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(α, β)-Reich Contraction Mapping Theorem in Multiplicative Metric Space

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Abstract

In this paper, we introduce the notion of (α, β)-Reich contraction, and obtain a fixed point theorem for such mappings in the setting of multiplicative metric space. Moreover, we give a Corollary as a consequence of the main result.

1 Introduction and Preliminaries

Theorem 1.1. *[1] Let (X, d) be a complete metric space and $T : X \mapsto X$ be a Banach contraction mapping, that is,*

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$, where $k \in [0, 1)$. Then T has a unique fixed point.

Theorem 1.2. *[2, 3] Let (X, d) be a complete metric space and $T : X \mapsto X$ be a Kannan contraction mapping, that is,*

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$, where $k \in [0, \frac{1}{2})$. Then T has a unique fixed point.

Theorem 1.3. *[4] Let (X, d) be a complete metric space and $T : X \mapsto X$ be a Chatterjea contraction mapping, that is,*

$$d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)]$$

for all $x, y \in X$, where $k \in [0, \frac{1}{2})$. Then T has a unique fixed point.

Definition 1.4. *[5] Let X be a nonempty set. A mapping $d : X \times X \mapsto \mathbb{R}$ is said to be a multiplicative metric if it satisfies the following conditions*

(a) *$d(x, y) \geq 1$ for all $x, y \in X$ and $d(x, y) = 1$ if and ony if $x = y$;*

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(b) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(c) $d(x, z) \leq d(x, y) \cdot d(y, z)$ *for all* $x, y, z \in X$.

The pair (X, d) *is called a multiplicative metric space.*

Definition 1.5. *[6] Let* (X, d) *be a multipicative metric space,* $x \in X$ *and* $\epsilon > 1$. *Then:*

(a) *The multiplicative open ball of radius* ϵ *with center* x *is defined by the set*

$$B_\epsilon(x) := \{y \in X | d(x, y) < \epsilon\}.$$

(b) *The multiplicative closed ball of radius* ϵ *with center* x *is defined by the set*

$$\overline{B}_\epsilon(x) := \{y \in X | d(x, y) \leq \epsilon\}.$$

Definition 1.6. *[6] Let* (X, d) *be a multiplicative metric space,* $\{x_n\}$ *be a sequence in* X *and* $x \in X$. *If for every multiplicative open ball* $B_\epsilon(x)$, *there exists a natural number* N *such that* $n \geq N \Longrightarrow x_n \in B_\epsilon(x)$, *then the sequence* $\{x_n\}$ *is said to be multiplicative convergent to* x, *denoted by* $x_n \to_* x$ *as* $n \to \infty$.

Lemma 1.7. *[6] Let* (X, d) *be a multiplicative metric space,* $\{x_n\}$ *be a sequence in* X *and* $x \in X$. *Then*

$$x_n \to_* x \text{ as } n \to \infty \text{ if and only if } d(x_n, x) \to_* 1 \text{ as } n \to \infty.$$

Lemma 1.8. *[6] Let* (X, d) *be a multiplicative metric space and* $\{x_n\}$ *be a sequence in* X. *If the sequence* $\{x_n\}$ *is multiplicative convergent, then the multiplicative limit point is unique.*

Definition 1.9. *[6] Let* (X, d) *be a multiplicative metric space and* $\{x_n\}$ *be a sequence in* X. *The sequence* $\{x_n\}$ *is called multiplicative Cauchy, if for all* $\epsilon > 1$, *there exists* $N \in \mathbb{N}$ *such that* $d(x_n, x_m) < \epsilon$ *for* $m, n \geq N$.

Lemma 1.10. *[6] Let* (X, d) *be a multiplicative metric space and* $\{x_n\}$ *be a sequence in* X. *Then* $\{x_n\}$ *is a multiplicative Cauchy sequence if and only if* $d(x_n, x_m) \to_* 1$ *as* $n, m \to \infty$.

Definition 1.11. *[6] Let* (X, d) *be a multiplicative metric space. The multiplicative metric space* X *is said to be complete if and only if every Cauchy sequence* $\{x_n\}$ *in* X *for all* $n \in \mathbb{N}$ *converges in* X.

Definition 1.12. *[6] Let* (X, d) *be a multiplicative metric space. A point* $x \in X$ *is said to a multiplicative limit point of* $S \subseteq X$ *if and only if* $(B_\epsilon(x) - \{x\}) \cap S \neq \emptyset$ *for every* $\epsilon > 1$.

Definition 1.13. *[6] Let* (X, d) *be a multiplicative metric space. We call a set* $S \subseteq X$ *multiplicative closed in* (X, d), *if* S *contains all of its multiplicative limit points.*

Definition 1.14. *[6] Let* (X, d) *be a multiplicative metric space. A self-mapping* f *is said to be a multiplicaitve Banach contraction if*

$$d(fx, fy) \leq d(x, y)^\lambda$$

for all $x, y \in X$ *where* $\lambda \in [0, 1)$.

Definition 1.15. *[6] Let (X,d) be a multiplicative metric space. A self-mapping f is said to be a multiplicaitve Kannan contraction if*

$$d(fx,fy) \leq (d(fx,x)\cdot d(fy,y))^{\lambda}$$

for all $x,y \in X$ where $\lambda \in [0,\frac{1}{2})$.

Definition 1.16. *[6] Let (X,d) be a multiplicative metric space. A self-mapping f is said to be a multiplicaitve Chatterjea contraction if*

$$d(fx,fy) \leq (d(fx,y)\cdot d(fy,x))^{\lambda}$$

for all $x,y \in X$ where $\lambda \in [0,\frac{1}{2})$.

Definition 1.17. *[7] Let X be a nonempty set, f be a self-mapping on X, and $\alpha,\beta : X \mapsto [0,\infty)$ be two mappings. We say that f is a cyclic (α,β)-admissible mapping if*

$$x \in X, \alpha(x) \geq 1 \Longrightarrow \beta(fx) \geq 1$$

and

$$x \in X, \beta(x) \geq 1 \Longrightarrow \alpha(fx) \geq 1.$$

Definition 1.18. *[8] Let (X,d) be a multiplicative metric space, and let $\alpha,\beta : X \mapsto [0,\infty)$ be two mappings. The mapping $f : X \mapsto X$ is said to be a multiplicative (α,β)-Banach contraction if*

$$\alpha(x)\beta(y)\cdot d(fx,fy) \leq d(x,y)^{\lambda}$$

for all $x,y \in X$, where $\lambda \in [0,1)$.

Theorem 1.19. *[8] Let (X,d) be a complete multiplicative metric space and $f : X \mapsto X$ be a multiplicative (α,β)-Banach contraction mapping. Suppose that the following condtions hold:*

(a) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$;

(b) f is cyclic (α,β)-admissible mapping;

(c) one of the following conditions holds:

[(i)] f is continuous;

[(ii)] if $\{x_n\}$ is a sequence in X such that $x_n \rightarrow_* x \in X$ as $n \rightarrow \infty$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$, then, $\beta(x) \geq 1$.

Then f has a fixed point. Furthermore, if $\alpha(x) \geq 1$ and $\beta(x) \geq 1$ for all fixed point $x \in X$, then f has a unique fixed point.

Definition 1.20. *[8] Let (X,d) be a multiplicative metric space, and let $\alpha,\beta : X \mapsto [0,\infty)$ be two mappings. The mapping $f : X \mapsto X$ is said to be a multiplicative (α,β)-Kannan contraction if*

$$\alpha(x)\beta(y) \cdot d(fx, fy) \leq (d(fx, x) \cdot d(fy, y))^{\lambda}$$

for all $x, y \in X$, where $\lambda \in [0, \frac{1}{2})$.

Definition 1.21. *[8] Let (X,d) be a multiplicative metric space, and let $\alpha,\beta : X \mapsto [0,\infty)$ be two mappings. The mapping $f : X \mapsto X$ is said to be a multiplicative (α,β)-Chatterjea contraction if*

$$\alpha(x)\beta(y) \cdot d(fx, fy) \leq (d(fx, y) \cdot d(fy, x))^{\lambda}$$

for all $x, y \in X$, where $\lambda \in [0, \frac{1}{2})$.

Theorem 1.22. *[8] Let (X,d) be a complete multiplicative metric space and $f : X \mapsto X$ be a multiplicative (α,β)-Kannan contraction mapping. Suppose that the following conditions hold:*

(a) *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$;*

(b) *f is cyclic (α,β)-admissible mapping;*

(c) *one of the following conditions holds:*

   [(i)] f is continuous;

   [(ii)] if $\{x_n\}$ is a sequence in X such that $x_n \to_* x \in X$ as $n \to \infty$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$, then, $\beta(x) \geq 1$.

Then f has a fixed point. Furthermore, if $\alpha(x) \geq 1$ and $\beta(x) \geq 1$ for all fixed point $x \in X$, then f has a unique fixed point.

Theorem 1.23. *[8] Let (X,d) be a complete multiplicative metric space and $f : X \mapsto X$ be a multiplicative (α,β)-Chatterjea contraction mapping. Suppose that the following conditions hold:*

(a) *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$;*

(b) *f is cyclic (α,β)-admissible mapping;*

(c) *one of the following conditions holds:*

   [(i)] f is continuous;

   [(ii)] if $\{x_n\}$ is a sequence in X such that $x_n \to_* x \in X$ as $n \to \infty$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$, then, $\beta(x) \geq 1$.

Then f has a fixed point. Furthermore, if $\alpha(x) \geq 1$ and $\beta(x) \geq 1$ for all fixed point $x \in X$, then f has a unique fixed point.

Theorem 1.24. *[9] Let (X,d) be a complete metric space and let $f : X \mapsto X$ be a Reich type single-valued (a,b,c)-contraction, that is, there exists nonnegative numbers a,b,c with $a+b+c<1$ such that*

$$d(f(x),f(y)) \leq ad(x,y) + bd(x,f(x)) + cd(y,f(y))$$

for each $x,y \in X$. Then T has a unique fixed point.

2 Main Result

Definition 2.1. *Let (X,d) be a multiplicative metric space, and let $\alpha,\beta : X \mapsto [0,\infty)$ be two mappings. The mapping $f : X \mapsto X$ will be called a multiplicative (α,β)-Reich contraction if*

$$\alpha(x)\beta(y) \cdot d(fx,fy) \leq (d(fx,x) \cdot d(fy,y) \cdot d(x,y))^{\lambda}$$

for all $x,y \in X$, where $\lambda \in [0,\frac{1}{3})$.

Theorem 2.2. *Let (X,d) be a complete multiplicative metric space and $f : X \mapsto X$ be a multiplicative (α,β)-Reich contraction mapping. Suppose that the following conditions hold:*

(a) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$;

(b) f is cyclic (α,β)-admissible mapping;

(c) one of the following conditions holds:

[(i)] f is continuous;

[(ii)] if $\{x_n\}$ is a sequence in X such that $x_n \to_* x \in X$ as $n \to \infty$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$, then, $\beta(x) \geq 1$.

Then f has a fixed point. Furthermore, if $\alpha(x) \geq 1$ and $\beta(x) \geq 1$ for all fixed point $x \in X$, then f has a unique fixed point.

Proof. Starting from a point $x_0 \in X$ in condition (a), we get $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$. We will construct the iterative sequence $\{x_n\}$, where $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$. Since f is cyclic (α,β)-admissible mapping, we have

$$\alpha(x_0) \geq 1 \Longrightarrow \beta(x_1) = \beta(fx_0) \geq 1$$

and

$$\beta(x_0) \geq 1 \Longrightarrow \alpha(x_1) = \alpha(fx_0) \geq 1.$$

By a similar method, we get

$$\alpha(x_n) \geq 1 \text{ and } \beta(x_n) \geq 1$$

for all $n \in \mathbb{N}$. This implies that

$$\alpha(x_{n-1})\beta(x_n) \geq 1$$

for all $n \in \mathbb{N}$. From the (α, β) Reich multiplicative contractive condition of f, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\leq \alpha(x_{n-1})\beta(x_n) \cdot d(fx_{n-1}, fx_n) \\ &\leq (d(fx_{n-1}, x_{n-1}) \cdot d(fx_n, x_n) \cdot d(x_{n-1}, x_n))^{\lambda} \end{aligned}$$

for all $n \in \mathbb{N}$. Thus we have

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)^h$$

for all $n \in \mathbb{N}$, where $h = \frac{2\lambda}{1-\lambda}$. Let $m, n \in \mathbb{N}$ such that $m < n$, then we get

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) \cdot d(x_{m+1}, x_{m+2}) \cdots d(x_{n-1}, x_n) \\ &\leq d(x_0, x_1)^{h^m + h^{m+1} + \cdots + h^{n-1}} \\ &\leq d(x_0, x_1)^{\frac{h^m}{1-h}}. \end{aligned}$$

Letting $m, n \to \infty$ we get $d(x_m, x_n) \to_* 1$, and so the sequence $\{x_n\}$ is multiplicative Cauchy. From the completeness of X, there exists $z \in X$ such that $x_n \to_* z$ as $n \to \infty$. Now we assume that f is continuous. Hence we obtain

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = f\left(\lim_{n\to\infty} x_n\right) = fz.$$

Next, we will assume that condition $[(ii)]$ holds. Hence, $\beta(z) \geq 1$. Then we have

$$\begin{aligned} d(fz, z) &\leq d(fz, fx_n) \cdot d(fx_n, z) \\ &\leq \alpha(z)\beta(x_n) \cdot d(fz, fx_n) \cdot d(fx_n, z) \\ &\leq (d(fz, z) \cdot d(fx_n, x_n) \cdot d(z, x_n))^{\lambda} \cdot d(fx_n, z) \\ &= (d(fz, z) \cdot d(x_{n+1}, x_n) \cdot d(z, x_n))^{\lambda} \cdot d(x_{n+1}, z). \end{aligned}$$

Letting $n \to \infty$ in the above inequality we get $d(fz, z) \leq d(fz, z)^{\lambda}$, which implies that $d(fz, z) = 1$, that is $fz = z$. This shows that z is a fixed point of f. Now we show that z is the unique fixed point of f. Assume that y is another fixed point of f. From the hypothesis, we find that $\alpha(z) \geq 1$ and $\beta(y) \geq 1$, and hence we have

$$\begin{aligned} d(z, y) &= d(fz, fy) \\ &\leq \alpha(z)\beta(y) \cdot d(fz, fy) \\ &\leq (d(fz, z) \cdot d(fy, y) \cdot d(z, y))^{\lambda} \\ &= (d(z, z) \cdot d(y, y) \cdot d(z, y))^{\lambda} \\ &= d(z, y)^{\lambda}. \end{aligned}$$

This shows that $d(z, y) = 1$, and so $z = y$. Therefore z is the unique fixed point of f. This completes the proof. □

Corollary 2.3. *Let* (X, d) *be a complete multiplicative metric space, and* $f : X \mapsto X$ *be a multiplicative Reich contraction mapping. Then* f *has a unique fixed point. Moreover, for any* $x \in X$*, the iterative sequence* $\{f^n x\}$ *converges to the fixed point.*

Proof. Setting $\alpha(x) = 1$ and $\beta(x) = 1$ for all $x \in X$ in the above theorem, we get the result. □

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