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# Extension of Banach Contraction Mapping Principle in Multiplicative Cone Pentagonal Metric Space to a Pair of Two Self Mappings

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**Abstract**

In this paper we combine the notions of multiplicative metric space [6] and cone pentagonal metric space [5] to form multiplicative cone pentagonal metric space. We prove a variant of the Banach contraction mapping theorem under two self-maps in this new space. Some corollaries are consequences of the main result, and some conjectures conclude the paper.

## 1 Introduction and Preliminaries

**Definition 1.1.** *[1] Let $P$ be a subset of $E$, where $E$ is a real Banach space. Then $P$ is called a multiplicative cone if the following conditions are satisfied:*

(a) *$P$ is closed, nonempty, and $P \neq \{1\}$;*

(b) *$a, b \in \mathbb{R}$, $a, b \geq 1$, and $x, y \in P$ imply that $x^a \cdot y^b \in P$;*

(c) *$P \cap \frac{1}{P} = \{1\}$.*

**Definition 1.2.** *[1] Given a multiplicative cone $P \subset E$, we define a partial ordering $\leq$ with respect to $P$ by $x \leq y$ iff $\frac{y}{x} \in P$.*

**Notation 1.3.** *[1] We write $x < y$ to indicate $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $\frac{y}{x} \in int(P)$, where $int(P)$ denotes the interior of $P$.*

**Definition 1.4.** *[1] We say the multiplicative cone $P$ is multiplicative normal if there exists a constant $K > 0$ such that for all $x, y \in E$, $1 \leq x \leq y$ implies*

$$\|x\| \leq \|y\|^K.$$

*The least positive number satisfying the above inequality is called the multiplicative constant of $P$.*

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**Definition 1.5.** *[1] Let $X$ be a nonempty set. Suppose that the map $m : X^2 \mapsto E$ satisfies*

(a) $m(x, y) \geq 1$ *for all* $x, y \in X$ *and* $m(x, y) = 1$ *if and only if* $x = y$;

(b) $m(x, y) = m(y, x)$;

(c) $m(x, y) \leq m(x, z) \cdot m(z, y)$ *for all* $x, y, z \in X$.

*Then $m$ is called a multiplicative cone metric on $X$ and $(X, m)$ is called a multiplicative cone metric space.*

**Example 1.6.** *[1] Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 1\} \subset \mathbb{R}^2$, and $m : X^2 \mapsto E$ be defined as $m(x, y) = (a^{|x-y|}, a^{\alpha|x-y|})$, where $\alpha \geq 0$ is a constant and $a > 1$ is a constant. Then $(X, m)$ is a multiplicative cone metric space.*

**Definition 1.7.** *[4] Let $X$ be a nonempty set and the mapping $m : X^2 \mapsto E$ satisfies*

(a) $m(x, y) \geq 1$ *for all* $x, y \in X$ *and* $m(x, y) = 1$ *if and only if* $x = y$;

(b) $m(x, y) = m(y, x)$ *for all* $x, y \in X$;

(c) $m(x, y) \leq m(x, z) \cdot m(z, w) \cdot m(w, y)$ *for all* $x, y \in X$ *and all distinct points* $z, w \in X - \{x, y\}$ *(multiplicative rectangular inequality).*

*Then $m$ is called a multiplicative cone rectangular metric and $(X, m)$ is called a multiplicative cone rectangular metric space.*

**Example 1.8.** *[4] Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 1\}$, $X = \mathbb{R}$, and $m : X^2 \mapsto E$ be defined as*

$$m(x, y) = \begin{cases} (1, 1) & \text{if } x = y \\ (a^{3\alpha}, a^3) & \text{if } x \text{ and } y \text{ are in } \{1, 2\},\ x \neq y \\ (a^{\alpha}, a) & \text{if } x \text{ and } y \text{ cannot both at a time in } \{1, 2\},\ x \neq y \end{cases}$$

*where $\alpha > 0$ is a constant and $a > 1$ is a constant. Then $(X, m)$ is a multiplicative cone rectangular metric space, but it is not a multiplicative cone metric space since we have $m(1, 2) = (a^{3\alpha}, a^3) > m(1, 3) \cdot m(3, 2) = (a^{2\alpha}, a^2)$.*

Now we introduce the definition of multiplicative cone pentagonal metric space as follows

**Definition 1.9.** *Let $X$ be a nonempty set. Suppose the mapping $m : X^2 \mapsto E$ satisfies*

(a) $1 < m(x, y)$ *for all* $x, y \in X$ *and* $m(x, y) = 1$ *if and only if* $x = y$;

(b) $m(x, y) = m(y, x)$ *for* $x, y \in X$;

(c) $m(x,y) \leq m(x,z) \cdot m(z,w) \cdot m(w,u) \cdot m(u,y)$ *for all* $x,y,z,w,u \in X$ *and for all distinct points* $z,w,u \in X - \{x,y\}$ *(multiplicative pentagonal property).*

*Then* $m$ *is called a multiplicative cone pentagonal metric on* $X$*, and* $(X,m)$ *is called a multiplicative cone pentagonal metric space.*

**Definition 1.10.** *Let* $(X,m)$ *be a multiplicative cone pentagonal metric space and* $\{x_n\}$ *be a sequence in* $(X,m)$*. Then*

(a) $\{x_n\}$ *multiplicative converges to* $x \in X$ *whenever for every* $c \in E$ *with* $1 \ll c$*, there is a natural number* $n_0$ *such that* $m(x_n,x) \ll c$ *for all* $n \geq n_0$*, we denote this by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x$*.*

(b) $\{x_n\}$ *is a multiplicative Cauchy sequence whenever for every* $c \in E$ *with* $1 \ll c$ *there is a natural number* $n_0$ *such that* $m(x_n, x_{n+r}) \ll c$ *for all* $n \geq n_0$*.*

(c) $(X,m)$ *is called mutliplicative complete cone pentagonal metric space if every multiplicative Cauchy sequence in* $(X,m)$ *is multiplicative convergent in* $(X,m)$*.*

**Definition 1.11.** *[4] Let* $P$ *be a multiplicative cone defined as above and let* $\Phi$ *be the set of all non-decreasing continuous functions* $\varphi : P \mapsto P$ *satisfying*

(a) $1 < \varphi(t) < t$ *for all* $t \in P - \{1\}$*;*

(b) *the series* $\prod_{n\geq 0} \varphi^n(t)$ *converges for all* $t \in P - \{1\}$*.*

*From (a) we have* $\varphi(1) = 1$ *and from (b) we have* $\lim \varphi^n(t) = 1$ *for all* $t \in P - \{1\}$*.*

**Definition 1.12.** *[2] Let* $T$ *and* $S$ *be self maps of a nonempty set* $X$*. If* $w = Tx = Sx$ *for some* $x \in X$*, then* $x$ *is called a coincidence point of* $T$ *and* $S$*, and* $w$ *is called a point of coincidence of* $T$ *and* $S$*.*

**Definition 1.13.** *[2] Let* $T$ *and* $S$ *be self maps of a nonempty set* $X$*.* $T$ *and* $S$ *are said to be weakly compatible if they commute at their coincidence point, that is,* $Tx = Sx$ *implies that* $TSx = STx$*.*

**Lemma 1.14.** *[3] Let* $T$ *and* $S$ *be weakly compatible self mappings of a nonempty set* $X$*. If* $T$ *and* $S$ *have a unique point of coincidence* $w = Tx = Sx$*, then* $w$ *is the unique common fixed point of* $T$ *and* $S$*.*

**Lemma 1.15.** *Let* $(X,m)$ *be a complete multiplicative cone pentagonal metric space. Let* $\{x_n\}$ *be a multiplicative Cauchy sequence in* $X$*, and suppose there is a natural number* $N$ *such that*

(a) $x_n \neq x_m$ *for all* $n,m > N$*;*

(b) $x_n, x$ *are distinct points in* $X$ *for all* $n > N$*;*

(c) $x_n, y$ *are distinct points in* $X$ *for all* $n > N$*;*

(d) $x_n \to x$ *and* $x_n \to y$ *as* $n \to \infty$*.*

*Then* $x = y$*.*

## 2 Main Result

Our main result is as follows

**Theorem 2.1.** *Let $(X, m)$ be a multiplicative cone pentagonal metric space. Suppose the mappings $S, f : X \mapsto X$ satisfy the contractive condition*

$$m(Sx, Sy) \leq \varphi(m(fx, fy))$$

*for all $x, y \in X$, where $\varphi \in \Phi$. Suppose that $S(X) \subseteq f(X)$, and $f(X)$ or $S(X)$ is a complete subspace of $X$, then the mappings $S$ and $f$ have a unique point of coincidence in $X$. Moreover, if $S$ and $f$ are weakly compatible, then $S$ and $f$ have a unique common fixed point in $X$.*

*Proof.* Let $x_0$ be an arbitrary point in $X$. Since $S(X) \subseteq f(X)$, we can choose $x_1 \in X$ such that $Sx_0 = fx_1$. Continuing this process, having chosen $x_n$ in $X$, we obtain $x_{n+1}$ such that $Sx_n = fx_{n+1}$ for all $n = 0, 1, 2, \cdots$. We assume that $Sx_n \neq Sx_{n-1}$ for all $n \in \mathbb{N}$. Then from the contractive definition of the theorem, we have

$$\begin{aligned} m(Sx_n, Sx_{n+1}) &\leq \varphi(m(fx_n, fx_{n+1})) \\ &= \varphi(m(Sx_{n-1}, Sx_n)) \\ &\leq \varphi^2(m(fx_{n-1}, fx_n)) \\ &\vdots \\ &\leq \varphi^n(m(Sx_0, Sx_1)). \end{aligned}$$

In a similar way it follows that

$$m(Sx_n, Sx_{n+2}) \leq \varphi^n(m(Sx_0, Sx_2))$$

$$m(Sx_n, Sx_{n+3}) \leq \varphi^n(m(Sx_0, Sx_3)).$$

Similarly for $k = 1, 2, 3, \cdots$, it further follows that

$$m(Sx_n, Sx_{n+3k+1}) \leq \varphi^n(m(Sx_0, Sx_{3k+1}))$$

$$m(Sx_n, Sx_{n+3k+2}) \leq \varphi^n(m(Sx_0, Sx_{3k+2}))$$

$$m(Sx_n, Sx_{n+3k+3}) \leq \varphi^n(m(Sx_0, Sx_{3k+3})).$$

Since $m(Sx_n, Sx_{n+1}) \leq \varphi^n(m(Sx_0, Sx_1))$, by multiplicative pentagonal property, we have

$$\begin{aligned} m(Sx_0, Sx_4) &\leq m(Sx_0, Sx_1) \cdot m(Sx_1, Sx_2) \cdot m(Sx_2, Sx_3) \cdot m(Sx_3, Sx_4) \\ &\leq m(Sx_0, Sx_1) \cdot \varphi(m(Sx_0, Sx_1)) \cdot \varphi^2(m(Sx_0, Sx_1)) \cdot \varphi^3(m(Sx_0, Sx_1)) \\ &\leq \prod_{i=0}^{3} \varphi^i(m(Sx_0, Sx_1)) \end{aligned}$$

and

$$\begin{aligned} m(Sx_0,Sx_7) &\leq m(Sx_0,Sx_1)\cdot m(Sx_1,Sx_2)\cdot m(Sx_2,Sx_3)\cdot m(Sx_3,Sx_4)\cdot m(Sx_4,Sx_5)\cdot m(Sx_5,Sx_6)\\ &\quad \cdot m(Sx_6,Sx_7)\\ &\leq \prod_{i=0}^{6}\varphi^i(m(Sx_0,Sx_1)). \end{aligned}$$

By induction, we have for each $k=1,2,3,\cdots$

$$m(Sx_0,Sx_{3k+1}) \leq \prod_{i=0}^{3k}\varphi^i(m(Sx_0,Sx_1)).$$

Also using $m(Sx_n,Sx_{n+1}) \leq \varphi^n(m(Sx_0,Sx_1))$, $m(Sx_n,Sx_{n+2}) \leq \varphi^n(m(Sx_0,Sx_2))$, and multiplicative pentagonal property, we have that

$$m(Sx_0,Sx_5) \leq \prod_{i=0}^{2}\varphi^i(m(Sx_0,Sx_1))\cdot\varphi^3(m(Sx_0,Sx_2))$$

and

$$m(Sx_0,Sx_8) \leq \prod_{i=0}^{5}\varphi^i(m(Sx_0,Sx_1))\cdot\varphi^6(m(Sx_0,Sx_2)).$$

By induction, we have for each $k=1,2,3,\cdots$

$$m(Sx_0,Sx_{3k+2}) \leq \prod_{i=0}^{3k-1}\varphi^i(m(Sx_0,Sx_1))\cdot\varphi^{3k}(m(Sx_0,Sx_2)).$$

Again using $m(Sx_n,Sx_{n+1}) \leq \varphi^n(m(Sx_0,Sx_1))$, $m(Sx_n,Sx_{n+3}) \leq \varphi^n(m(Sx_0,Sx_3))$, and multiplicative pentagonal property, we have that

$$m(Sx_0,Sx_6) \leq \prod_{i=0}^{2}\varphi^i(m(Sx_0,Sx_1))\cdot\varphi^3(m(Sx_0,Sx_3))$$

and

$$m(Sx_0,Sx_9) \leq \prod_{i=0}^{5}\varphi^i(m(Sx_0,Sx_1))\cdot\varphi^6(m(Sx_0,Sx_3)).$$

By induction, we have for each $k=1,2,3,\cdots$

$$m(Sx_0,Sx_{3k+3}) \leq \prod_{i=0}^{3k-1}\varphi^i(m(Sx_0,Sx_1))\cdot\varphi^{3k}(m(Sx_0,Sx_3)).$$

Now using $m(Sx_n,Sx_{n+3k+1}) \leq \varphi^n(m(Sx_0,Sx_{3k+1}))$, and $m(Sx_0,Sx_{3k+1}) \leq \prod_{i=0}^{3k}\varphi^i(m(Sx_0,Sx_1))$, for each $k = 1, 2, 3, \cdots$, we have that

$$\begin{aligned} m(Sx_n, Sx_{n+3k+1}) &\leq \varphi^n(\prod_{i=0}^{3k} \varphi^i(m(Sx_0, Sx_1))) \\ &\leq \varphi^n(\prod_{i=0}^{3k} \varphi^i(m(Sx_0, Sx_1) \cdot m(Sx_0, Sx_2) \cdot m(Sx_0, Sx_3))) \\ &\leq \varphi^n(\prod_{i=0}^{\infty} \varphi^i(m(Sx_0, Sx_1) \cdot m(Sx_0, Sx_2) \cdot m(Sx_0, Sx_3))). \end{aligned}$$

Now using $m(Sx_n, Sx_{n+3k+2}) \leq \varphi^n(m(Sx_0, Sx_{3k+2}))$, and $m(Sx_0, Sx_{3k+2}) \leq \prod_{i=0}^{3k-1} \varphi^i(m(Sx_0, Sx_1)) \cdot \varphi^{3k}(m(Sx_0, Sx_2))$, for each $k = 1, 2, 3, \cdots$, we have that

$$\begin{aligned} m(Sx_n, Sx_{n+3k+2}) &\leq \varphi^n(\prod_{i=0}^{3k-1} \varphi^i(m(Sx_0, Sx_1)) \cdot \varphi^{3k}(m(Sx_0, Sx_2))) \\ &\leq \varphi^n(\prod_{i=0}^{\infty} \varphi^i(m(Sx_0, Sx_1) \cdot m(Sx_0, Sx_2) \cdot m(Sx_0, Sx_3))). \end{aligned}$$

Now using $m(Sx_n, Sx_{n+3k+3}) \leq \varphi^n(m(Sx_0, Sx_{3k+3}))$, and $m(Sx_0, Sx_{3k+3}) \leq \prod_{i=0}^{3k-1} \varphi^i(m(Sx_0, Sx_1)) \cdot \varphi^{3k}(m(Sx_0, Sx_3))$, for each $k = 1, 2, 3, \cdots$, we have that

$$m(Sx_n, Sx_{n+3k+3}) \leq \varphi^n(\prod_{i=0}^{\infty} \varphi^i(m(Sx_0, Sx_1) \cdot m(Sx_0, Sx_2) \cdot m(Sx_0, Sx_3))).$$

Thus, for each $m$ we have

$$m(Sx_n, Sx_{n+m}) \leq \varphi^n(\prod_{i=0}^{\infty} \varphi^i(m(Sx_0, Sx_1) \cdot m(Sx_0, Sx_2) \cdot m(Sx_0, Sx_3))).$$

Since $\prod_{i=0}^{\infty} \varphi^i(m(Sx_0, Sx_1) \cdot m(Sx_0, Sx_2) \cdot m(Sx_0, Sx_3))$ converges (by Definition 1.11), where $m(Sx_0, Sx_1) \cdot m(Sx_0, Sx_2) \cdot m(Sx_0, Sx_3) \in P - \{1\}$, and $P$ is closed, $\prod_{i=0}^{\infty} \varphi^i(m(Sx_0, Sx_1) \cdot m(Sx_0, Sx_2) \cdot m(Sx_0, Sx_3)) \in P - \{1\}$. Hence

$$\lim_{n\to\infty} \varphi^n(\prod_{i=0}^{\infty} \varphi^i(m(Sx_0, Sx_1) \cdot m(Sx_0, Sx_2) \cdot m(Sx_0, Sx_3))) = 1.$$

Then for given $c \gg 1$, there is a natural number $N_1$ such that

$$\varphi^n(\prod_{i=0}^{\infty} \varphi^i(m(Sx_0, Sx_1) \cdot m(Sx_0, Sx_2) \cdot m(Sx_0, Sx_3))) \ll c$$

for all $n \geq N_1$. It follows that $m(Sx_n, Sx_{n+m}) \ll c$, for all $n \geq N_1$. Therefore $\{Sx_n\}$ is a multiplicative Cauchy sequence in $X$. Suppose $S(X)$ is a complete subspace of $X$, then there exists a point $z \in S(X)$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} fx_{n+1} = z$. Also we can find a point $y \in X$ such that $fy = z$. Now we show that $Sy = z$. Given $c \gg 1$, we choose natural numbers $N_2, N_3$ such that $m(z, fx_n) \ll c^{\frac{1}{4}}$ for all

$n \geq N_2$, and $m(Sx_n, Sx_{n-1}) \ll c^{\frac{1}{4}}$ for all $n \geq N_3$. Since $x_n \neq x_m$ for $n \neq m$, by multiplicative pentagonal property we have that

$$\begin{aligned} m(Sy, z) &\leq m(Sy, Sx_n) \cdot m(Sx_n, fx_n) \cdot m(fx_n, fx_{n-1}) \cdot m(fx_{n-1}, z) \\ &\leq \varphi(m(fy, fx_n)) \cdot m(Sx_n, Sx_{n-1}) \cdot m(Sx_{n-1}, Sx_{n-2}) \cdot m(fx_{n-1}, z) \\ &< m(fy, fx_n) \cdot m(Sx_n, Sx_{n-1}) \cdot m(Sx_{n-1}, Sx_{n-2}) \cdot m(fx_{n-1}, z) \\ &\ll c^{\frac{1}{4}} \cdot c^{\frac{1}{4}} \cdot c^{\frac{1}{4}} \cdot c^{\frac{1}{4}} \\ &= c \end{aligned}$$

for all $n > N$, where $N = \max\{N_2, N_3\}$. Since $c$ is arbitrary, we have $m(Sy, z) \ll c^{\frac{1}{m}}$ for all $m \in \mathbb{N}$. Since $c^{\frac{1}{m}} \to 1$ as $m \to \infty$, we conclude that $c^{\frac{1}{m}} \cdot \frac{1}{m(Sy,z)} \to \frac{1}{m(Sy,z)}$ as $m \to \infty$. Since $P$ is closed, $\frac{1}{m(Sy,z)} \in P$. Hence $m(Sy, z) \in P \cap \frac{1}{P}$. By definition of multiplicative cone, we get that $m(Sy, z) = 1$, and so $Sy = fy = z$. Hence $z$ is a point of coincidence of $S$ and $f$.

Next, we show that $z$ is unique. Suppose $z'$ is another point of coincidence of $S$ and $f$, that is, $Sx = fx = z'$, for some $x \in X$. Then

$$m(z, z') = m(Sy, Sx) \leq \varphi(m(fy, fx)) = \varphi(m(z, z')) < m(z, z').$$

Hence $z = z'$. Since $S$ and $f$ are weakly compatible, by Lemma 1.14, $z$ is the unique common fixed point of $S$ and $f$, and the proof is finished. □

**Corollary 2.2.** *Let $(X, m)$ be a multiplicative cone pentagonal metric space and $P$ be a multiplicative normal cone with multiplicative normal constant $k$. Suppose the mappings $S, f : X \mapsto X$ satisfy the contractive condition*

$$m(Sx, Sy) \leq m(fx, fy)^{\lambda}$$

*for all $x, y \in X$, where $\lambda \in [0, 1)$. Suppose that $S(X) \subseteq f(X)$ and $f(X)$ or $S(X)$ is a complete subspace of $X$, then the mappings $S$ and $f$ have a unique point of coincidence in $X$. Moreover, if $S$ and $f$ are weakly compatible, then $S$ and $f$ have a unique common fixed point in $X$.*

*Proof.* Define $\varphi : P \mapsto P$ by $\varphi(t) = t^{\lambda}$. Then it is clear that $\varphi$ satisfies the conditions in Definition 1.11. Hence the result follows from the above theorem. □

**Corollary 2.3.** *Let $(X, m)$ be a multiplicative cone pentagonal metric space. Suppose the mapping $S : X \mapsto X$ satisfy the following*

$$m(Sx, Sy) \leq \varphi(m(x, y))$$

*for all $x, y \in X$, where $\varphi \in \Phi$. Then $S$ has a unique fixed point in $X$.*

*Proof.* Put $f = I$ in the above theorem, where $I$ is the identity mapping. This completes the proof. □

**Corollary 2.4.** *Let $(X, m)$ be a multiplicative cone pentagonal metric space and $P$ be a multiplicative normal cone with multiplicative normal constant $k$. Suppose the mapping $S : X \mapsto X$ satisfies the contractive condition*

$$m(Sx, Sy) \leq m(x, y)^{\lambda}$$

*for all $x, y \in X$, where $\lambda \in [0, 1)$. Then $S$ has a unique fixed point in $X$.*

*Proof.* Put $f = I$ in Corollary 2.2, where $I$ is the identity mapping. This completes the proof. □

**Example 2.5.** *Let $X = \{r, s, t, u, v\}$, $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 1\}$ be a multiplicative cone in $E$, and $a > 1$ be a constant. Define $m : X^2 \mapsto E$ by*

$$m(x, x) = 1,$$

$$m(r, s) = m(s, r) = (a^4, a^8),$$

$$m(r, t) = m(t, r) = m(t, u) = m(u, t) = m(s, t) = m(t, s) = m(s, u) = m(u, s) = m(r, u) = m(u, r) = (a, a^2),$$

$$m(r, v) = m(v, r) = m(s, v) = m(v, s) = m(t, v) = m(v, t) = m(u, v) = m(v, u) = (a^3, a^6).$$

*Then $(X, m)$ is a complete multiplicative cone pentagonal metric space, but $(X, m)$ is not a complete cone multiplicative rectangular metric space because it lacks the multiplicative rectangular property:*

$$(a^4, a^8) = m(r, s) > m(r, t) \cdot m(t, u) \cdot m(u, s) = (a^3, a^6).$$

*Now we define mappings $S, f : X \mapsto X$ as follows:*

$$S(x) = \begin{cases} u & \text{if } x \neq v \\ s & \text{if } x = v \end{cases}$$

$$f(x) = \begin{cases} t & \text{if } x = r \\ r & \text{if } x = s \\ s & \text{if } x = t \\ u & \text{if } x = u \\ v & \text{if } x = v \end{cases}$$

*Clearly $S(X) \subseteq f(X)$, $f(X)$ is a complete subspace of $X$ and the pair $(S, f)$ is weakly compatible. The inequality $m(Sx, Sy) \leq \varphi(m(fx, fy))$ holds for all $x, y \in X$, where $\varphi(t) = t^{\frac{1}{3}}$ and $u \in X$ is the unique common fixed point of the mappings $S$ and $f$.*

## 3 Open Problems

The open problem is to prove or disprove the following

**Conjecture 3.1.** *Let* $(X, m)$ *be a multiplicative cone pentagonal metric space. Suppose the mappings* $S, f : X \mapsto X$ *satisfy the contractive condition*

$$m(Sfx, Sfy) \leq \varphi(m(Sx, Sy))$$

*for all* $x, y \in X$*, where* $\varphi \in \Phi$*. Suppose that* $S$ *is one-to-one,* $S(X)$ *is a complete subspace of* $X$*, then the mapping* $f$ *have a unique fixed point in* $X$*. Moreover, if* $S$ *and* $f$ *are commuting at the fixed point of* $f$*, then* $S$ *and* $f$ *have a unique common fixed point in* $X$*.*

**Conjecture 3.2.** *Conjecture 3.1 holds in multiplicative cone rectangular metric space [4].*

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