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Reich Contraction Mapping Theorem of Integral Type in Metric Spaces with w-Distance

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Abstract

In this paper, we introduce the notion of Reich contraction of integral type in metric spaces with w-distance and prove a fixed point theorem. Some conjectures conclude the paper.

1 Introduction and Preliminaries

Theorem 1.1. *[1] Let T be a mapping from a complete metric space (X, d) into itself satisfying*

$$d(Tx, Ty) \leq c[d(x, Tx) + d(y, Ty)], \quad \forall x, y \in X, \tag{2.1}$$

where $c \in (0, \frac{1}{2})$ is a constant. Then T has a unique fixed point in X.

Theorem 1.2. *[2] Let T be a mapping from a complete metric space (X, d) into itself satisfying*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq c \int_0^{d(x,y)} \varphi(t)dt, \quad \forall x, y \in X, \tag{2.2}$$

where $c \in (0, 1)$ is a constant and $\varphi : [0, \infty) \mapsto [0, \infty)$ is Lebesgue integrable, summable on each compact subset of $[0, \infty)$ and $\int_0^\epsilon \varphi(t)dt > 0$ for each $\epsilon > 0$. Then T has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} T^n x = a$ for each $x \in X$.

Notation 1.3. *Throughout this paper, we assume the following*

(a) $\mathbb{R} = (-\infty, +\infty)$.

(b) $\mathbb{R}^+ = [0, +\infty)$.

(c) $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, *where $\mathbb{N}$ denotes the set of all positive integers.*

(d) *Φ will denote the class of all functions $\varphi : [0, \infty) \mapsto [0, \infty)$ which are Lebesgue integrable, summable on each compact subset of $\mathbb{R}^+$ and $\int_0^\epsilon \varphi(t)dt > 0$ for each $\epsilon > 0$.*

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Definition 1.4. *[3] Let (X, d) be a metric space. A function $p : X \times X \mapsto \mathbb{R}^+$ is called a w-distance in X if it satisfies the following*

(w_1) $p(x, z) \leq p(x, y) + p(y, z)$, $\forall x, y, z \in X$,

(w_2) *for each $x \in X$, a mapping $p(x, \cdot) : X \mapsto \mathbb{R}^+$ is lower semi-continuous, that is, if $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in X with $\lim_{n\to\infty} y_n = y \in X$, then, $p(x, y) \leq \liminf_{n\to\infty} p(x, y_n)$,*

(w_3) *for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.*

Definition 1.5. *[4] A self-mapping T in a metric space (X, d) is called orbitally continuous at $u \in X$ if $\lim_{n\to\infty} T^n x = u$, $x \in X$, implies that $\lim_{n\to\infty} TT^n x = Tu$. The mapping T is orbitally continuous in X if T is orbitally continuous at each $u \in X$.*

Lemma 1.6. *[3] Let X be a metric space with metric d and let p be a w-distance in X. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences in X, let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}^+$ converging to 0, and let $x, y, z \in X$, then the following hold:*

(a) *If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$.*

(b) *If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}_{n\in\mathbb{N}}$ converges to z.*

(c) *If $p(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $n > m$, then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.*

(d) *If $p(x, x_n) \leq \alpha_n$, for any $n \in \mathbb{N}$, then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.*

Lemma 1.7. *[5] Let $\varphi \in \Phi$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence with $\lim_{n\to\infty} r_n = a$. Then*

$$\lim_{n\to\infty} \int_0^{r_n} \varphi(t)dt = \int_0^a \varphi(t)dt.$$

Lemma 1.8. *[5] Let $\varphi \in \Phi$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence. Then $\lim_{n\to\infty} \int_0^{r_n} \varphi(t)dt = 0$, if and only if $\lim_{n\to\infty} r_n = 0$.*

Theorem 1.9. *[6] Let (X, d) be a complete metric space and let $f : X \mapsto X$ be a Reich type single-valued (a, b, c)-contraction, that is, there exists nonnegative numbers a, b, c with $a + b + c < 1$ such that*

$$d(f(x), f(y)) \leq ad(x, y) + bd(x, f(x)) + cd(y, f(y))$$

for each $x, y \in X$. Then T has a unique fixed point.

2 Main Results

Theorem 2.1. *Let (X, d) be a complete metric space and let p be a w-distance in X. Assume that $T : X \mapsto X$ satisfies*

$$\int_0^{p(Tx,Ty)} \varphi(t)dt \leq c \int_0^{\frac{1}{3}[p(x,y)+p(x,Tx)+p(y,Ty)]} \varphi(t)dt, \ \forall x, y \in X, \tag{3.1}$$

where $c \in [0, 1)$ *is a constant and* $\varphi \in \Phi$*. Then* T *has a unique fixed point* $u \in X$*,* $p(u, u) = 0$*, and* $\lim_{n\to\infty} p(T^n x_0, u) = 0$ *for each* $x_0 \in X$*.*

Proof. Pick an arbitrary point $x_0 \in X$ and define $x_n = T^n x_0$ for each $n \in \mathbb{N}_0$. Now we consider the following two cases:

Case 1. Assume that $x_{n_0} = x_{n_0-1}$ for some $n_0 \in \mathbb{N}$. It is easy to see that x_{n_0-1} is a fixed point of T, $x_n = x_{n_0-1}$ for each $n \geq n_0$, and $\lim_{n\to\infty} T^n x_0 = x_{n_0-1}$. Suppose that $p(x_{n_0-1}, x_{n_0-1}) > 0$. It follows from (3.1) and $\varphi \in \Phi$ that

$$\begin{aligned}
\int_0^{p(x_{n_0-1},x_{n_0-1})} \varphi(t)dt &= \int_0^{p(Tx_{n_0-1},Tx_{n_0-1})} \varphi(t)dt \\
&\leq c \int_0^{\frac{1}{3}[p(x_{n_0-1},x_{n_0-1})+p(x_{n_0-1},Tx_{n_0-1})+p(x_{n_0-1},Tx_{n_0-1})]} \varphi(t)dt \\
&\leq c \int_0^{\frac{1}{3}[p(x_{n_0-1},x_{n_0-1})+p(x_{n_0-1},x_{n_0-1})+p(x_{n_0-1},x_{n_0-1})]} \varphi(t)dt \\
&< \int_0^{p(x_{n_0-1},x_{n_0-1})} \varphi(t)dt
\end{aligned}$$

which is a contradiction. Hence $p(x_{n_0-1}, x_{n_0-1}) = 0$ which yields that

$$\lim_{n\to\infty} p(x_n, x_{n_0-1}) = p(x_{n_0-1}, x_{n_0-1}) = 0.$$

Case 2. Assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. Suppose that

$$p(x_{n_0-1}, x_{n_0}) = 0 \ for \ some \ n_0 \in \mathbb{N}. \tag{3.2}$$

From (3.1), (3.2), and $\varphi \in \Phi$, we deduce the following

$$\begin{aligned}
0 &\leq \int_0^{p(x_{n_0},x_{n_0+1})} \varphi(t)dt \\
&= \int_0^{p(Tx_{n_0-1},Tx_{n_0})} \varphi(t)dt \\
&\leq c \int_0^{\frac{1}{3}[p(x_{n_0-1},x_{n_0})+p(x_{n_0-1},Tx_{n_0-1})+p(x_{n_0},Tx_{n_0})]} \varphi(t)dt \\
&= c \int_0^{\frac{1}{3}[p(x_{n_0-1},x_{n_0})+p(x_{n_0-1},x_{n_0})+p(x_{n_0},x_{n_0+1})]} \varphi(t)dt \\
&= 0
\end{aligned}$$

which means that

$$\int_0^{p(x_{n_0},x_{n_0+1})} \varphi(t)dt = 0$$

which together with $\varphi \in \Phi$ gives that

$$p(x_{n_0}, x_{n_0+1}) = 0. \tag{3.3}$$

Note that (3.2), (3.3) and (w_1) gives that

$$0 \leq p(x_{n_0-1}, x_{n_0+1}) \leq p(x_{n_0-1}, x_{n_0}) + p(x_{n_0}, x_{n_0+1}) = 0$$

that is

$$p(x_{n_0-1}, x_{n_0+1}) = 0. \tag{3.4}$$

It follows from (3.2),(3.4), and Lemma 1.6, that $x_{n_0} = x_{n_0+1}$, which is absurd and hence

$$p(x_{n-1}, x_n) > 0, \ \forall n \in \mathbb{N}. \tag{3.5}$$

From (3.1), (3.5) and $\varphi \in \Phi$, we deduce the following

$$\begin{aligned}
\int_0^{p(x_n,x_{n+1})} \varphi(t)dt &= \int_0^{p(Tx_{n-1},Tx_n)} \varphi(t)dt \\
&\leq c \int_0^{\frac{1}{3}[p(x_{n-1},x_n)+p(x_{n-1},Tx_{n-1})+p(x_n,Tx_n)]} \varphi(t)dt \\
&= c \int_0^{\frac{1}{3}[p(x_{n-1},x_n)+p(x_{n-1},x_n)+p(x_n,x_{n+1})]} \varphi(t)dt \\
&\leq c \int_0^{\frac{1}{3}[3p(x_{n-1},x_n)]} \varphi(t)dt \\
&< \int_0^{p(x_{n-1},x_n)} \varphi(t)dt, \ \forall n \in \mathbb{N}
\end{aligned}$$

which together with (3.5) implies that

$$0 < p(x_n, x_{n+1}) < p(x_{n-1}, x_n), \qquad \forall n \in \mathbb{N}. \tag{3.6}$$

Note that (3.6) yields that the sequence $\{p(x_n, x_{n+1})\}_{n \in \mathbb{N}_0}$ is positive and strictly decreasing. Thus there exists a constant $v \geq 0$ with

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = v. \tag{3.7}$$

Suppose that $v > 0$. By means of (3.1), (3.7), $\varphi \in \Phi$ and Lemma 1.7, we conclude that

$$\begin{aligned}
\int_0^v \varphi(t)dt &= \lim_{n\to\infty} \int_0^{p(x_n,x_{n+1})} \varphi(t)dt \\
&= \lim_{n\to\infty} \int_0^{p(Tx_{n-1},Tx_n)} \varphi(t)dt \\
&\leq \lim_{n\to\infty} c \int_0^{\frac{1}{3}[p(x_{n-1},x_n)+p(x_{n-1},Tx_{n-1})+p(x_n,Tx_n)]} \varphi(t)dt \\
&= \lim_{n\to\infty} c \int_0^{\frac{1}{3}[p(x_{n-1},x_n)+p(x_{n-1},x_n)+p(x_n,x_{n+1})]} \varphi(t)dt \\
&\leq \lim_{n\to\infty} c \int_0^{\frac{1}{3}[3p(x_{n-1},x_n)]} \varphi(t)dt \\
&< \int_0^v \varphi(t)dt
\end{aligned}$$

which is impossible and hence $v = 0$, that is,

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{3.8}$$

Similarly, we get that

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = 0. \tag{3.9}$$

Now we show that

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{3.10}$$

Otherwise there is a constant $\epsilon > 0$ such that for each positive integer k, there are positive integers $m(k)$ and $n(k)$ with $m(k) > n(k) > k$ such that

$$p(x_{n(k)}, x_{m(k)}) > \epsilon.$$

For each positive integer k, let $m(k)$ denote the least integer exceeding $n(k)$ and satisfying the above inequality. It follows that

$$p(x_{n(k)}, x_{m(k)}) > \epsilon \textit{ and } p(x_{n(k)}, x_{m(k)-1}) \leq \epsilon, \quad \forall k \in \mathbb{N}. \tag{3.11}$$

Note that

$$\begin{aligned} \epsilon &< p(x_{n(k)}, x_{m(k)}) \\ &\leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{m(k)}) \\ &\leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{m(k)}) \\ &\leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{n(k)}) + \epsilon + p(x_{m(k)-1}, x_{m(k)}). \end{aligned} \tag{3.12}$$

Letting $k \to \infty$ in (3.12) and using (3.8), (3.9) and (3.11), we get

$$\lim_{k\to\infty} p(x_{n(k)}, x_{m(k)}) = \lim_{k\to\infty} p(x_{n(k)-1}, x_{m(k)-1}) = \epsilon. \tag{3.13}$$

By virtue of (3.1), (3.13), $\varphi \in \Phi$, and Lemma 1.7, we deduce that

$$\begin{aligned} \int_0^{\epsilon} \varphi(t)dt &= \lim_{k\to\infty} \int_0^{p(x_{n(k)}, x_{m(k)})} \varphi(t)dt \\ &= \lim_{k\to\infty} \int_0^{p(Tx_{n(k)-1}, Tx_{m(k)-1})} \varphi(t)dt \\ &\leq \lim_{k\to\infty} c \int_0^{\frac{1}{3}[p(x_{n(k)-1}, x_{m(k)-1}) + p(x_{n(k)-1}, Tx_{n(k)-1}) + p(x_{m(k)-1}, Tx_{m(k)-1})]} \varphi(t)dt \\ &= \lim_{k\to\infty} c \int_0^{\frac{1}{3}[p(x_{n(k)-1}, x_{m(k)-1}) + p(x_{n(k)-1}, x_{n(k)}) + p(x_{m(k)-1}, x_{m(k)})]} \varphi(t)dt \\ &\leq \lim_{k\to\infty} c \int_0^{\frac{1}{3}[3p(x_{n(k)-1}, x_{m(k)-1})]} \varphi(t)dt \\ &< \int_0^{\epsilon} \varphi(t)dt \end{aligned}$$

which is a contradiction. Thus, (3.10) holds. Let $\epsilon > 0$ and δ denote the number in (w_3). It follows from (3.10) that there exists $N \in \mathbb{N}$ satisfying

$$p(x_N, x_n) < \delta \ and \ p(x_N, x_m) < \delta, \ \forall n, m \in \mathbb{N}$$

which together with (w_3) yields that

$$d(x_n, x_m) < \epsilon, \ \forall n, m \geq N$$

that is, $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. Since (X, d) is a complete metric space, it follows that there exists a point $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Observe that (3.10) guarantees that for each $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ satisfying

$$0 \leq p(x_n, x_m) < \epsilon, \ \forall n, m \geq N_\epsilon$$

which together with (w_2) and $\lim_{n\to\infty} x_n = u$ yields that

$$0 \leq p(x_n, u) \leq \lim_{m\to\infty} \inf p(x_n, x_m) \leq \epsilon, \ \forall n \geq N_\epsilon$$

which gives that

$$\lim_{n\to\infty} p(x_n, u) = 0. \tag{3.14}$$

Making use of (3.1),(3.14), $\varphi \in \Phi$, and Lemma 1.7, we obtain that

$$\begin{aligned}
0 &\leq \int_0^{p(Tx_n,Tu)} \varphi(t)dt \\
&\leq c \int_0^{\frac{1}{3}[p(x_n,u)+p(x_n,Tx_n)+p(u,Tu)]} \varphi(t)dt \\
&= c \int_0^{\frac{1}{3}[p(x_n,u)+p(x_n,x_{n+1})+p(u,Tu)]} \varphi(t)dt \\
&\leq c \int_0^{\frac{1}{3}[3p(x_n,u)]} \varphi(t)dt \to 0 \ as \ n \to \infty
\end{aligned}$$

that is

$$\lim_{n\to\infty} \int_0^{p(Tx_n,Tu)} \varphi(t) = 0$$

which together with Lemma 1.8 means that

$$\lim_{n\to\infty} p(x_{n+1}, Tu) = \lim_{n\to\infty} p(Tx_n, Tu) = 0$$

which together with (w_1) and (3.8) yields that

$$0 \leq p(x_n, Tu) \leq p(x_n, x_{n+1}) + p(x_{n+1}, Tu) \to 0 \ as \ n \to \infty$$

that is

$$\lim_{n\to\infty} p(x_n, Tu) = 0. \tag{3.15}$$

Combining (3.14) and (3.15) and using Lemma 1.6, we deduce that $u = Tu$. Next we show that $p(u, u) = 0$. Suppose that $p(u, u) > 0$. It follows from (3.1) and $\varphi \in \Phi$ that

$$\begin{aligned} 0 &< \int_0^{p(u,u)} \varphi(t)dt \\ &= \int_0^{p(Tu,Tu)} \varphi(t)dt \\ &\leq c \int_0^{\frac{1}{3}[p(u,u)+p(u,Tu)+p(u,Tu)]} \varphi(t)dt \\ &= c \int_0^{\frac{1}{3}[p(u,u)+p(u,u)+p(u,u)]} \varphi(t)dt \\ &< \int_0^{p(u,u)} \varphi(t)dt \end{aligned} \tag{3.16}$$

which is impossible. That is, $p(u, u) = 0$. Finally, we show that T possesses a unique fixed point in X. Suppose that α and β are two fixed points of T in X. Similar to the proof of (3.16), we infer that $p(\alpha, \alpha) = p(\beta, \beta) = 0$. Suppose that $p(\beta, \alpha) > 0$. It follows from (3.1) and $\varphi \in \Phi$ that

$$\begin{aligned} 0 &< \int_0^{p(\beta,\alpha)} \varphi(t)dt \\ &= \int_0^{p(T\beta,T\alpha)} \varphi(t)dt \\ &\leq c \int_0^{\frac{1}{3}[p(\beta,\alpha)+p(\beta,T\beta)+p(\alpha,T\alpha)]} \varphi(t)dt \\ &= c \int_0^{\frac{1}{3}[p(\beta,\alpha)+p(\beta,\beta)+p(\alpha,\alpha)]} \varphi(t)dt \\ &= c \int_0^{\frac{1}{3}p(\beta,\alpha)} \varphi(t)dt \\ &< \int_0^{p(\beta,\alpha)} \varphi(t)dt \end{aligned}$$

which is absurd. Consequently, $p(\beta, \alpha) = 0$, which together with $p(\beta, \beta) = 0$ and Lemma 1.6 implies that $\beta = \alpha$. This completes the proof. □

3 Conjectures

The open problems are to prove or disprove the following

Conjecture 3.1. *Let (X, d) be a complete metric space and let p be a w-distance in X. Assume that $T : X \mapsto X$ satisfies*

$$\int_0^{p(Tx,Ty)} \varphi(t)dt \leq a \int_0^{p(Tx,x)} \varphi(t)dt + b \int_0^{p(Ty,y)} \varphi(t)dt + c \int_0^{p(x,y)} \varphi(t)dt, \ \forall x, y \in X,$$

where $\varphi \in \Phi$ *and*

$$a, b, \text{ and } c \text{ are nonnegative and } a + b + c < 1.$$

Then T *has a unique fixed point* $u \in X$, $p(u, u) = 0$, *and* $\lim_{n\to\infty} p(T^n x_0, u) = 0$ *for each* $x_0 \in X$.

Conjecture 3.2. *Let* (X, d) *be a complete metric space and let* p *be a* w*-distance in* X. *Assume that* $T : X \mapsto X$ *is an orbitally continuous mapping satisfying*

$$\int_0^{p(Tx,Ty)} \varphi(t)dt \le a \int_0^{p(Tx,x)} \varphi(t)dt + b \int_0^{p(Ty,y)} \varphi(t)dt + c \int_0^{p(x,y)} \varphi(t)dt, \ \forall x, y \in X,$$

where $\varphi \in \Phi$ *and*

$$a, b, \text{ and } c \text{ are nonnegative and } a + b + c < 1.$$

Then T *has a unique fixed point* $u \in X$, $p(u, u) = 0$, *and* $\lim_{n\to\infty} p(T^n x_0, u) = 0$ *for each* $x_0 \in X$.

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