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On New Sulaiman-type Hardy Integral Inequalities

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Abstract

In this article, we present new integral inequalities that relate a function to its primitive in the context of L^p-spaces on finite intervals. These inequalities can be presented as variations of the Sulaiman-type Hardy integral inequalities. One of our approach combines a new rearrangement of key integrals with the Chebyshev integral inequality. We then derive reverse forms of these inequalities to demonstrate the flexibility and broad applicability of the method.

1 Introduction

The Hardy integral inequality is a fundamental result in analysis. It provides a comprehensive upper bound relating a function to its primitive in the context of L^p-spaces. For completeness, we recall it in detail in the theorem below.

Theorem 1.1. *[4] Let $p > 1$, $f : [0, +\infty) \to [0, +\infty)$ be a function and, for any $x \geq 0$,*

$$F(x) = \int_0^x f(t)dt,$$

provided that it exists. Then we have

$$\int_0^{+\infty} \left(\frac{F(x)}{x}\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} f^p(x)dx,$$

provided that the integral in the upper bound exists.

The technical details can be found in [4, 5]. The Hardy integral inequality has many applications in functional analysis and the study of differential and integral equations. When the interval of integration is finite rather than $[0, +\infty)$, several variations of this inequality have been established. See [2, 7, 9], and the references therein. Among these, a version proposed by W. T. Sulaiman in [9] is particularly relevant to this article. For completeness, we recall it in detail in the theorem below.

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Theorem 1.2. *[9, Theorem 3.1, part 1] Let $p > 1$, $a, b \in \mathbb{R}$ with $a < b$, $f : [a, b] \to [0, +\infty)$ be a function and, for any $x \in [a, b]$,*

$$F(x) = \int_a^x f(t)dt,$$

provided that it exists. Then we have

$$\int_a^b \left(\frac{F(x)}{x}\right)^p dx \leq \frac{(b-a)^p}{p} \int_a^b \left(\frac{f(x)}{x}\right)^p dx - \frac{1}{p} \int_a^b \left(1 - \frac{a}{x}\right)^p f^p(x)dx,$$

provided that the integrals in the upper bound exist.

This theorem is interesting because it establishes a precise relationship between a function and its primitive in the context of weighted L^p-spaces. However, it should be noted that the upper bound may fail to be sharp when $(b-a)^p$ is large. Important developments inspired by the Sulaiman-type Hardy integral inequality can be found in [1, 6, 8, 10].

An analysis of the proof of [9, Theorem 3.1, part 1] reveals that it is sufficiently flexible to allow for new inequalities to be envisaged. Building upon this, the present article aims to establish two new variations. A common feature of these variations is their potential to overcome the limitation of large values of $(b-a)^p$. The proofs mainly rely on a new arrangement of a key integral in the original argument. The application of the Chebyshev integral inequality, as presented in [3], is also innovative in this context. Additionally, we demonstrate that our inequalities can be reversed for $p \in (0, 1)$ with only slight modification.

The remainder of the paper is organized as follows: Section 2 presents the new Sulaiman-type Hardy integral inequalities. Their reverse forms are established in Section 3. Finally, Section 4 provides concluding remarks.

2 Sulaiman-type Integral Inequalities

2.1 First new integral inequality

Our first integral inequality provides an alternative to [9, Theorem 3.1, part 1].

Theorem 2.1. *Let $p > 1$, $a \in \mathbb{R}\backslash\{0\}$, $b \in \mathbb{R}$ with $a < b$, $f : [a, b] \to [0, +\infty)$ be a function and, for any $x \in [a, b]$,*

$$F(x) = \int_a^x f(t)dt,$$

provided that it exists. Then we have

$$\int_a^b \left(\frac{F(x)}{x}\right)^p dx \leq \frac{b}{ap}\left(1 - \frac{a}{b}\right)^p \int_a^b f^p(x)dx - \frac{b}{ap} \int_a^b \left(1 - \frac{a}{x}\right)^p f^p(x)dx,$$

provided that the integrals in the upper bound exist.

Proof. We follow the first steps to [9, Proof of Theorem 3.1, part 1]. Applying the Hölder integral inequality with $p > 1$ and the Fubini-Tonelli integral theorem, we obtain

$$\begin{aligned}\int_a^b \left(\frac{F(x)}{x}\right)^p dx &= \int_a^b x^{-p} \left(\int_a^x f(t)dt\right)^p dx \\ &\le \int_a^b x^{-p} \left(\int_a^x f^p(t)dt\right) \left(\int_a^x dt\right)^{p-1} dx \\ &= \int_a^b x^{-p}(x-a)^{p-1} \left(\int_a^x f^p(t)dt\right) dx \\ &= \int_a^b f^p(t) \left(\int_t^b x^{-p}(x-a)^{p-1}dx\right) dt. \qquad (1)\end{aligned}$$

Let us evaluate the integral with respect to x, proceeding differently to [9, Proof of Theorem 3.1, part 1]. By suitably arranging the integrand, applying a simple upper bound for x with $x \in [t, b]$, and recognizing a standard primitive, we obtain

$$\begin{aligned}\int_t^b x^{-p}(x-a)^{p-1}dx &= \int_t^b x\frac{1}{x^2} \left(1-\frac{a}{x}\right)^{p-1} dx \\ &\le b\int_t^b \frac{1}{x^2} \left(1-\frac{a}{x}\right)^{p-1} dx \\ &= b\left[\frac{1}{ap}\left(1-\frac{a}{x}\right)^p\right]_t^b \\ &= \frac{b}{ap}\left(\left(1-\frac{a}{b}\right)^p - \left(1-\frac{a}{t}\right)^p\right).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\int_a^b \left(\frac{F(x)}{x}\right)^p dx &\le \int_a^b f^p(t)\frac{b}{ap}\left(\left(1-\frac{a}{b}\right)^p - \left(1-\frac{a}{t}\right)^p\right) dt \\ &= \frac{b}{ap}\left(1-\frac{a}{b}\right)^p \int_a^b f^p(t)dt - \frac{b}{ap}\int_a^b \left(1-\frac{a}{t}\right)^p f^p(t)dt.\end{aligned}$$

After a minor change in notation, we can write

$$\int_a^b \left(\frac{F(x)}{x}\right)^p dx \le \frac{b}{ap}\left(1-\frac{a}{b}\right)^p \int_a^b f^p(x)dx - \frac{b}{ap}\int_a^b \left(1-\frac{a}{x}\right)^p f^p(x)dx.$$

This completes the proof. □

In contrast to [9, Theorem 3.1, part 1], the obtained inequality remains robust even when $(b-a)^p$ is large. In particular, p can take large values without compromising the sharpness of the upper bound, since $(1-a/b)^p \in [0,1]$ and $(1-a/x)^p \in [0,1]$ for any $x \in [a,b]$. However, the inequality requires that $a \neq 0$, which can be considered a limitation.

2.2 Second new integral inequality

The theorem below provides another alternative to [9, Theorem 3.1, part 1]. The proof innovates by the use of the Chebyshev integral inequality.

Theorem 2.2. *Let $p > 1$, $a, b \in \mathbb{R}$ with $a < b$, $f : [a, b] \to [0, +\infty)$ be a function and, for any $x \in [a, b]$,*

$$F(x) = \int_a^x f(t)dt,$$

provided that it exists. Then we have

$$\int_a^b \left(\frac{F(x)}{x}\right)^p dx \leq \frac{1}{p(p-1)} \int_a^b \frac{1}{b-x} \left(x^{1-p} - b^{1-p}\right) \left((b-a)^p - (x-a)^p\right) f^p(x)dx,$$

provided that the integral in the upper bound exists.

Proof. We can reuse Equation (1), i.e.,

$$\int_a^b \left(\frac{F(x)}{x}\right)^p dx \leq \int_a^b f^p(t) \left(\int_t^b x^{-p}(x-a)^{p-1}dx\right) dt.$$

Let us evaluate the integral with respect to x. Applying the Chebyshev integral inequality by noting that, for $p > 1$, x^{-p} and $(x-a)^{p-1}$ are of opposite monotonicity, we have

$$\begin{aligned}
\int_t^b x^{-p}(x-a)^{p-1}dx &\leq \frac{1}{b-t} \left(\int_t^b x^{-p}dx\right) \left(\int_t^b (x-a)^{p-1}dx\right) \\
&= \frac{1}{b-t} \left[\frac{1}{1-p} x^{1-p}\right]_t^b \left[\frac{1}{p}(x-a)^p\right]_t^b \\
&= \frac{1}{p(p-1)} \frac{1}{b-t} \left(t^{1-p} - b^{1-p}\right) \left((b-a)^p - (t-a)^p\right).
\end{aligned}$$

Therefore, we have

$$\int_a^b \left(\frac{F(x)}{x}\right)^p dx \leq \frac{1}{p(p-1)} \int_a^b f^p(t) \frac{1}{b-t} \left(t^{1-p} - b^{1-p}\right) \left((b-a)^p - (t-a)^p\right) dt.$$

After a minor change in notation, we can write

$$\int_a^b \left(\frac{F(x)}{x}\right)^p dx \leq \frac{1}{p(p-1)} \int_a^b \frac{1}{b-x} \left(x^{1-p} - b^{1-p}\right) \left((b-a)^p - (x-a)^p\right) f^p(x)dx.$$

This ends the proof. □

In comparison with Theorem 2.1, the upper bound exhibits a higher degree of complexity owing to the functional structure of the integrand. Nevertheless, it can be linearized with some additional mathematical effort. Since it is derived through a sequence of highly precise intermediate inequalities, the resulting bound is sharp. Note that, contrary to Theorem 2.1, we can have $a = 0$. In this case, the inequality becomes

$$\int_0^b \left(\frac{F(x)}{x}\right)^p dx \leq \frac{1}{p(p-1)} \int_0^b \frac{1}{b-x} \left(x^{1-p} - b^{1-p}\right)\left(b^p - x^p\right) f^p(x)dx.$$

Furthermore, taking $p = 2$, we obtain the elegant

$$\int_0^b \left(\frac{F(x)}{x}\right)^2 dx \leq \frac{1}{2} \int_0^b \left(\frac{b}{x} - \frac{x}{b}\right) f^2(x)dx.$$

3 Reverse Sulaiman-type Integral Inequalities

This section focuses on the case when $p \in (0, 1)$. It presents reverse Sulaiman-type integral inequalities based on Theorems 2.1 and 2.2.

3.1 A key result

The reverse inequality for the case $p \in (0, 1)$ of [9, Theorem 3.1, part 1] is established in [9, Theorem 3.1, part 2]. We recall this result in the theorem below.

Theorem 3.1. *[9, Theorem 3.1, part 2] Let $p \in (0, 1)$, $a, b \in \mathbb{R}$ with $a < b$, $f : [a, b] \to [0, +\infty)$ be a function and, for any $x \in [a, b]$,*

$$F(x) = \int_a^x f(t)dt,$$

provided that it exists. Then we have

$$\int_a^b \left(\frac{F(x)}{x}\right)^p dx \geq \frac{1}{p}\left(1 - \frac{a}{b}\right)^p \int_a^b f^p(x)dx - \frac{1}{pb^p} \int_a^b (x-a)^p f^p(x)dx,$$

provided that the integrals in the upper bound exist.

The lower bound is a slight modification of the upper bound obtained for the case $p > 1$, in [9, Theorem 3.1, part 1].

3.2 A new integral inequality

Build on Theorem 2.2, a new reverse Sulaiman integral inequality is provided in the theorem below.

Theorem 3.2. *Let $p \in (0, 1)$, $a \in \mathbb{R}\backslash\{0\}$, $b \in \mathbb{R}$ with $a < b$, $f : [a, b] \to [0, +\infty)$ be a function and, for any $x \in [a, b]$,*

$$F(x) = \int_a^x f(t)dt,$$

provided that it exists. Then we have

$$\int_a^b \left(\frac{F(x)}{x}\right)^p dx \geq \frac{1}{ap}\left(1-\frac{a}{b}\right)^p \int_a^b x f^p(x)dx - \frac{1}{ap}\int_a^b x\left(1-\frac{a}{x}\right)^p f^p(x)dx,$$

provided that the integrals in the upper bound exist.

Proof. We follow the first steps to [9, Proof of Theorem 3.1, part 2]. Applying the reverse Hölder integral inequality with $p \in (0,1)$ and the Fubini-Tonelli integral theorem, we obtain

$$\begin{aligned}\int_a^b \left(\frac{F(x)}{x}\right)^p dx &= \int_a^b x^{-p}\left(\int_a^x f(t)dt\right)^p dx \\ &\geq \int_a^b x^{-p}\left(\int_a^x f^p(t)dt\right)\left(\int_a^x dt\right)^{p-1} dx \\ &= \int_a^b x^{-p}(x-a)^{p-1}\left(\int_a^x f^p(t)dt\right) dx \\ &= \int_a^b f^p(t)\left(\int_t^b x^{-p}(x-a)^{p-1}dx\right) dt. \qquad (2)\end{aligned}$$

Let us evaluate the integral with respect to x, proceeding differently to [9, Proof of Theorem 3.1, part 2]. By suitably arranging the integrand, applying a simple lower bound for x with $x \in [t,b]$, and recognizing a standard primitive, we obtain

$$\begin{aligned}\int_t^b x^{-p}(x-a)^{p-1}dx &= \int_t^b x\frac{1}{x^2}\left(1-\frac{a}{x}\right)^{p-1} dx \\ &\geq t\int_t^b \frac{1}{x^2}\left(1-\frac{a}{x}\right)^{p-1} dx \\ &= t\left[\frac{1}{ap}\left(1-\frac{a}{x}\right)^p\right]_t^b \\ &= \frac{1}{ap}t\left(\left(1-\frac{a}{b}\right)^p - \left(1-\frac{a}{t}\right)^p\right).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\int_a^b \left(\frac{F(x)}{x}\right)^p dx &\geq \int_a^b f^p(t)\frac{1}{ap}t\left(\left(1-\frac{a}{b}\right)^p - \left(1-\frac{a}{t}\right)^p\right) dt \\ &= \frac{1}{ap}\left(1-\frac{a}{b}\right)^p \int_a^b t f^p(t)dt - \frac{1}{ap}\int_a^b t\left(1-\frac{a}{t}\right)^p f^p(t)dt.\end{aligned}$$

After a minor change in notation, we can write

$$\int_a^b \left(\frac{F(x)}{x}\right)^p dx \geq \frac{1}{ap}\left(1-\frac{a}{b}\right)^p \int_a^b x f^p(x)dx - \frac{1}{ap}\int_a^b x\left(1-\frac{a}{x}\right)^p f^p(x)dx.$$

This completes the proof. □

In comparison with the upper bound for the case $p > 1$ in Theorem 2.1, we observe that the weight functions of the weighted L^p-norms have changed, and that the factor b no longer appears explicitly.

3.3 Second new integral inequality

The theorem below provides another alternative to [9, Theorem 3.1, part 2].

Theorem 3.3. *Let $p \in (0,1)$, $a, b \in \mathbb{R}$ with $a < b$, $f : [a,b] \to [0,+\infty)$ be a function and, for any $x \in [a,b]$,*

$$F(x) = \int_a^x f(t)dt,$$

provided that it exists. Then we have

$$\int_a^b \left(\frac{F(x)}{x}\right)^p dx \geq \frac{1}{p(p-1)} \int_a^b \frac{1}{b-x}\left(x^{1-p} - b^{1-p}\right)\left((b-a)^p - (x-a)^p\right) f^p(x)dx,$$

provided that the integral in the upper bound exists.

Proof. We can reuse Equation (2), i.e.,

$$\int_a^b \left(\frac{F(x)}{x}\right)^p dx \geq \int_a^b f^p(t)\left(\int_t^b x^{-p}(x-a)^{p-1}dx\right)dt.$$

Let us evaluate the integral with respect to x. Applying the Chebyshev integral inequality by noting that, for $p \in (0,1)$, x^{-p} and $(x-a)^{p-1}$ are of same monotonicity, we have

$$\begin{aligned}\int_t^b x^{-p}(x-a)^{p-1}dx &\geq \frac{1}{b-t}\left(\int_t^b x^{-p}dx\right)\left(\int_t^b (x-a)^{p-1}dx\right)\\ &= \frac{1}{b-t}\left[\frac{1}{1-p}x^{1-p}\right]_t^b\left[\frac{1}{p}(x-a)^p\right]_t^b\\ &= \frac{1}{p(p-1)}\frac{1}{b-t}\left(t^{1-p} - b^{1-p}\right)\left((b-a)^p - (t-a)^p\right).\end{aligned}$$

Therefore, we have

$$\int_a^b \left(\frac{F(x)}{x}\right)^p dx \geq \frac{1}{p(p-1)} \int_a^b f^p(t)\frac{1}{b-t}\left(t^{1-p} - b^{1-p}\right)\left((b-a)^p - (t-a)^p\right)dt.$$

After a minor change in notation, we can write

$$\int_a^b \left(\frac{F(x)}{x}\right)^p dx \geq \frac{1}{p(p-1)} \int_a^b \frac{1}{b-x}\left(x^{1-p} - b^{1-p}\right)\left((b-a)^p - (x-a)^p\right) f^p(x)dx.$$

This ends the proof. □

Note that the obtained lower bound coincides exactly with the upper bound established for the case $p > 1$ in Theorem 2.2. Hence, Theorems 2.2 and 3.3 are mutually consistent and together provide a unified framework for both ranges of the parameter p. In particular, they yield complementary inequalities that remain valid for all $p > 0$. This highlights the symmetry between the direct and reverse cases.

4 Conclusion

In this article, we establish new Sulaiman-type Hardy integral inequalities, together with their reverse forms. Our proposed approach, which is based on a rearrangement of integrals and the application of the Chebyshev inequality, could yield more precise and flexible results than existing formulations. These inequalities further our understanding of the relationships between functions and their primitives in L^p-spaces. Future research will focus on extending these results to weighted settings and multidimensional analogues.

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