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Frequency-dependent Symmetric Hybrid Multistep Method with Bounded Amplitude Error for Stiff and Oscillatory ODEs

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Abstract

This paper considers a new fourth-order frequency-dependent symmetric hybrid linear multistep method for oscillatory second-order differential equations. The proposed method achieves minimal phase-lag and bounded amplitude error, which accurately reproduces orbital trajectories, and requires fewer function evaluations. Numerical experiment confirms improved efficiency, accuracy, and long-term stability over some existing methods in the literature.

1 Introduction

Second-order ordinary differential equations (ODEs) in which the first derivative is explicitly absent arise frequently in many areas of science and engineering. A general form of such equations is given by

$$\begin{cases} y'' = f(x, y(x)), \quad x \in [x_0, X], \\ y(x_0) = y_0, \\ y'(x_0) = y'_0, \end{cases} \tag{1.1}$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ is assumed to be sufficiently smooth to ensure the existence and uniqueness of the solution. In many practical applications, solutions of (1.1) exhibit oscillatory behavior, as is common in orbital dynamics, quantum mechanics, and wave propagation [1–4].

Classical linear multistep methods have long been applied to problems of the form (1.1), but their performance is fundamentally limited for oscillatory solutions. Lambert and Watson [1] showed that no constant-coefficient P-stable multistep method can achieve an algebraic order higher than two, regardless of the step size. This limitation has spurred the development of extended frameworks that go beyond classical approaches. Among these, hybrid multistage-multistep methods [2, 5–8] have proven particularly effective. By introducing additional free parameters, these methods allow for higher-order accuracy while

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preserving desirable stability properties, making them well-suited for long-term integration of oscillatory problems.

A key challenge in numerical integration of oscillatory ODEs is the control of phase errors. Brusa and Nigro [9] introduced the concept of *phase-lag*, which quantifies the difference in oscillation frequency between the exact and numerical solutions. Subsequent studies formalized the notion of *phase-lag order* [10, 11] and developed methods aimed at minimizing or eliminating phase-lag at selected frequencies. For problems exhibiting periodic or nearly periodic behavior, controlling phase-lag is often more important than increasing the algebraic order of accuracy, as dispersion errors can dominate the global solution over long integration intervals [10].

Another important consideration is the symmetry of the numerical scheme. Symmetric formulas tend to preserve qualitative properties of the exact solution, especially in Hamiltonian or near-Hamiltonian systems. They suppress secular growth of invariants and maintain bounded parasitic oscillations over long time integrations [7, 12, 13]. By combining high algebraic order, phase-lag control, and symmetry, hybrid methods achieve both stability and accuracy, making them a powerful tool for solving oscillatory second-order ODEs.

In this work, we aim to develop and analyze a new frequency-dependent symmetric hybrid linear multistep method with enhanced phase properties, capable of accurately integrating oscillatory second-order differential equations while minimizing computational effort.

2 Preliminary Results

The k-step LMMs for solving problem (1.1) numerically is given by

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j}, \quad k \geq 2 \tag{2.1}$$

where it is assumed that $\alpha_k = 1$ and $\beta_k \neq 1$. For simplicity, the LMM (2.1) can be written as $\rho(E)y_n = h^2 \sigma E f_n$, where the shift operator is $E^j y_n = yn + j, \quad j \in \mathbb{Z}$. The first and second characteristic polynomials are $\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j$ and $\sigma(\xi) = \sum_{j=0}^{k} \beta_j \xi^j$, respectively.

To bypass the order barrier imposed on LMMs (2.1) [1, 14], Hybrid methods incorporating additional off-step points have been introduced to construct high order hybrid methods for the direct numerical solution of (1.1). For such methods, see [5, 15, 16] and the references therein.

Let the corresponding difference operator for the LMM (2.1) be defined by

$$L[y(x); h] = \sum_{j=0}^{k} \left[\alpha_j y(x + jh) - h^2 \beta_j y''(x + jh) \right], \tag{2.2}$$

where $y(x)$ is an arbitrary continuously differentiable function on the the interval $[a, b]$. Without loss of generality, the Taylor expansion of $y(x)$ about the point x yields

$$\sum_{j=0}^{k} \left[\alpha_j y(x + jh) - h^2 \beta_j y''(x + jh)\right] = C_{p+2} h^{p+2} y^{(p+2)}(x) + O(h^{(p+3)}), \tag{2.3}$$

with an error constant given by

$$C_q = \frac{1}{q!} \sum_{j=0}^{k} j^{q-2} \left(j^2 \alpha_j - q(q-1)\beta_j\right) - \sum_{j=0}^{k} \frac{j^{q-2}}{(q-2)!} \beta_j, \quad q > 2. \tag{2.4}$$

From (2.3), C_{p+2} is the error constant, $C_{p+2} h^{p+2} y^{(p+2)}(x)$ represent the LTE at the point x_n and p is the algebraic order of the method. The application of method (2.1) on the scalar test equation (2.5) (see [1]),

$$y'' + \lambda^2 y = 0, \quad \lambda \in \mathbb{R} \tag{2.5}$$

will yield the finite difference equation of the form

$$\sum_{r=0} A_r(\xi^2) y_{n+r} = 0, \quad \xi = i\lambda h, \tag{2.6}$$

where $A_r(\xi^2)$ are polynomials in ξ^2. It follows that the characteristic equation is

$$\Phi(R, \xi) = \sum_{j=0}^{k} A_r(\xi^2) R^2, \quad k > 1. \tag{2.7}$$

which motivates the following definitions

Definition 2.1. *(Zero-stability) The method* (2.1) *is zero-stable if the roots of the first charateristic polynomial has modulus less than or equal to one, and if every root of modulus one has multiplicity not greater than two.*

Definition 2.2. *(Consistency) The LMM* (2.1) *is consistent if it has order* $p \geq 1$.

Definition 2.3. *(Convergence) The LMM* (2.1) *is convergent if it is zero-stable and consistent.*

Definition 2.4. *(Interval of Periodicity) The LMM* (2.1) *is said to have periodicity interval* $(0, H^2)$ *if for all* $H^2 \in (0, H^2)$, *the roots of the characteristic polynomial have at most two compex conjugate roots of modulus one while other roots are less than or equal to one.*

Definition 2.5. *(P-stability) The LMM* (2.1) *is p-stable if* $(0, H^2) = (0, \infty)$.

Definition 2.6. *(Phase-lag) For any symmetric multistep methods, the phase-lag (frequency distortion) of order* q *is given by*

$$t(z) = z - \theta(z) = Cz^{q+1} + O(z^{q+2}), \tag{2.8}$$

where C *is the phase lag constant and* q *is the phase lag order.*

3 Derivation of Method

For the numerical integration of (1.1) we consider the symmetric hybrid linear multistep method,

$$y_{n+1} - 2y_n + y_{n-1} = h^2 \left[\beta_0(u) f_n + \beta_1(u)(f_{n+1} + f_{n-1})\right] + h^2 \beta_{c_1}(u)(f_{n+c_1} + f_{n-c_1}), \tag{3.1}$$

whose corresponding hybrid methods are given by

$$y_{n\pm c_1} = \alpha_2(u) y_{n+1} + \alpha_1(u) y_n + \alpha_0(u) y_{n-1} + h^2(\phi_0(u) f_n + \phi_1(u) f_{n+1}). \tag{3.2}$$

Let $u = \omega h$. The coefficients $\beta_j(u)$, $\beta_{c_1}(u)$, $\alpha_l(u)$, and $\phi_j(u)$, with $j = 0, 1$ and $l = 0, 1, 2$, are unknown functions of the frequency u and the stepsize h. The numerical solution at shifted nodes is denoted by $y_{n\pm c_1} \approx y(x_{n\pm c_1})$ and the corresponding function evaluations are $f_{n\pm c_1} = f(x_{n\pm c_1}, y_{n\pm c_1})$, $f_{n\pm j} = f(x_{n\pm j}, y_{n\pm j})$, $j = 0, 1$. Substituting the trigonometrically fitted basis function (3.3)

$$U(x) = a_0 + a_1 x + a_2 x^2 + a_3 sin(wx) + a_4 cos(wx) \tag{3.3}$$

and its second derivative (3.4)

$$U''(x) = 2 - w^2(a_1 sin(wx) + a_2 cos(wx)) \tag{3.4}$$

into the collocation equations allows us to solve for the unknown coefficients. Once determined, these coefficients define a trigonometrically fitted hybrid collocation method that is consistent with the underlying oscillatory problem. Now we assume that the interpolating basis function (3.3) coincides with the exact solution at the point $x_{n\pm j}$ to get

$$U(x_{n\pm j}) = y_{n\pm j}, \quad j = 0, 1. \tag{3.5}$$

Also, collocating (3.4) at x_{n+v_1}, $x_{n\pm j}$, $j = 0, 1$ we obtain

$$U''(x_{n+v_1}) = f_{n+v_1}, \quad U''(x_{n\pm j}) = f_{n\pm j}, \; j = 0, 1. \tag{3.6}$$

Observe that (3.5) and (3.6) result to a system of five equations with five unknowns imposed by symmetry. Solving this resulting system of equation for $a_j, j = 0(1)4$ and substituting the resulting values of a_j's with $x = x_{n+1} + th$ (without loss of generality, we set $x_n = 0$ so that $x = h + ht$ into (3.3), we obtain a continuous hybrid method with continuous coefficients which is then used to generate the output method and its corresponding hybrid methods. It follows that for $t = 1$ in the continuous method and by carefully choosing $v_1 = \frac{1}{2}$ we obtain

$$
\begin{cases}
\beta_0(u) = \dfrac{\left(u^2 + 2\cos(u) - 2\right)\sin\left(\frac{u}{2}\right)}{u^2\left(\sin\left(\frac{u}{2}\right) - \sin\left(\frac{3u}{2}\right) + \sin(u)\right)}, \\[2ex]
\beta_1(u) = \dfrac{4\sin^2\left(\frac{u}{2}\right)\left(\sin\left(\frac{u}{2}\right) + \sin(u)\right) - u^2\sin\left(\frac{3u}{2}\right)}{u^2\left(\sin\left(\frac{u}{2}\right) - \sin\left(\frac{3u}{2}\right) + \sin(u)\right)}, \\[2ex]
\beta_2(u) = \dfrac{\sin(u)\left(u^2 + 2\cos(u) - 2\right)}{u^2\left(\sin\left(\frac{u}{2}\right) - \sin\left(\frac{3u}{2}\right) + \sin(u)\right)}.
\end{cases}
\tag{3.7}
$$

Similarly, for the hybrid pairs we let $t = [-\frac{1}{2}, \frac{1}{2}]$ in the continuous scheme to obtain

$$
\begin{cases}
\hat{\phi}_0(u) = -\dfrac{(u^2 + 12)\ \sin\left(\frac{u}{2}\right) + 4\ \sin\left(\frac{3u}{2}\right) - 12\ \sin(u)}{8u^2\left(\sin\left(\frac{u}{2}\right) - \sin\left(\frac{3u}{2}\right) + \sin(u)\right)}, \\[2ex]
\hat{\phi}_1(u) = \dfrac{(u^2 + 4)\ \sin\left(\frac{3u}{2}\right) - 4\ \sin\left(\frac{u}{2}\right) - 4\ \sin(u)}{8u^2\left(\sin\left(\frac{u}{2}\right) - \sin\left(\frac{3u}{2}\right) + \sin(u)\right)} \\[2ex]
\hat{\alpha}_0(u) = \dfrac{3u^2\cos\left(\frac{u}{2}\right) + 8\cos\left(\frac{u}{2}\right) + 4\cos(u) - 12}{8\left(u^2\cos\left(\frac{u}{2}\right) + 2\cos(u) - 2\right)}, \\[2ex]
\hat{\alpha}_1(u) = \dfrac{3u^2\cos\left(\frac{u}{2}\right) - 8\cos\left(\frac{u}{2}\right) + 8\cos(u)}{4\left(u^2\cos\left(\frac{u}{2}\right) + 2\cos(u) - 2\right)}, \\[2ex]
\hat{\alpha}_2(u) = \dfrac{-u^2\cos\left(\frac{u}{2}\right) + 8\cos\left(\frac{u}{2}\right) - 4\cos(u) - 4}{8\left(u^2\cos\left(\frac{u}{2}\right) + 2\cos(u) - 2\right)}.
\end{cases}
\tag{3.8}
$$

and

$$
\begin{cases}
\phi_0(u) = \dfrac{3\left(u^2 - 4\right)\sin\left(\frac{u}{2}\right) - 8\sin(0) + 4\sin\left(\frac{3u}{2}\right) + 8\sin\left(\frac{u}{2}\right) - 4\sin(u)}{8u^2\left(\sin\left(\frac{u}{2}\right) - \sin\left(\frac{3u}{2}\right) + \sin(u)\right)}, \\[2ex]
\phi_1(u) = \dfrac{-\left(3u^2 + 4\right)\sin\left(\frac{3u}{2}\right) + 8\sin(0) + 12\sin\left(\frac{u}{2}\right) - 8\sin\left(\frac{3u}{2}\right) + 12\sin(u)}{8u^2\left(\sin\left(\frac{u}{2}\right) - \sin\left(\frac{3u}{2}\right) + \sin(u)\right)}, \\[2ex]
\alpha_0(u) = \dfrac{-u^2\cos\left(\frac{u}{2}\right) + 8\cos\left(\frac{u}{2}\right) - 4\cos u - 4}{8\left(u^2\cos\left(\frac{u}{2}\right) + 2\cos u - 2\right)}, \\[2ex]
\alpha_1(u) = \dfrac{3u^2\cos\left(\frac{u}{2}\right) - 8\cos\left(\frac{u}{2}\right) + 8\cos u}{4\left(u^2\cos\left(\frac{u}{2}\right) + 2\cos u - 2\right)}, \\[2ex]
\alpha_2(u) = \dfrac{3u^2\cos\left(\frac{u}{2}\right) + 8\cos\left(\frac{u}{2}\right) + 4\cos u - 12}{8\left(u^2\cos\left(\frac{u}{2}\right) + 2\cos u - 2\right)},
\end{cases}
\tag{3.9}
$$

respectively. We observe that for small values of u, the above coefficients are subject to heavy cancellation; hence, we employ the Taylor series expansion. See [17] and the references. Thus, the coefficients in (3.7), (3.8) and (3.9) are respectively expressed as follows;

$$\beta_0 = \frac{1}{24} + \frac{17u^2}{2160} + \frac{349u^4}{725760} + \frac{2437u^6}{87091200} + \frac{147527u^8}{91968307200} + \frac{91958879u^{10}}{1004293914624000} + \cdots$$
$$\beta_1 = \frac{7}{12} + \frac{u^2}{240} - \frac{13u^4}{241920} + \frac{u^6}{9676800} - \frac{13u^8}{10218700800} - \frac{1559u^{10}}{334764638208000} + \cdots$$
$$\beta_2 = \frac{1}{6} - \frac{13u^2}{1080} - \frac{31u^4}{72576} - \frac{1223u^6}{43545600} - \frac{14741u^8}{9196830720} - \frac{45977101u^{10}}{502146957312000} + \cdots$$

whose local truncation error is $LTE = -\frac{1}{640}\left(h^6 y^{(6)}(x_n)\right) + O(h^8)$. The hybrid methods and their corresponding local truncation errors are

$$\hat{\phi}_0 = -\frac{13}{320} - \frac{203u^2}{69120} - \frac{2021u^4}{11612160} - \frac{55901u^6}{5573836800} - \frac{5033u^8}{6370099200} - \frac{17482567u^{10}}{224737099776000} + \cdots$$
$$\hat{\phi}_1 = \frac{7}{320} + \frac{u^2}{7680} + \frac{u^4}{774144} + \frac{u^6}{88473600} + \frac{1181u^8}{44590694400} + \frac{308767u^{10}}{74912366592000} + \cdots$$
$$\hat{\alpha}_0 = \frac{63}{160} - \frac{11u^2}{1280} + \frac{331u^4}{1075200} - \frac{791u^6}{73728000} + \frac{1786537u^8}{4768727040000} - \frac{4923229u^{10}}{381498163200000} + \cdots$$
$$\hat{\alpha}_1 = \frac{57}{80} + \frac{11u^2}{640} - \frac{331u^4}{537600} + \frac{791u^6}{36864000} - \frac{1786537u^8}{2384363520000} + \frac{4923229u^{10}}{190749081600000} + \cdots$$
$$\hat{\alpha}_2 = -\frac{17}{160} - \frac{11u^2}{1280} + \frac{331u^4}{1075200} - \frac{791u^6}{73728000} + \frac{1786537u^8}{4768727040000} - \frac{4923229u^{10}}{381498163200000} + \cdots$$

$LTE = \frac{5}{768}\left(h^5 y^{(5)}(x_n)\right) + O(h^7)$

$$\begin{aligned}
\phi_0 &= \frac{7}{320} - \frac{u^2}{480} + \frac{u^4}{3840} - \frac{u^6}{645120} + \frac{u^8}{18579456} - \frac{u^{10}}{689889024} + \frac{u^{12}}{32011828352}\cdots, \\
\phi_1 &= -\frac{13}{320} + \frac{u^2}{480} - \frac{u^4}{3840} + \frac{u^6}{645120} - \frac{u^8}{18579456} + \frac{u^{10}}{689889024} - \frac{u^{12}}{32011828352}\cdots, \\
\alpha_0 &= -\frac{17}{160} - \frac{u^2}{480} + \frac{u^4}{2304} - \frac{7u^6}{552960} + \frac{u^8}{258048} - \frac{13u^{10}}{185794560} + \frac{23u^{12}}{3632428800}\cdots, \\
\alpha_1 &= \frac{57}{80} - \frac{u^2}{160} + \frac{u^4}{384} - \frac{u^6}{13824} + \frac{u^8}{8257536} - \frac{5u^{10}}{185794560} + \frac{7u^{12}}{5189184000}\cdots, \\
\alpha_2 &= \frac{63}{160} + \frac{u^2}{248} - \frac{u^4}{1152} + \frac{u^6}{55296} - \frac{u^8}{3317760} + \frac{u^{10}}{258048000} - \frac{u^{12}}{25821120000}\cdots.
\end{aligned} \tag{3.10}$$

$LTE = -\frac{5}{768}\left(h^5 y^{(5)}(x_n)\right) + O(h^7)$

The behaviour of the coefficients are given in Figures (1), (2), and (3),

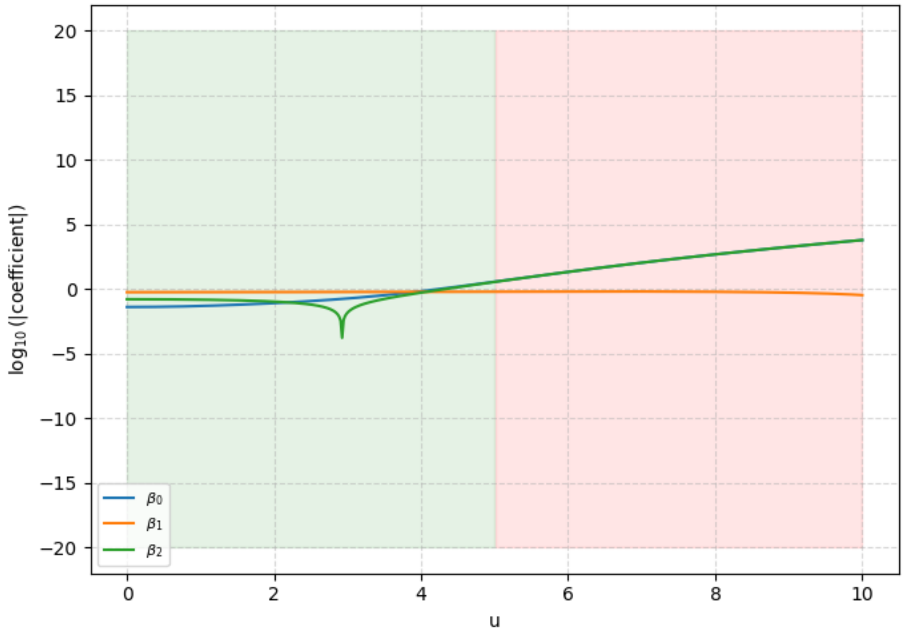

Figure 1: Logarithmic behaviour of β coefficients versus u. Green and red zones indicate stable and risky regions, respectively.

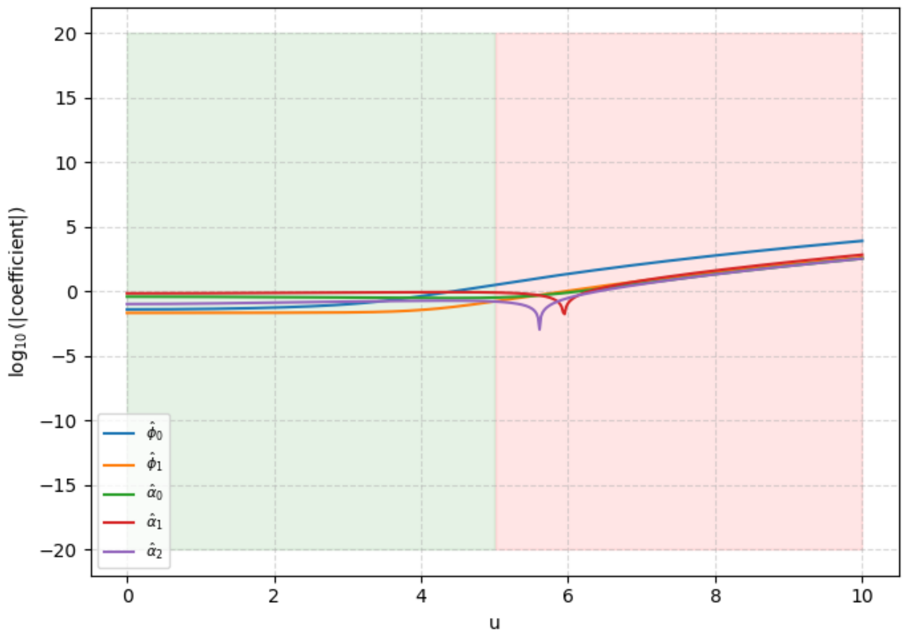

Figure 2: Logarithmic behaviour of $\hat{\phi}$ and $\hat{\alpha}$ coefficients versus u.

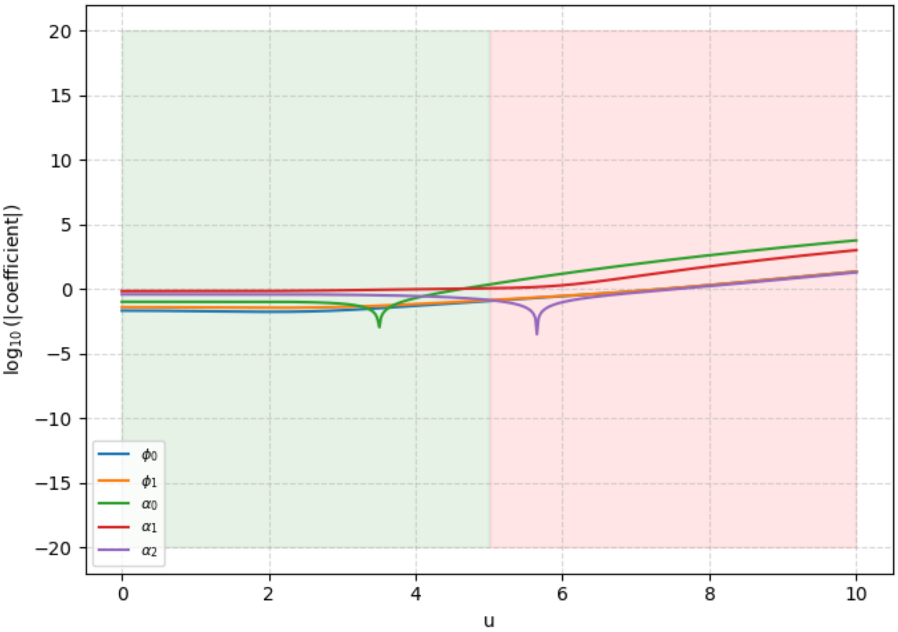

Figure 3: Logarithmic behaviour of ϕ and α coefficients versus u.

4 Numerical Experiment and Discussion

In this section, we carry out a numerical experiment to illustrate the efficiency of the derived scheme.

4.1 Implementation Procedure

The implementation follows four main stages: initialization, prediction, correction, and update. At the initialization stage, the first step value is obtained through a Taylor expansion based on the given initial conditions and the right-hand side function $f(t, y)$. The predictor stage employs $y_{n+1}^{(0)} = 2y_n - y_{n-1} + h^2 f(t_n, y_n)$, which provides a first approximation of the next step. The corrector stage refines this prediction using a Newton iteration process applied to the nonlinear residual equation $R(y_{n+1}) = y_{n+1} - 2y_n + y_{n-1} - h^2 \left[\frac{1}{24}(f_{n+1} + f_{n-1}) + \frac{7}{12} f_n + \frac{1}{6}(f_{n+\frac{1}{2}} + f_{n-\frac{1}{2}})\right]$. The midpoint function evaluations $f_{n\pm\frac{1}{2}}$ are approximated using symmetric interpolations that depend on y_{n+1}. Convergence of the Newton iteration is controlled by a prescribed tolerance of 10^{-12}. The final update replaces (y_{n-1}, y_n) by (y_n, y_{n+1}), and the process is repeated until the final time T is reached. The steps are carefully illustrated below

Algorithm 1 Implementation Procedure of the Hybrid Method

Require: Step size h, final time T, initial values t_0, y_0, and right-hand side function $f(t, y)$
Ensure: Numerical solution $\{y_n\}$ for $t_n \in [t_0, T]$

Step 1: Initialization
Compute the first approximation using a Taylor expansion:

$$y_1 = y_0 + hf(t_0, y_0) + \frac{h^2}{2} f_t(t_0, y_0) + O(h^3)$$

Set $n \leftarrow 1$.
Step 2: Prediction
Predict y_{n+1} using the explicit formula:

$$y_{n+1}^{(0)} = 2y_n - y_{n-1} + h^2 f(t_n, y_n)$$

Step 3: Correction
repeat
  Evaluate midpoint functions:

$$f_{n\pm\frac{1}{2}} = f\left(t_n \pm \frac{h}{2}, \frac{y_n + y_{n\pm1}^{(k)}}{2}\right)$$

  Form the residual:

$$R(y_{n+1}^{(k)}) = y_{n+1}^{(k)} - 2y_n + y_{n-1} - h^2\left[\frac{1}{24}(f_{n+1}^{(k)} + f_{n-1}) + \frac{7}{12} f_n + \frac{1}{6}(f_{n+\frac{1}{2}}^{(k)} + f_{n-\frac{1}{2}})\right]$$

  Update using Newton's iteration:

$$y_{n+1}^{(k+1)} = y_{n+1}^{(k)} - \left(\frac{\partial R}{\partial y_{n+1}}(y_{n+1}^{(k)})\right)^{-1} R(y_{n+1}^{(k)})$$

until $\|R(y_{n+1}^{(k+1)})\| < 10^{-12}$
Step 4: Update
Set $y_{n+1} \leftarrow y_{n+1}^{(k+1)}$
Update indices: $(y_{n-1}, y_n) \leftarrow (y_n, y_{n+1})$
Increment $n \leftarrow n + 1$
Step 5: Termination
Continue the iteration until $t_n \geq T$.

4.2 Numerical Results

We consider the following orbital problem studied by Stiefel and Bettis [18]:

$$\begin{aligned} &y'' = -y + 0.001e^{it}, \\ &y(0) = 1, \quad y'(0) = 0.9995i, \quad i^2 = -1 \end{aligned} \tag{4.1}$$

whose exact solution is given by

$$\begin{aligned} y(t) &= u(t) + v(t), \\ u(t) &= cos(t) + 0.0005t\sin(t), \\ v(t) &= sin(t) - 0.0005t\cos(t) \end{aligned} \tag{4.2}$$

representing a motion on a perturbed circular orbit in the complex plane in which the point $y(t)$ slowly spirals outward such that its distance from the origin at any given time t is described by

$$\Phi(t) = \sqrt{u^2(t) + v^2(t)} \tag{4.3}$$

Using the predictor $y_{n+2} - 2y_{n+1} + y_n = h^2 f_{n+1}$, we approximate the problem (4.1) in the interval $[0, 40]$ which corresponds to 20 orbits of the points $y(t)$ on the uniform step size

$$h = \frac{\pi}{2^k}, \quad k = 3(1)10 \tag{4.4}$$

The problem is well known to have a solution $\Phi(t) = 1.0019719765345$.

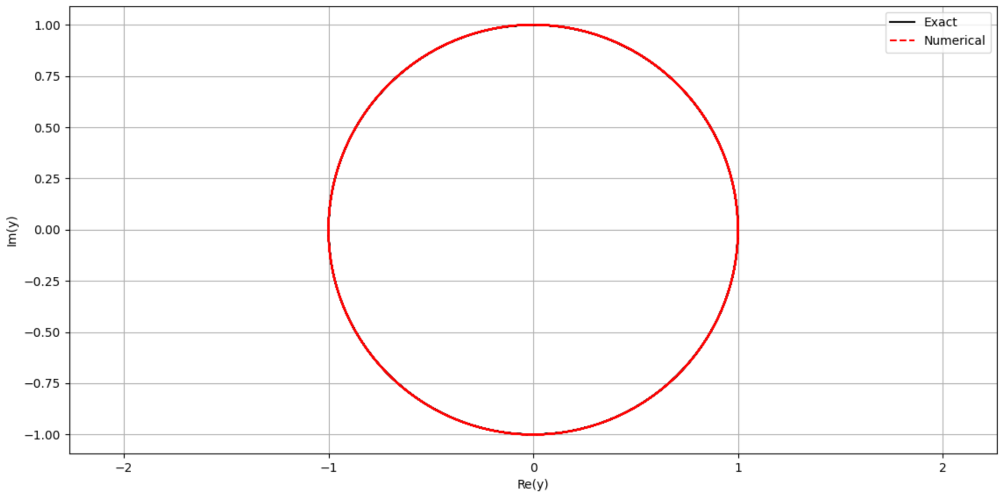

Figure 4: Numerical solution comparing with exact solution

Table 1: Numerical results for the hybrid method at $T = 40\pi$.

k	y_{exact}	y_{num}	$\lvert\Phi_{error}\rvert$	$\lvert err(40\pi)\rvert$	nfe
3	1.00197	1.01481	1.461e-02	4.735e-02	3535
4	1.00197	1.00365	3.445e-03	5.254e-02	4669
5	1.00197	1.00119	9.940e-04	4.362e-02	6902
6	1.00197	1.00043	2.270e-04	6.030e-03	11396
7	1.00197	1.00026	5.600e-05	6.169e-03	17919
8	1.00197	1.00021	1.400e-05	6.063e-03	26064

Table 2: Results at $T = 40\pi$ comparing the proposed method with some existing methods

k	y_{exact}	New Method	Method in [4]	Method in [8]
3	1.00197	1.01481	0.99486	1.000549
4	1.00197	1.00365	0.99722	1.000502
5	1.00197	1.00119	0.99758	1.000497
6	1.00197	1.00043	0.99769	1.000495
7	1.00197	1.00026	0.99773	1.000494
8	1.00197	1.00021	0.99775	1.000494

Figure (1) demonstrates the variation of the main multistep weights with respect to u. At low frequencies ($u \leq 5$, shaded green), the coefficients remain nearly constant, indicating a stable and well-conditioned regime. For higher frequencies ($u > 5$, shaded red), the coefficients exhibit increasing magnitude and divergence among them, suggesting heightened sensitivity to numerical errors. This behavior informs the practical stepsize selection. Figure (2), which also arises from the trigonometric fitting and collocation. Their logarithmic behavior reveals that the coefficients remain small and nearly uniform in the low-frequency region, further indicating minimal phase-lag and amplitude error. Figure (3) shows the remaining coefficients, similar to the previous sets. The logarithmic scale highlights the boundedness of coefficients at low frequencies and their rapid growth at higher frequencies, which corresponds to regions where truncation and rounding errors become more significant. Figure (4) shows the numerical solution that compares accurately with the theoretical solution. Also, from Tables (1) and (2), we observe that the new method outperforms methods in [4] and [8].

4.3 Error Behavior

Table 1 and Figure 4 illustrate the error behavior of the proposed method for the test problem at $T = 40\pi$. Two error modes are observed: a rapidly diminishing local amplitude error and a global component that stabilizes after several oscillations due to residual phase drift. The amplitude error $|\Phi_{\text{error}}|$ decreases almost geometrically with increasing k, from 1.46×10^{-2} at $k = 3$ to 1.4×10^{-5} at $k = 8$. This confirms the high local accuracy, strong phase preservation, and long-term stability. In contrast, the global error $|err(40\pi)|$ remains moderate ($k = 3$–5) due to accumulated phase effects, but stabilizes around 6×10^{-3} for $k \geq 6$.

5 Conclusion

In this paper, we propose a new family of frequency-dependent symmetric hybrid linear multistep methods designed for the efficient numerical integration of periodic initial value problems (IVPs). The methods attain fourth-order accuracy while exhibiting highly favorable phase properties, a feature that ensures stability and reliability over long-term integration. Such characteristics make them particularly well-suited for oscillatory and orbital problems, where conventional schemes often suffer from phase errors and reduced efficiency. To illustrate the effectiveness of the proposed approach, we present a numerical experiment on an orbital oscillatory problem, which highlights both the superior accuracy and computational efficiency of the method in comparison with existing techniques (see Figure 4 and Tables 1–2).

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