mathcal{H}$ such that

$$g(\mu) \in \Omega, \quad \mathcal{T}(\mu) \in \Omega^*, \quad \langle \mathcal{T}(\mu), g(\mu)\rangle = 0,$$

is known as the general complementarity problem, introduced and studied by Noor [26] in 1988, which include the nonlinear complementarity problem as a special case.

For the applications, formulations and generalizations of the complementarity problems, see [9, 26, 31, 39, 48, 49, 52].

For special choices of the operators $\mathcal{T}, h, g, \mathcal{A}(.,.)$ the bifunction $E(.,.)$ and the closed convex set Ω, one can obtain a large number of complementarity problems and variational inequality problems as special cases of the extended general equilibrium problem (2.1). Thus it is clear that the problem (2.1) is very general and unifying one and has numerous applications in pure and applied sciences.

We now recall some well known results and notions.

Definition 2.1. If A is a maximal monotone operator on $\mathcal{H}$, then, for a constant $\rho > 0$, the resolvent operator associated with $\mathcal{T}$ is defined by

$$\mathcal{J}_A = (I + \rho A)^{-1}(\mu), \quad \forall \mu \in \mathcal{H},$$

where I is the identity operator.

It is known that the resolvent operator $\mathcal{J}_A$ is single-valued and nonexpansive, that is,

Assumption 1. The resolvent operator J_A is nonexpansive.

$$\|J_A(\mu) - J_A(\nu)\| \leq \|\mu - \nu\|, \quad \forall \mu, \nu \in \mathcal{H}. \tag{2.8}$$

Assumption 1 is used to prove the existence of a solution of extended general equilibrium inclusions as well as in analyzing convergence of the iterative methods.

Definition 2.2. The bifunction $E(.,.) : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is said to be:

1. Strongly monotone, if there exist a constant $\alpha > 0$, such that
$$E(\mu, \nu) - E(\eta, \nu) \geq \alpha \|\mu - \eta\|^2, \quad \forall \eta, \mu, \nu \in \mathcal{H}.$$
2. Lipschitz continuous, if there exist a constant $\beta > 0$, such that
$$\|E(\mu, \nu) - E(\eta, \nu)\| \leq \beta \|\mu - \eta\|, \quad \forall \eta, \mu, \nu \in \mathcal{H}.$$
3. Monotone, if
$$\langle E(\mu, \nu) - E(\eta, \nu), \nu \rangle \geq 0, \quad \forall \mu, \nu \in \mathcal{H}.$$
4. Pseudo monotone, if
$$E(\mu, \nu) \geq 0 \quad \Rightarrow \quad -E(\nu, \mu \geq 0, \quad \forall \mu, \nu \in \mathcal{H}.$$

Definition 2.3. An operator $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is said to be:

1. Strongly monotone, if there exist a constant $\alpha > 0$, such that
$$\langle \mathcal{T}(\mu) - \mathcal{T}(\nu), \mu - \nu \rangle \geq \alpha \|\mu - \nu\|^2, \quad \forall \mu, \nu \in \mathcal{H}.$$
2. Lipschitz continuous, if there exist a constant $\beta > 0$, such that
$$\|\mathcal{T}(\mu) - \mathcal{T}(\nu)\| \leq \beta \|\mu - \nu\|, \quad \forall \mu, \nu \in \mathcal{H}.$$
3. Monotone, if
$$\langle \mathcal{T}(\mu) - \mathcal{T}(\nu), \mu - \nu \rangle \geq 0, \quad \forall \mu, \nu \in \mathcal{H}.$$
4. Pseudo monotone, if
$$\langle \mathcal{T}(\mu), \nu - \mu \rangle \geq 0 \quad \Rightarrow \quad \langle \mathcal{T}(\nu), \nu - \mu \rangle \geq 0, \quad \forall \mu, \nu \in \mathcal{H}.$$

Remark 2.4. Every strongly monotone operator is a monotone operator and monotone operator is a pseudo monotone operator, but the converse is not true.

3 Resolvent Methods

In this section, we use the fixed point formulation to suggest and analyze some new implicit methods for solving the general equilibrium inclusions. First of all, we establish the equivalence between the extended general equilibrium inclusions and the fixed point problem.

Lemma 3.1. *The function $\mu \in \mathcal{H}$ is a solution of the extended general equilibrium inclusion (2.1), if and only if, $\mu \in \mathcal{H}$ satisfies the relation*

$$g(\mu) = \mathcal{J}_A[h(\mu) - \rho E(\mu, \nu)], \quad \forall \nu \in \mathcal{H}, \tag{3.1}$$

where $\mathcal{J}_{\mathcal{A}}$ is the resolvent operator and $\rho > 0$ is a constant.

Proof. Let $\mu \in \mathcal{H}$ be a solution of (2.1), then, for a constant ρ and $\forall \nu \in \mathcal{H}$,

$$\begin{aligned} \rho E(\mu, \nu) + g(\mu) \quad & - \quad h(\mu) + \rho \mathcal{A}(g(\mu)) \ni 0, \\ & \Longleftrightarrow \\ -h(\mu) + \rho E(\mu, \nu) \quad & + \quad g(\mu) + \rho A(g(\mu)) \ni 0 \\ & \Longleftrightarrow \\ g(\mu) = & \quad \mathcal{J}_A[h(\mu) - \rho E(\mu, \nu)], \end{aligned}$$

the required (3.1). □

Lemma 3.1 implies that the general equilibrium inclusion (2.1) is equivalent to the fixed point problem (3.1). This equivalent fixed point formulation (3.1) plays an important role in deriving the main results.

From the equation (3.1), we have

$$\mu = \mu - g(\mu) + J_A\big[h(\mu) - \rho E(\mu, \nu))\big].$$

We define the function F associated with (3.1) as

$$F(\mu) = \mu - g(\mu) + J_A\big[h(\mu) - \rho E(\mu, \nu))\big]. \tag{3.2}$$

To prove the unique existence of the solution of the problem (2.1), it is enough to show that the map F defined by (3.2) has a fixed point.

Theorem 3.2. *Let the operator g be strongly monotone with constant $\sigma > 0$ and Lipschitz continuous with constant $\zeta > 0$, respectively. Let the bifunction $E(.,.)$ and the operator h be Lipschitz continuous with constants β, ζ_1. If there exists a parameter $\rho > 0$, such that*

$$\rho < \frac{1-k}{\beta}, \quad k < 1, \tag{3.3}$$

where

$$\theta = \rho\beta + k \tag{3.4}$$
$$k = \sqrt{1 - 2\sigma + \zeta^2} + \zeta_1, \tag{3.5}$$

then there exists a unique solution of the problem (2.1).

Proof. From Lemma 3.1, it follows that problems (3.1) and (2.1) are equivalent. Thus it is enough to show that the map $F(u)$, defined by (3.2) has a fixed point.

For all $\eta \neq \mu \in \mathcal{H}$, we have

$$\begin{aligned}
\Big\|F(\mu) - F(\eta)\Big\| &= \Big\|\mu - \eta - (g(\mu) - g(\eta))\Big\| \\
&\quad + J_A\Big\|\Big[h(\mu) - \rho E(\mu,\nu))\Big] - J_A\Big[h(\eta) - \rho E(\eta,\nu)\Big]\Big\| \\
&\leq \Big\|\mu - \eta - (g(\mu) - g(\eta))\Big\| + \Big\|h(\eta) - h(\mu) - \rho(E(\eta,\nu) - E(\mu,\nu))\Big\| \\
&\leq \Big\|\mu - \eta - (g(\mu) - g(\eta))\Big\| + \Big\|h(\eta) - h(\mu)\Big\| + \rho\Big\|(E(\eta,\nu) - E(\mu,\nu)\Big\| \\
&\leq \Big\|\mu - \eta - (g(\mu) - g(\eta)\Big\| + \zeta_1\Big\|\eta - \mu\Big\| + \rho\beta\Big\|\eta - \mu\Big\|.
\end{aligned} \tag{3.6}$$

Since the operator g is strongly monotone with constants $\sigma > 0$ and Lipschitz continuous with constant $\zeta > 0$, it follows that

$$\begin{aligned}
\|\mu - \eta - (g(\mu) - g(\eta))\|^2 &\leq \|\mu - \eta\|^2 - 2\langle g(\mu) - g(\eta), \mu - \eta\rangle \\
&\quad + \zeta^2\|g(\mu) - g(\eta)\|^2 \\
&\leq (1 - 2\sigma + \zeta^2)\|\mu - \eta\|^2.
\end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we have

$$\begin{aligned}
\|F(\mu) - F(\nu)\| &\leq 2\left\{\sqrt{(1 - 2\sigma + \zeta^2)} + \zeta_1 + \rho\beta\right\}\|\mu - \nu\| \\
&= \theta\|\mu - \eta\|,
\end{aligned}$$

where

$$\begin{aligned}
\theta &= \rho\beta + k \\
k &= 2\sqrt{1 - 2\sigma + \zeta^2} + \zeta_1.
\end{aligned}$$

From (3.3), it follows that $\theta < 1$, which implies that the map $F(u)$ defined by (3.2) has a fixed point, which is the unique solution of (2.1). □

The fixed point formulation (3.1) is applied to propose and suggest the iterative methods for solving the problem (2.1). We now suggest and analyze the three step iterative methods for solving the general equilibrium inclusion (2.1).

Algorithm 1. For a given μ_0, compute the approximate solution $\{\mu_{n+1}\}$ by the iterative schemes

$$y_n = (1-\gamma_n)\mu_n + \gamma_n\{\mu_n - g(\mu_n) + J_A[h(\mu_n) - \rho E(\mu,\nu)]\} \quad (3.8)$$

$$z_n = (1-\beta_n)\mu_n + \beta_n\{y_n - g(y_n) + J_A[h(y_n) - \rho E(y_n,\nu)]\} \quad (3.9)$$

$$\mu_{n+1} = (1-\alpha_n)\mu_n + \alpha_n\{z_n - g(z_n) + J_A[h(z_n) - \rho E(z_n,\nu]\}, \quad (3.10)$$

which are known as modified Noor iterations.

We now study the convergence analysis of Algorithm 1, which is the main motivation of our next result.

Theorem 3.3. *Let the operators g, h and the bifunction $E(.,.)$ satisfy all the assumptions of Theorem 3.2. If the condition (3.3) holds, then the approximate solution $\{u_n\}$ obtained from Algorithm 1 converges to the exact solution $\mu \in \mathcal{H}$ of the general equilibrium inclusion (2.1) strongly in $\mathcal{H}$.*

Proof. From Theorem 3.2, we see that there exists a unique solution $\mu \in \mathcal{H}$ of the general equilibrium inclusions (2.1). Let $\mu \in H$ be the unique solution of (2.1). Then, using Lemma 3.1, we have

$$\begin{aligned}\mu &= (1-\alpha_n)\mu + \alpha_n\{\mu - g(\mu) + J_A[h(\mu) - \rho E(\mu,\nu)]\} &(3.11)\\ &= (1-\beta_n)\mu + \beta_n\{\mu - g(\mu) + J_A[h(\mu) - \rho E(\mu,\nu)]\} &(3.12)\\ &= (1-\gamma_n)\mu + \gamma_n\{\mu - g(\mu) + J_A[h(\mu) - \rho E(\mu,\nu)]\}. &(3.13)\end{aligned}$$

From (3.10), (3.11), we have

$$\begin{aligned}\|\mu_{n+1} - \mu\| &= \|(1-\alpha_n)(\mu_n - \mu) + \alpha_n(z_n - \mu - (g(z_n) - g(\mu)))\\ &\quad + \alpha_n\{A[h(\mu_n) - \rho E(\mu_n,\nu)] - J_{A(\mu)}[h(\mu) - \rho E(\mu,\nu)\}\|\\ &\leq (1-\alpha_n)\|\mu_n - \mu\| + \alpha_n\|z_n - \mu - (g(z_n) - g(\mu))\|\\ &\quad + \alpha_n\|h(w_n) - h(\mu) - \rho(E(z_n,\nu) - E(\mu,\nu)\|\\ &\leq (1-\alpha_n)\|\mu_n - \mu\| + \alpha_n(k+\rho\beta)||w_n - \mu||\\ &= (1-\alpha_n)\|u_n - \mu\| + \alpha_n\theta\|w_n - \mu\|, &(3.14)\end{aligned}$$

where θ is defined by (3.4).

In a similar way, from (3.8) and (3.12), we have

$$\begin{aligned}\|z_n - \mu\| &\leq (1-\beta_n)\|\mu_n - \mu\| + 2\beta_n\theta\|y_n - \mu - (g(y_n) - g(\mu))\|\\ &\quad + \beta_n\|g(y_n) - g(\mu) - \rho(y_n - \mu)\| + \beta_n\eta\|y_n - \mu\|\\ &\leq (1-\beta_n)\|\mu_n - \mu\| + \beta_n(k+\rho)\|y_n - \mu\|,\\ &\leq (1-\beta_n)\|\mu_n - \mu\| + \beta_n\theta\|y_n - \mu\|, &(3.15)\end{aligned}$$

where θ is defined by (3.3).

From (3.8) and (3.13), we obtain

$$\begin{aligned}\|y_n-\mu\| &\le (1-\gamma_n)\|\mu_n-\mu\|+\gamma_n\theta\|\mu_n-\mu\| \\ &\le (1-(1-\theta)\gamma_n)\|\mu_n-\mu\|\le\|\mu_n-\mu\|. \end{aligned} \tag{3.16}$$

From (3.15) and (3.16), we obtain

$$\begin{aligned}\|z_n-\mu\| &\le (1-\beta_n)\|\mu_n-\mu\|+\beta_n\theta\|\mu_n-\mu\| \\ &= (1-(1-\theta)\beta_n)\|\mu_n-\mu\|\le\|\mu_n-\mu\|. \end{aligned} \tag{3.17}$$

Form the above equations, we have

$$\begin{aligned}\|\mu_{n+1}-\mu\| &\le (1-\alpha_n)\|\mu_n-\mu\|+\alpha_n\theta\|\mu_n-\mu\| \\ &= [1-(1-\theta)\alpha_n]\|\mu_n-\mu\| \\ &\le \prod_{i=0}^{n}[1-(1-\theta)\alpha_i]\|\mu_0-\mu\|. \end{aligned}$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1-\theta>0$, we have $\prod_{i=0}^{n}[1-(1-\theta)\alpha_i]=0$. Consequently the sequence $\{u_n\}$ convergence strongly to μ. From (3.16) and (3.17), it follows that the sequences $\{y_n\}$ and $\{w_n\}$ also converge to μ strongly in $\mathcal{H}$. This completes the proof. □

We suggest new perturbed iterative schemes for solving the extended general equilibrium inclusion (2.1).

Algorithm 2. For a given μ_0, compute the approximate solution $\{\mu_n\}$ by the iterative schemes

$$\begin{aligned} y_n &= (1-\gamma_n)\mu_n+\gamma_n\{\mu_n-g(\mu_n)+J_A[h(\mu_n)-\rho E(\mu_n,\nu)]\}+\gamma_n h_n \\ z_n &= (1-\beta_n)\mu_n+\beta_n\{y_n-g(y_n)+J_A[h(y_n)-\rho E(y_n,\nu)]\}+\beta_n f_n \\ \mu_{n+1} &= (1-\alpha_n)\mu_n+\alpha_n\{z_n-g(z_n)+J_A[h(z_n)-\rho E(z_n,\nu))\}+\alpha_n e_n, \end{aligned}$$

where $\{e_n\}$, $\{f_n\}$, and $\{h_n\}$ are the sequences of the elements of $\mathcal{H}$ introduced to take into account possible inexact computations and $J_{A(\mu_n)}$ is the corresponding perturbed resolvent operator and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy

$$0\le\alpha_n,\beta_n,\gamma_n\le 1;\quad \forall n\ge 0,\quad \sum_{n=0}^{\infty}\alpha_n=\infty.$$

For $\gamma_n=0$, we obtain the perturbed Ishikawa iterative method and for $\gamma_n=0$ and $\beta_n=0$, we obtain the perturbed Mann iterative schemes for solving general equilibrium inclusion (2.1).

Also, we can suggest the following iterative methods for solving the general equilibrium inclusions.

Algorithm 3. For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1}=\mu_n-g(\mu_n)+J_A[h(\mu_n)-\rho E(\mu_n,\nu))], \tag{3.18}$$

which is known as the resolvent method.

Algorithm 4. For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \mu_n - g(\mu_n) + J_{A(}[h(\mu_n) - \rho E(\mu_{n+1}, \nu))], \tag{3.19}$$

which is an implicit resolvent method and is equivalent to the following two-step method.

Algorithm 5. For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} z_n &= \mu_n - g(\mu_n) + J_A[h(\mu_n) - \rho E(\mu_n, \nu)] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + J_A[h(\mu_n) - \rho E(z_n, \nu))]. \end{aligned}$$

Algorithm 6. For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \mu_n - g(\mu_n) + J_A[h(\mu_{n+1}) - \rho E(\mu_{n+1}, \nu)],$$

which is known as the modified resolvent method and is equivalent to the iterative method.

Algorithm 7. For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} z_n &= \mu_n - g(\mu_n) + J_{A(\mu_n)}[g(\mu_n) - \rho E(\mu_n, \nu)] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + J_A[h(z_n) - \rho E(z_n, \nu)], \end{aligned}$$

which is two-step predictor-corrector method for solving the problem (2.1).

We can rewrite the equation (3.1) as:

$$\mu = \mu - g(\mu) + J_A[h(\frac{\mu + \mu}{2}) - \rho E(\frac{\mu + \mu}{2}, \nu)].$$

This fixed point formulation is used to suggest the following implicit method.

Algorithm 8. For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \mu_n - g(\mu_n) + J_A[h(\frac{\mu_n + \mu_{n+1}}{2}) - \rho E(\frac{\mu_n + \mu_{n+1}}{2}, \nu)].$$

To implement the implicit method, one uses the predictor-corrector technique. We obtain a new two-step method for solving the problem (2.1).

Algorithm 9. For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} z_n &= \mu_n - g(\mu_n) + J_A[h(\mu_n) - \rho E(\mu_n, \nu)] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + J_A\Big[h(\Big(\frac{z_n + \mu_n}{2}\Big) - \rho E(\frac{z_n + \mu_n}{2}, \nu)\Big], \end{aligned}$$

which is a new predictor-corrector two-step method.

For a parameter ξ, one can rewrite the (3.1) as

$$\mu = \mu - g(\mu) + J_A\Big[h((1-\xi)\mu + \xi\mu) - \rho E((1-\xi)\mu + \xi\mu, \nu)\Big].$$

This equivalent fixed point formulation enables to suggest the following inertial method for solving the problem (2.1).

Algorithm 10. For a given $\mu_0, \mu_1 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \mu_n - g(\mu_n) + J_A\Big[h((1-\xi)\mu_n + \xi\mu_{n-1}) - \rho E((1-\xi)\mu_n + \xi\mu_{n-1}), \nu)\Big],$$

which is equivalent to the following two-step inertial method.

Algorithm 11. For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} z_n &= (1-\xi)u_n + \xi u_{n-1} \\ \mu_{n+1} &= \mu_n - g(\mu_n) + J_A[h(z_n) - \rho E(z_n, \nu)]. \end{aligned}$$

We now suggest multi-step inertial methods for solving the extended general equilibrium inclusions (2.1).

Algorithm 12. For given μ_0, μ_1, compute μ_{n+1} by the recurrence relation

$$\begin{aligned} & z_n = \mu_n - \theta_n(\mu_n - \mu_{n-1}), \\ & y_n = (1-\gamma_n)z_n + \gamma_n\left\{z_n - g(z_n) + J_A\left[h\left(\frac{z_n + \mu_n}{2}\right) - \rho E\left(\frac{z_n + \mu_n}{2}, \nu\right)\right]\right\}, \\ & t_n = (1-\beta_n)y_n + \beta_n\left\{y_n - g(y_n) + J_A\left[h\left(\frac{y_n + z_n + \mu_n}{3}\right) - \rho E\left(\frac{y_n + z_n + \mu_n}{3}, \nu\right)\right]\right\}, \\ & \mu_{n+1} = (1-\alpha_n)z_n + \alpha_n\left\{t_n - g(t_n) + J_A\left[h\left(\frac{z_n + y_n + t_n + \mu_n}{4}\right) - \rho E\left(\frac{y_n + z_n + t_n + \mu_n}{4}, \nu\right)\right]\right\}, \end{aligned}$$

where $\alpha_n, \beta_n, \gamma_n, \theta_n \in [0,1], \quad \forall n \geq 1$.

For $g = h$, Algorithm 12 reduces to:

Algorithm 13. For given μ_0, μ_1, compute μ_{n+1} by the recurrence relation

$$\begin{aligned} & z_n = \mu_n - \theta_n(\mu_n - \mu_{n-1}), \\ & y_n = (1-\gamma_n)z_n + \gamma_n\left\{z_n - g(z_n) + J_A\left[g\left(\frac{z_n + \mu_n}{2}\right) - \rho E\left(\frac{z_n + \mu_n}{2}, \nu\right)\right]\right\}, \\ & t_n = (1-\beta_n)y_n + \beta_n\left\{y_n - g(y_n) + J_A\left[g\left(\frac{y_n + z_n + \mu_n}{3}\right) - \rho E\left(\frac{y_n + z_n + \mu_n}{3}, \nu\right)\right]\right\}, \\ & \mu_{n+1} = (1-\alpha_n)z_n + \alpha_n\left\{t_n - g(t_n) + J_A\left[g\left(\frac{z_n + y_n + t_n + \mu_n}{4}\right) - \rho E\left(\frac{y_n + z_n + t_n + \mu_n}{4}, \nu\right)\right]\right\}, \end{aligned}$$

where $\alpha_n, \beta_n, \gamma_n, \theta_n \in [0,1], \quad \forall n \geq 1$.

for solving the general equilibrium inclusions (2.2), where $\alpha_n, \beta_n, \gamma_n, \theta_n \in [0,1], \quad \forall n \geq 1$.

Remark 3.4. For different and suitable choice of the parameters ρ, η, α, operators g, h, the bifunction $E(.,.)$ and convex-valued sets, one can recover new and known iterative methods for solving general equilibrium inclusions, equilibrium complementarity problems and related optimization problems. Using the technique and ideas of Theorem 3.2 and Theorem 3.3, one can analyze the convergence of Algorithm 12 and its special cases.

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4 Dynamical Systems Technique

In this section, we consider the dynamical systems technique for solving the extended general equilibrium inclusions. The projected dynamical systems associated with variational inequalities were considered by Dupuis and Nagurney [13]. It is worth mentioning that the dynamical systems are the initial value and boundary value problems. Consequently, variational inequalities and nonlinear problems arising in various branches in pure and applied sciences can now be studied via the differential equations. It has been shown that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems, see [13,22,31,39,41,45,48,49,59,60]. We consider some new iterative methods for solving the extended general variational inclusions. We investigate the convergence analysis of these new methods involving only the monotonicity of the operators.

We now define the residue vector $R(\mu)$ by the relation

$$R(\mu) = \Big\{ J_A[h(\mu) - \rho E(\mu,\nu)] - g(\mu) \Big\}, \quad \forall \nu \in \mathcal{H}. \tag{4.1}$$

Invoking Lemma 3.1, one can easily conclude that $\mu \in \mathcal{H}$ is a solution of the problem (2.1), if and only if, $\mu \in \mathcal{H}$ is a zero of the equation

$$R(\mu) = 0. \tag{4.2}$$

We now consider a dynamical system associated with the extended general equilibrium inclusions. Using the equivalent formulation (3.1), we suggest a class of resolvent dynamical systems as

$$\frac{d\mu}{dt} = \lambda \Big\{ J_A[h(\mu) - \rho E(\mu,\nu)] - g(\mu) \Big\}, \quad \mu(t_0) = \alpha, \tag{4.3}$$

where λ is a parameter. The system of type (4.3) is called the resolvent dynamical system associated with the problem (2.1). Here the right hand is related to the resolvent and is discontinuous on the boundary. From the definition, it is clear that the solution of the dynamical system always stays in $\mathcal{H}$. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution of (2.1) can be studied.

The equilibrium point of the dynamical system (4.3) is defined as follows.

Definition 4.1. An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (4.3), if,

$$\frac{d\mu}{dx} = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the extended general equilibrium inclusion (2.1), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point. This implies that $\mu \in \mathcal{H}$ is a solution of the general equilibrium inclusion (2.1), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.

Definition 4.2. [15] The dynamical system is said to converge to the solution set S^* of (4.3), if , irrespective of the initial point, the trajectory of the dynamical system satisfies

$$\lim_{t\to\infty} \text{dist}(\mu(t), S^*) = 0, \tag{4.4}$$

where

$$\text{dist}(\mu, S^*) = \inf_{\nu \in S^*} \|\mu - \nu\|.$$

It is easy to see, if the set S^* has a unique point μ^*, then (4.4) implies that

$$\lim_{t\to\infty} \mu(t) = \mu^*.$$

If the dynamical system is still stable at μ^* in the Lyapunov sense, then the dynamical system is globally asymptotically stable at μ^*.

Definition 4.3. The dynamical system is said to be globally exponentially stable with degree η at μ^*, if, irrespective of the initial point, the trajectory of the system satisfies

$$\|\mu(t) - \mu^*\| \leq u_1\|\mu(t_0) - \mu^*\| exp(-\eta(t - t_0)), \quad \forall t \geq t_0,$$

where u_1 and η are positive constants independent of the initial point.

It is clear that the globally exponentially stability is necessarily globally asymptotically stable and the dynamical system converges arbitrarily fast.

Lemma 4.4. *(Gronwall Lemma) [22] Let $\hat{\mu}$ and $\hat{\nu}$ be real-valued nonnegative continuous functions with domain $\{t : t \leq t_0\}$ and let $\alpha(t) = \alpha_0(|t - t_0|)$, where α_0 is a monotone increasing function. If, for $t \geq t_0$,*

$$\hat{\mu} \leq \alpha(t) + \int_{t_0}^{t} \hat{\mu}(s)\hat{\nu}(s)ds,$$

then

$$\hat{\mu}(s) \leq \alpha(t) exp\{\int_{t_0}^{t} \hat{\nu}(s)ds\}.$$

We now establish that the trajectory of the solution of the resolvent dynamical system (4.3) converges to the unique solution of the extended general equilibrium inclusions (2.1).

Theorem 4.5. *Let the bifunction $E(.,.)$ and the operators $g, h : H \longrightarrow H$ be Lipschitz continuous with constants $\beta > 0, \zeta > 0, \zeta_1 > 0$ respectively. If $\lambda(\zeta + \zeta_1 + \rho\beta) < 1$, then, for each $\mu_0 \in \mathcal{H}$, there exists a unique continuous solution $\mu(t)$ of the dynamical system (4.3) with $\mu(t_0) = \mu_0$ over $[t_0, \infty)$.*

Proof. Let

$$G(\mu) = \{J_A[h(\mu) - \rho E(\mu,\nu)] - g(\mu)\}, \quad \forall \mu \in H.$$

where $\lambda > 0$ is a constant and $G(\mu) = \frac{d\mu}{dt}$.

$\forall \mu, \nu \in H$, we have

$$\begin{aligned}
\|G(\mu) - G(\eta)\| &\leq \lambda\{J_A[h(\mu) - \rho E(\mu,\nu)] - J_A[h(\eta) - \rho E(\eta,\nu)]\|\} \\
&\quad +\lambda\|g(\mu) - g(\eta)\| \\
&= \lambda\{\|g(\mu) - g(\eta)\| + \|J_A[h(\mu) - \rho E(\mu,\nu)] - J_A[h(\eta) - \rho E(\eta,\nu)]\| \\
&\quad +\|J_A[h(\eta) - \rho E(\eta,\nu)] - J_A[h(\eta) - \rho E(\eta,\nu)]\|\} \\
&\leq \lambda\{\|g(\mu) - g(\eta)\| + \|h(\mu) - h(\eta) - \rho(E(\mu,\nu) - E(\eta,\nu))\} \\
&\leq \lambda\{\|g(\mu) - g(\eta)\| + \|h(\mu) - h(\eta)\| + \rho\|E(\mu,\nu) - E(\eta,\nu)\|\} \\
&\leq \lambda\{(\zeta + \zeta_1 + \beta\rho)\}\|\mu - \eta\|.
\end{aligned}$$

This implies that the operator $G(\mu)$ is a Lipschitz continuous with constant $\lambda\{(\zeta + \zeta_1 + \rho\beta)\} < 1$ and for each $\mu \in \mathcal{H}$, there exists a unique and continuous solution $\mu(t)$ of the dynamical system (4.3), defined on an interval $t_0 \leq t < T_1$ with the initial condition $\mu(t_0) = \mu_0$. Let $[t_0, T_1)$ be its maximal interval of existence. Then we have to show that $T_1 = \infty$. Consider, for any $\mu \in \Omega(\mu)$,

$$\begin{aligned}
\|G(\mu)\| &= \|\frac{d\mu}{dt}\| = \lambda\|[h(\mu) - \rho E(\mu,\nu)] - g(\mu)\| \\
&\leq \lambda\{\|J_A[h(\mu) - \rho E(\mu,\nu)] - J_A[0]\| + \|J_A[0] - g(\mu)\|\} \\
&\leq \lambda\{\delta\|\{g(\mu) - \rho E(\mu,\nu)\| + \|J_A[h(\mu)] - J_A[0]\| + \|J_A[0] - g(u)\|\} \\
&\leq \lambda\{(\rho\beta + \zeta_1 + \zeta)\|u\| + 2\|J_{A(\mu)}[0]\|\}.
\end{aligned}$$

Then

$$\begin{aligned}
\|\mu(t)\| &\leq \|\mu_0\| + \int_{t_0}^{t} \|\mu(s)\| ds \\
&\leq (\|\mu_0\| + k_1(t - t_0)) + k_2 \int_{t_0}^{t} \|\mu(s)\| ds,
\end{aligned}$$

where $k_1 = 2\lambda\|J_{A(\mu)}[0]\|$ and $k_2 = \delta\lambda(\rho\beta + \zeta_1 + \zeta)$. Hence by the Gronwall Lemma 4.4, we have

$$\|\mu(t)\| \leq \{\|u_0\| + k_1(t - t_0)\}e^{k_2(t-t_0)}, \quad t \in [t_0, T_1).$$

This shows that the solution is bounded on $[t_0, T_1)$. So $T_1 = \infty$. □

Theorem 4.6. *If the assumptions of Theorem 4.5 hold, then the dynamical system (4.3) converges globally exponentially to the unique solution of the extended general equilibrium inclusion (2.1).*

Proof. Since the bifunction $E(.,.)$ and the operators h, g are Lipschitz continuous, it follows from Theorem 4.5 that the dynamical system (4.3) has unique solution $\mu(t)$ over $[t_0, T_1)$ for any fixed $\mu_0 \in H$. Let $\mu(t)$ be a solution of the initial value problem (4.3). For a given $\mu^* \in H$ satisfying (2.1), consider the Lyapunov function

$$L(\mu) = \lambda\|\mu(t) - \mu^*\|^2, \quad u(t) \in \mathcal{H}. \tag{4.5}$$

From (4.3) and (4.5), we have

$$\begin{aligned}
\frac{dL}{dt} &= 2\lambda\langle \mu(t) - \mu^*, \frac{d\mu}{dt}\rangle \\
&= 2\lambda\langle \mu(t) - \mu^*, J_A[h(\mu(t)) - \rho E(\mu(t), \nu)] - g(\mu(t))\rangle \\
&= 2\lambda\langle \mu(t) - \mu^*, J_A[h(\mu(t)) - \rho E(\mu(t), \nu)] - g(\mu^*) \\
&\quad +g(\mu^*) - g(\mu(t))\rangle \\
&= -2\lambda\langle \mu(t) - \mu^*, g(\mu(t)) - g(\mu^*)\rangle \\
&\quad +2\lambda\langle \mu(t) - \mu^*, J_A[h(\mu(t)) - \rho E(\mu(t), \nu)] - g(\mu^*)\rangle \\
&\leq -2\lambda\langle \rho(E(\mu(t), \nu) - E(\mu^*(t), v)), g(\mu(t)) - g(\mu^*)\rangle \\
&\quad +2\lambda\langle \mu(t) - \mu^*(t), J_A[g(\mu(t)) - \rho E(\mu(t), \nu)] \\
&\quad -J_A[h(\mu^*(t)) - \rho E(\mu^*(t), \nu)]\rangle, \\
&\leq -2\lambda\sigma\|\mu(t) - \mu^*\|^2 + \lambda\|g(\mu(t)) - g(\mu^*)\|^2 \\
&\quad +\lambda\|J_A[h(\mu(t)) - \rho E(\mu(t), \nu)] - J_A[h(\mu^*(t)) - \rho E(\mu^*(t), \nu)\|^2
\end{aligned} \tag{4.6}$$

Using the Lipschitz continuity of the operators $\mathcal{T}, h$, we have

$$\begin{aligned}
& \|J_A[h(\mu(t)) - \rho E(\mu(t), \nu)] - J_A[h(\mu^*(t)) - \rho E(\mu^*(t), \nu))]\| \\
\leq\ & \|h(\mu(t)) - h(\mu^*(t)) - \rho(E(\mu(t), \nu) - E(\mu^*(t), \nu))\| \\
\leq\ & (\zeta_1 + \rho\beta)\|\mu(t) - \mu^*(t)\|.
\end{aligned} \tag{4.7}$$

From (4.6) and (4.7), we have

$$\frac{d}{dt}\|\mu(t) - \mu^*(t)\| \leq 2\xi\lambda\|\mu(t) - \mu^*(t)\|,$$

where

$$\xi = ((\zeta_1 + \rho\beta) - 2\sigma).$$

Thus, for $\lambda = -\lambda_1$, where λ_1 is a positive constant, we have

$$\|\mu(t) - \mu^*\| \leq \|\mu(t_0) - \mu^*\|e^{-\xi\lambda_1(t-t_0)},$$

which shows that the trajectory of the solution of the dynamical system (4.3) converges globally exponentially to the unique solution of the extended general equilibrium inclusions (2.1). □

We use the dynamical system (4.3) to suggest some iterative for solving the extended general equilibrium inclusion (2.1). These methods can be viewed in the sense of Noor [29–31] involving the double resolvent operator.

For simplicity, we take $\lambda = 1$. Thus the dynamical system(4.3) becomes

$$\frac{d\mu}{dt} + g(\mu) = J_{A(\mu)}\big[h(\mu) - \rho E(\mu,\nu)\big], \quad \mu(t_0) = \alpha. \tag{4.8}$$

The forward difference scheme is used to construct the implicit iterative method. Discretizing (4.8), we have

$$\frac{\mu_{n+1} - \mu_n}{h_1} + g(\mu_n) = J_A[h(\mu_n) - \rho E(\mu_{n+1},\nu)], \tag{4.9}$$

where h_1 is the step size.

Now, we can suggest the following implicit iterative method for solving the problem (2.1).

Algorithm 14. For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \mu_n - g(\mu_n) + J_A\Big[h(\mu_n) - \rho E(\mu_{n+1},\nu) - \frac{\mu_{n+1} - \mu_n}{h_1}\Big].$$

This is an implicit method and is equivalent to the following two-step method.

Algorithm 15. For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= \mu_n - g(\mu_n) + J_A[h(\mu_n) - \rho E(\mu_n,\nu)] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + J_A\Big[h(\mu_n) - \rho E(y_n,\nu) - \frac{y_n - \mu_n}{h_1}\Big]. \end{aligned}$$

Discretizing (4.8), we now suggest an other implicit iterative method for solving the extended general equilibrium inclusion (2.1).

$$\frac{\mu_{n+1} - \mu_n}{h} + g(\mu_n) = J_A[h(\mu_{n+1}) - \rho E(\mu_{n+1},\nu)], \tag{4.10}$$

where h is the step size.

This formulation enables us to suggest the two-step iterative method.

Algorithm 16. For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= \mu_n - g(\mu_n) + J_A\Big[h(\mu_n) - \rho E(\mu_n,\nu)\Big] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + J_A\Big[h(y_n) - \rho E(y_n,\nu) - \frac{y_n - \mu_n}{h}\Big]. \end{aligned}$$

Discretizing (4.8), we propose another implicit iterative method.

$$\frac{\mu_{n+1} - \mu_n}{h_1} + g(\mu_n) = J_A\Big[h(\mu_n) - \rho E(\mu_{n+1}, \nu)\Big]$$

where h_1 is the step size.

For $h_1 = 1$, we can suggest an implicit iterative method for solving the problem (2.1).

Algorithm 17. For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \mu_n - g(\mu_n) + J_A\Big[h(\mu_n) - \rho E(\mu_{n+1}, \nu)\Big].$$

From (4.8), we have

$$\frac{d\mu}{dt} + g(\mu) \quad = \quad J_A\Big[h((1-\alpha)\mu + \alpha\mu) - \rho E((1-\alpha)\mu + \alpha\mu)), \nu)\Big], \tag{4.11}$$

where $\alpha \in [0, 1]$ is a constant.

Discretization (4.11) and taking $h_1 = 1$, we have

$$\mu_{n+1} \quad = \quad \mu_n - g(\mu_n) + J_A\Big[h\big((1-\alpha)\mu_n + \alpha\mu_{n-1}\big) - \rho E((1-\alpha)\mu_n + \alpha\mu_{n-1}), \nu)\Big],$$

which is an inertial type iterative method for solving the extended general equilibrium inclusion (2.1). Using the predictor-corrector techniques, we have

Algorithm 18. For a given μ_0, μ_1, compute μ_{n+1} by the iterative schemes

$$\begin{aligned} y_n &= (1-\alpha)\mu_n + \alpha\mu_{n-1} \\ \mu_{n+1} &= \mu_n - g(\mu_n) + J_A\Big[h(y_n) - \rho E(y_n, \nu)\Big], \end{aligned}$$

which is known as the inertial two-step iterative method.

We now introduce the second order dynamical system associated with the extended general equilibrium inclusion (2.1). To be more precise, we consider the problem of finding $\mu \in \mathtt{H}$ such that

$$\gamma\frac{d^2\mu}{dx^2} + \frac{d\mu}{dx} = \lambda\Big\{J_A\Big[h(\mu) - \rho E(\mu, \nu)\Big] - g(\mu)\Big\}, \tag{4.12}$$
$$\mu(a) = \alpha, \mu(b) = \beta,$$

where $\gamma > 0, \lambda > 0$ and $\rho > 0$ are constants. We would like to emphasize that the problem (4.12) is indeed a second order boundary vale problem. In a similar way, we can define the second order initial value problem associated with the dynamical system.

The equilibrium point of the dynamical system (4.12) is defined as follows.

Definition 4.7. An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (4.12), if,

$$\gamma \frac{d^2\mu}{dx^2} + \frac{d\mu}{dx} = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the general equilibrium inclusion (2.1), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.

From (4.12), we have

$$g(\mu) = J_A\Big[h(\mu) - \rho E(\mu, \nu)\Big].$$

Thus, we can rewrite (4.12) as follows:

$$g(\mu) = J_A\Big[h(\mu) - \rho E(\mu, \nu) + \gamma \frac{d^2\mu}{dx^2} + \frac{d\mu}{dx}\Big]. \tag{4.13}$$

For $\lambda = 1$, the problem (4.12) is equivalent to finding $\mu \in \mathcal{H}$ such that

$$\begin{aligned} \gamma \frac{d^2\mu}{dx^2} + \frac{d\mu}{dx} + g(\mu) &= J_A\Big[h(\mu) - \rho E(\mu, \nu)\Big], \\ &\mu(a) = \alpha, \mu(b) = \beta. \end{aligned} \tag{4.14}$$

The problem (4.14) is called the second dynamical system, which is in fact a second order boundary value problem. This interlink among various fields of mathematical and engineering sciences is fruitful in developing implementable numerical methods for finding the approximate solutions of the extended general equilibrium inclusions. Consequently, one can explore the ideas and techniques of the differential equations to suggest and propose hybrid proximal point methods for solving the extended general equilibrium variational inclusions and related optimization problems.

We discretize the second-order dynamical systems (4.14) using central finite difference and backward difference schemes to have

$$\gamma \frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h_1^2} + \frac{\mu_n - \mu_{n-1}}{h_1} + g(\mu_n) = J_A\Big[h(\mu_n) - \rho E(\mu_{n+1}, \nu)\Big], \tag{4.15}$$

where h_1 is the step size.

If $\gamma = 1, h_1 = 1$, then, from equation(4.15) we have

Algorithm 19. For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \mu_n + g(\mu_n) + J_A\Big[h(\mu_n) - \rho E(\mu_{n+1}, \nu)\Big],$$

which is the extragradient method for solving the extended general equilibrium inclusions (2.1).

Algorithm 19 is an implicit method. To implement the implicit method, we use the predictor-corrector technique to suggest the method.

Algorithm 20. For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n + \theta_n\mu_{n-1} \\ \mu_{n+1} &= \mu_n - g(\mu_n) + J_A\Big[h(\mu_n) - \rho E(y_n, \nu)\Big], \end{aligned}$$

is called the two-step inertial iterative method, where $\theta_n \in [0,1]$ is a constant.

In a similar way, we have the following two-step method.

Algorithm 21. For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n + \theta_n\mu_{n-1} \\ \mu_{n+1} &= \mu_n - g(\mu_n) + J_A\Big[h(y_n) - \rho E(y_n, \nu)\Big], \end{aligned}$$

which is also called the double inertial resolvent method for solving the extended general equilibrium inclusions (2.1).

We discretize the second-order dynamical systems (4.3) using central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h_1^2} + \frac{\mu_n - \mu_{n-1}}{h_1} + g(\mu_{n+1}) = J_A\Big[h(\mu_n) - \rho E(\mu_{n+1}, \nu)\Big],$$

where h_1 is the step size.

Using this discrete form, we can suggest the following an iterative method for solving the extended general equilibrium inclusions (2.1).

Algorithm 22. For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \mu_n - g(\mu_n) + J_A\Big[h(\mu_{n+1}) - \rho E(\mu_{n+1}, \nu) - \gamma\frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h_1^2} + \frac{\mu_n - \mu_{n-1}}{h_1}\Big].$$

Algorithm 22 is called the hybrid inertial proximal method for solving the extended general equilibrium inclusions and related optimization problems. This is a new proposed method.

Note that, for $\gamma = 1, h_1 = 1$, Algorithm 22 reduces to the following iterative method.

Algorithm 23. For given μ_0, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \mu_n - g(\mu_n) + J_A\Big[h(\mu_{n+1}) + \mu_{n+1} - \mu_n - \rho E(\mu_{n+1}, \nu)\Big],$$

which is called the resolvent method.

We now consider the third order dynamical systems associated with the extended general equilibrium inclusions of the type (2.1). To be more precise, we consider the problem of finding $\mu \in \mathcal{H}$, such that

$$\gamma \frac{d^3\mu}{dt^3} + \zeta \frac{d^2\mu}{dt^2} + \xi \frac{d\mu}{dt} + g(\mu) = J_A[h(\mu) - \rho E(\mu, \nu)], \tag{4.16}$$
$$\text{Bonudary Conditions} \qquad u(a) = \alpha, \dot{\mu}(a) = \beta, \dot{\mu}(b) = \beta_1,$$

where $\gamma > 0, \zeta, \xi, \beta, \alpha, \beta_1$ and $\rho > 0$ are constants. Problem (4.16) is called third order dynamical system associated with extended general equilibrium inclusions (2.1).

The equilibrium point of the dynamical system (4.16) is defined as follows.

Definition 4.8. An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (4.12), if,

$$\gamma \frac{d^3\mu}{dt^3} + \zeta \frac{d^2\mu}{dt^2} + \xi \frac{d\mu}{dt} = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the general equilibrium inclusion (2.1), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.

Consequently, the problem (4.3) can be written as

$$g(\mu) = J_A\Big[h(\mu) - \rho E(\mu, \nu) + \gamma \frac{d^3\mu}{dt^3} + \zeta \frac{d^2\mu}{dt^2} + \xi \frac{d\mu}{dt}\Big]. \tag{4.17}$$

We discretize the third-order dynamical systems (4.16) using central finite difference and backward difference schemes to have

Algorithm 24. For given μ_0, μ_1, μ_2, compute μ_{n+1} by the iterative scheme

$$\gamma \frac{u_{n+2} - 2u_{n+1} + 2u_{n-1} - u_{n-2}}{2h_1^3} + \zeta \frac{u_{n+1} - 2u_n + u_{n-1}}{h_1^2}$$
$$+\xi \frac{3\mu_n - 4\mu_{n-1} + \mu_{n-2}}{2h_1} + g(\mu_n) = J_A\Big[h(\mu_n) - \rho E(\mu_{n+1}, \nu)\Big], \tag{4.18}$$

where h_1 is the step size.

Similarly discretizing dynamical systems (4.17) using central finite difference and backward difference schemes, we have

$$\begin{aligned}\mu_{n+1} \;=\; & \mu_n - g(\mu_n) + J_A\Big[\Big\{h(\mu_n) - \rho E(\mu_{n+1}, \nu)\Big\} \\ & + \gamma \frac{u_{n+2} - 2u_{n+1} + 2u_{n-1} - u_{n-2}}{2h_1^3} + \zeta \frac{u_{n+1} - 2u_n + u_{n-1}}{h_1^2} \\ & + \xi \frac{3\mu_n - 4\mu_{n-1} + \mu_{n-2}}{2h_1}\Big],\end{aligned} \tag{4.19}$$

If $\gamma = 1, h_1 = 1, \zeta = 1, \xi = 1$, then, from equation(4.18) after adjustment, we have

Algorithm 25. For a given μ_0, compute u_{n+1} by the iterative scheme

$$u_{n+1} = \mu_n - g(\mu_n) + J_A\big[h(\mu_n) - \rho E(\mu_{n+1}, \nu) + \frac{\mu_{n+1} + 3\mu_n}{2}\big].$$

This is an inertial type hybrid iterative methods for solving the general equilibrium inclusions (2.1).

Remark 4.9. For appropriate and suitable choice of the operators $\mathcal{T}, g, h$, the bifunction $E(.,.)$, convex set, parameters and the spaces, one can suggest a wide class of implicit, explicit and inertial type methods for solving extended general equilibrium inclusions and related optimization problems.

5 Sensitivity Analysis

In recent years variational inequalities are being used as mathematical programming models to study a large number of equilibrium problems arising in finance, economics, transportation, operations research and engineering sciences. The behaviour of such problems as a result of changes in the problem data is always of concern, which is called sensitivity analysis. Dafermos [12] considered the sensitivity analysis considered the sensitivity of the variational inequalities uisng essentially the projection method. These results were extended for variational inequalities by Noor [28] and for variational inclusions by Noor et al. [40]. We like to mention that sensitivity analysis is important for several reasons. First, estimating problem data often introduces measurement errors, sensitivity analysis helps in identifying sensitive parameters that should be obtained with relatively high accuracy. Second, sensitivity analysis may help to predict the future changes of the equilibrium as a result of changes in the governing system. Third, sensitivity analysis provides useful information for designing or planning various equilibrium systems. Furthermore, from mathematical and engineering point of view, sensitivity analysis can provide new insight regarding problems being studied can stimulate new ideas and techniques for problem solving the problems due to these and other reasons. In this section, we study the sensitivity analysis of the general equilibrium inclusions , that is, examining how solutions of such problems change when the data of the problems are changed.

We now consider the parametric versions of the problem (2.1). To formulate the problem, let M be an open subset of $\mathcal{H}$ in which the parameter λ takes values. Let $g(\mu, \lambda)$ be given identity operator defined on $\mathcal{H} \times \mathcal{H} \times M$ and take value in $\mathcal{H} \times \mathcal{H}$. From now onward, we denote $g_\lambda(.) \equiv g(., \lambda)$ and $E_\lambda(.) \equiv E(., \lambda)$, respectively, unless otherwise specified.

The parametric extended general equilibrium inclusions problem is to find $\mu \in \mathcal{H}$ such that

$$0 \in \rho E_\lambda(\mu, \nu) + g_\lambda(\mu) - h_\lambda(\mu) + \rho A(g_\lambda(\mu)), \quad \forall \nu \in \mathcal{H} \times M. \tag{5.1}$$

We also assume that, for some $\overline{\lambda} \in M$, the problem (5.1) has a unique solution $\overline{\mu}$. From Lemma 3.1, we see that the parametric general equilibrium inclusion are equivalent to the fixed point problem:

$$g_\lambda(\mu) = J_A\Big[h_\lambda(\mu) - \rho E_\lambda(\mu, \nu)\Big],$$

or equivalently

$$\mu = \mu - g_\lambda(\mu) + J_A[h_\lambda(\mu) - \rho E_\lambda(\mu, \nu)].$$

We now define the mapping F_λ associated with the problem (5.1) as

$$F_\lambda(\mu) = \mu - g_\lambda(\mu) + J_A[h_\lambda(\mu) - \rho E_\lambda(\mu, \nu)], \quad \forall(\mu, \lambda) \in \mathcal{H} \times M. \tag{5.2}$$

We use this equivalence to study the sensitivity analysis of the extended general equilibrium inclusion. We assume that for some $\overline{\lambda} \in M$, problem (5.1) has a solution $\overline{\mu}$ and X is a closure of a ball in $\mathcal{H}$ centered at $\overline{\mu}$. We want to investigate those conditions under which, for each λ in a neighborhood of $\overline{\lambda}$, problem (5.1) has a unique solution $\mu(\lambda)$ near $\overline{u}$ and the function $u(\lambda)$ is (Lipschitz) continuous and differentiable.

Definition 5.1. Let $E_\lambda(.)$ be a bifunction on $X \times M$. Then, the bifunction $E_\lambda(.)$ is said to be:

(a) *Locally strongly monotone* with constant $\sigma > 0$, if

$$\langle E_\lambda(\mu, \nu) - E_\lambda(\eta, \nu), \nu\rangle \geq \sigma\|\mu - \eta\|^2, \quad \forall\lambda \in M, \eta, \mu, \nu \in X.$$

(b) *Locally Lipschitz continuous* with constant $\zeta > 0$, if

$$\|E_\lambda(\mu, \nu) - E_\lambda(\eta, \nu)\| \leq \zeta\|\mu - \eta\|, \quad \forall\lambda \in M, \eta, \mu, \nu \in X.$$

Definition 5.2. An operator $\mathcal{T}_\lambda : \mathcal{H} \to \mathcal{H}$ is said to be:

1. Locally strongly monotone, if there exist a constant $\alpha > 0$, such that

$$\langle \mathcal{T}_\lambda(\mu) - \mathcal{T}_\lambda(\nu), \mu - \nu\rangle \geq \alpha\|\mu - \nu\|^2, \quad \forall\mu, \nu \in \mathcal{H}.$$

2. Locally Lipschitz continuous, if there exist a constant $\beta > 0$, such that

$$\|\mathcal{T}_\lambda(\mu) - \mathcal{T}_\lambda(\nu)\| \leq \beta\|\mu - \nu\|, \quad \forall\mu, \nu \in \mathcal{H}.$$

3. Locally monotone, if

$$\langle \mathcal{T}_\lambda(\mu) - \mathcal{T}_\lambda(\nu), \mu - \nu\rangle \geq 0, \quad \forall\mu, \nu \in \mathcal{H}.$$

We consider the case, when the solutions of the parametric general equilibrium inclusion (5.1) lie in the interior of X. Following the ideas of Dafermos [13], Noor [28] and Noor et al. [35], we consider the map $F_\lambda(\mu)$ as defined by (5.2). We have to show that the map $F_\lambda(\mu)$ has a fixed point, which is a solution of the parametric extended general equilibrium inclusion (5.1). First of all, we prove that the map $F_\lambda(\mu)$, defined by (5.2), is a contraction map with respect to μ uniformly in $\lambda \in M$.

Lemma 5.3. *Let $g_\lambda(.)$ be a locally strongly monotone with constants $\sigma > 0$ and locally Lipschitz continuous with constants $\zeta > 0$ respectively. If Assumption 1 holds and the bifunction $E_\lambda(.,.)$ and operator h_λ be locally Lipchitz continous with constant $\beta > 0, \zeta_1$, we have*

$$\|F_\lambda(\mu_1) - F_\lambda(\mu_2)\| \leq \theta\|\mu_1 - \mu_2\|,$$

for

$$\rho < \frac{1-k}{\beta} \quad k < 1, \tag{5.3}$$

where

$$\theta = \left\{\sqrt{1 - 2\sigma + \zeta^2} + \zeta_1 + \rho\beta\right\} = \{k + \rho\beta\} \tag{5.4}$$

and

$$k = \sqrt{1 - 2\sigma + \zeta^2} + \zeta_1. \tag{5.5}$$

Proof. In order to prove the existence of a solution of (5.1), it is enough to show that the mapping $F_\lambda(\mu)$, defined by (5.2), is a contraction mapping.

For $\mu_1 \neq \mu_2 \in \mathcal{H}$, and using Assumption 1, we have

$$\begin{aligned}
\|F_\lambda(\mu_1) - F_\lambda(\mu_2)\| &\leq \|\mu_1 - \mu_2 - (g_\lambda(\mu_1) - g_\lambda(\mu_2))\| \\
&\quad + \|J_A[h_\lambda(\mu_1) - \rho E_\lambda(\mu_1,\nu)] - J_A[h_\lambda(\mu_2) - \rho E_\lambda(\mu_2,\nu)]\| \\
&\leq \|\mu_1 - \mu_2 - (g_\lambda(\mu_1) - g_\lambda(\mu_2))\| \\
&\quad + \|h_\lambda(\mu_) - h_\lambda(\mu_2) - \rho(E_\lambda(\mu_1,\nu) - E_\lambda(\mu_2,\nu)\| \\
&\leq \|\mu_1 - \mu_2 - (g_\lambda(\mu_1) - g_\lambda(\mu_2))\| + \rho\|E_\lambda(\mu_1,\nu) - E_\lambda(\mu_2,\nu)\| \\
&\quad + \|h_\lambda(\mu_1) - h_\lambda(\mu_2)\| + \rho\|E_\lambda(\mu_1,\nu) - E_\lambda(\mu_2,\nu)\| \\
&\leq \|\mu_1 - \mu_2 - (g_\lambda(\mu_1) - g_\lambda(\mu_2))\| + \zeta_1\|\mu - \nu\| + \rho\beta\|\mu_1 - \mu_2)\|.
\end{aligned} \tag{5.6}$$

Since the operator g_λ is a locally strongly monotone with constant $\sigma > 0$ and locally Lipschitz continuous with constant $\zeta > 0$, it follows that

$$\begin{aligned}
||\mu_1 - \mu_2 - (g_\lambda(\mu_1) - g_\lambda(\mu_2)||^2 &\leq ||u_1 - u_2||^2 - 2\langle g_\lambda(\mu_1) - g_\lambda(\mu_2), \mu_1 - \mu_2\rangle \\
&\quad + ||g_\lambda(\mu_1) - g_\lambda(\mu_2)||^2 \\
&\leq (1 - 2\sigma + \zeta^2)||\mu_1 - \mu_2||^2.
\end{aligned} \tag{5.7}$$

From (5.5), (5.6), (5.7) and using the locally Lipschitz continuity of the bifunction $E(.,.)$ and the operator h_λ, we have

$$\begin{aligned}
\|F_\lambda(\mu_1) - F_\lambda(\mu_2)\| &\leq \left\{\zeta_1 + \sqrt{(1 - 2\sigma + \zeta^2)} + \rho\beta\right\} \|\mu_1 - \mu_2\| \\
&= \theta\|\mu_1 - \mu_2\|,
\end{aligned}$$

where

$$\theta = k + \rho\beta.$$

From (5.3), it follows that $\theta < 1$. Thus it follows that the mapping $F_\lambda(\mu)$, defined by (5.2), is a contraction mapping and consequently it has a fixed point, which belongs to $\mathcal{H}$ satisfying extended quasi general equilibrium inclusion (5.1), the required result. □

Remark 5.4. From Lemma 5.3, we see that the map $F_\lambda(\mu)$ defined by (5.2) has a unique fixed point $\mu(\lambda)$, that is, $\mu(\lambda) = F_\lambda(\mu)$. Also, by assumption, the function $\overline{\mu}$, for $\lambda = \overline{\lambda}$ is a solution of the parametric extended general equilibrium inclusion (5.1). Again using Lemma 5.3, we see that $\overline{\mu}$, for $\lambda = \overline{\lambda}$, is a fixed point of $F_\lambda(\mu)$ and it is also a fixed point of $F_{\overline{\lambda}}(\mu)$. Consequently, we conclude that

$$\mu(\overline{\lambda}) = \overline{\mu} = F_{\overline{\lambda}}(\mu(\overline{\lambda})).$$

Using Lemma 5.3, we can prove the continuity of the solution $\mu(\lambda)$ of the parametric general equilibrium inclusion (5.1) using the technique of Noor [52].

Lemma 5.5. *Assume that the bifunction E_λ and the operator h_λ are locally Lipschitz continuous with respect to the parameter λ. If the operator $g_\lambda(.)$ is Locally Lipschitz continuous and the map $\lambda \to J_A$ is continuous (or Lipschitz continuous), then the function $u(\lambda)$ satisfying (5.2) is (Lipschitz) continuous at $\lambda = \overline{\lambda}$.*

We now state and prove the main result of this paper and is the motivation our next result.

Theorem 5.6. *Let $\overline{\mu}$ be the solution of the parametric general equilibrium inclusion (5.1) for $\lambda = \overline{\lambda}$. Let $E_\lambda, h_\lambda(\mu)$ be the locally strongly monotone Lipschitz continuous operators for all $\mu, \nu \in X$. If the map $\lambda \to J_{A_\mu}$ is (Lipschitz) continuous at $\lambda = \overline{\lambda}$ and the operator g_λ is locally strongly monotne Lipschitz continuous, then there exists a neighborhood $N \subset M$ of $\overline{\lambda}$ such that for $\lambda \in N$, the parametric general equilibrium inclusion (5.2) has a unique solution $\mu(\lambda)$ in the interior of $X, u(\overline{\lambda}) = \overline{u}$ and $u(\lambda)$ is (Lipschitz) continuous at $\lambda = \overline{\lambda}$.*

Proof. Its proof follows from Lemma 5.3, Lemma 5.5 and Remark 5.4. □

Conclusion

Some new classes of extended general equilibrium inclusions are introduced and investigated. We have proved that the extended general equilibrium inclusions are equivalent to the fixed point problem. We have applied the equivalence between the general equilibrium inclusions and fixed point problems to suggest some new multi step multi-step iterative methods for solving the general equilibrium inclusions. These new methods include extragradient methods, multi step hybrid resolvent methods as special cases. Convergence analysis of the proposed method is discussed for strongly monotone and Lipschitz continuous operators. Sensitivity analysis is also investigated for general equilibrium inclusions using the equivalent fixed point approach. Iterative methods suggested and analyzed in this paper for solving general equilibrium inclusions are the novel generalizations, improvements, refinements and modifications of Noor (three step) iterations [29–31], which include Ishikawa (two-step) iterations, Mann Iteration (one step) iteration and Picard method as special cases. Using the technique and ideas of Ashish et al. [2,3], Cho et al. [8], Cristescu et al. [10,11], Kwuni et al. [19], Mahato [21], Natrangan et al. [23], Noor et al. [41–44,49], Pamsang et al. [53], Rattanaseeha et al. [54], Suantai et al. [56], Tomar [57] and Yadav et al. [62], one can explore the applications of these multi step methods for solving the general equilibrium inclusions in the fixed point theory, fractal geometry, chaos theory, coding, number theory, spectral geometry, dynamical systems, complex analysis, nonlinear programming, graphics, artificial intelligence, control engineering, management sciences, stock exchange, regression and link prediction problems [58], financial mathematical [4], and computer aided design. Comparison of these new methods with other technique is an open problem, which need further research efforts.

Conflict interest:

Authors have no conflict of interest.

Authors contributions:

All authors contributed equally to the conception, design of the work, analysis, interpretation of data, reviewing it critically and final approval of the version for publication.

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