



Some Results on Subclasses of Multivalent and Meromorphic Functions Defined by Multiplier Transformations

Salah Mahdi Ali

College of Nursing, University of Al-Qadisiyah, Al-Diwaniyah, Iraq
e-mail: salah.mahdi@qu.edu.iq

Abstract

The objective of the present work is to investigate a specific family of uniformly meromorphic of multivalent functions defined in \mathbb{U}^* associated with multiplier transformations. We get some results for this class, like, coefficient estimates, distortion theorem, closure theorem and radii of starlike.

1. Introduction

Assume that \mathcal{U}_e be the family of functions that take the following type:

$$\mathbb{J}(\eta) = q_0 \eta^{-e} + \sum_{n=1-e}^{\infty} q_n \eta^n, \quad q_0 > 0, \quad q_n \geq 0, \quad e \in N = \{1, 2, \dots\}, \quad (1)$$

that are multivalent meromorphic in $\mathbb{U}^* = \{\eta \in \mathbb{C}, 0 < |\eta| < 1\} = U \setminus \{0\}$.

For $n \in \mathbb{Z}$, $\ell > 0$, $\mathfrak{X} \geq 0$ and $\mathbb{J} \in \mathcal{U}_e$, the multiplier transformations $\mathfrak{L}_e^m(\ell, \mathfrak{X}) : \mathcal{U}_e \rightarrow \mathcal{U}_e$ is defined by (see [4])

$$\mathfrak{L}_e^m(\ell, \mathfrak{X})\mathbb{J}(\eta) = q_0 \eta^{-e} + \sum_{n=1-e}^{\infty} \left(\frac{(\ell + \mathfrak{X}(n+e))}{\ell} \right)^m q_n \eta^n. \quad (2)$$

A function $\mathbb{J} \in \mathcal{U}_e$ is named multivalent meromorphic starlike function of order α whenever it satisfies the condition

$$-Re \left\{ \frac{\eta \mathbb{J}'(\eta)}{\mathbb{J}(\eta)} \right\} > \alpha, \quad (\eta \in \mathbb{U}^*; \quad 0 \leq \alpha < e), \quad (3)$$

and is named multivalent meromorphic convex functions of order α whenever it satisfies the condition

$$-Re \left\{ \frac{\eta \mathbb{J}''(\eta)}{\mathbb{J}'(\eta)} + 1 \right\} > \alpha, \quad (\eta \in \mathbb{U}^*; \quad 0 \leq \alpha < e). \quad (4)$$

We define $G(x, \ell, \mathfrak{X}, m, e)$ as the family of functions in \mathcal{U}_e that fulfill the condition stated below:

$$Re \left\{ - \left(\frac{\eta (\mathfrak{L}_e^m(\ell, \mathfrak{X})\mathbb{J}(\eta))''}{(1+e)(\mathfrak{L}_e^m(\ell, \mathfrak{X})\mathbb{J}(\eta))'} + x \right) \right\} \geq \left| \frac{\eta (\mathfrak{L}_e^m(\ell, \mathfrak{X})\mathbb{J}(\eta))''}{(1+e)(\mathfrak{L}_e^m(\ell, \mathfrak{X})\mathbb{J}(\eta))'} + x + 2 \right|, \quad (5)$$

where $0 < x < 1, n \in \mathbb{Z}, \ell > 0, \mathfrak{X} \geq 0$.

Received: December 6, 2025; Accepted: December 24, 2025; Published: January 27, 2026

2020 Mathematics Subject Classification: 30C45

Keywords and phrases: multivalent functions, meromorphic functions, distortion, convex, fixed point.

*Corresponding author

Copyright © 2026 the Author

A number of studies have lately focused on analyzing subclasses of meromorphic functions, as documented in references [1,2,3,5,6,7,8,9,10].

2. Coefficient Estimates

Theorem 1. Let $\mathbb{J} \in \mathcal{U}_e$. Then $\mathbb{J} \in G(x, \ell, \mathfrak{X}, m, e)$, iff

$$\sum_{n=1-e}^{\infty} n[(e+1)(x+1) + (n-1)] \left(\frac{(\ell + \mathfrak{X}(n+e))}{\ell} \right)^m q_n \leq q_0 e(e+1)x. \quad (6)$$

Sharpness of the result is attained for the function \mathbb{J} described as

$$\mathbb{J}(\mathfrak{y}) = q_0 \mathfrak{y}^{-e} + \frac{q_0 e(e+1)x}{n[(e+1)(x+1) + (n-1)] \left(\frac{(\ell + \mathfrak{X}(n+e))}{\ell} \right)^m \mathfrak{y}^{1-e}}, \quad n \geq 1-e. \quad (7)$$

Proof. Suppose that $\mathbb{J} \in G(x, \ell, \mathfrak{X}, m, e)$. Then

$$Re \left\{ - \left(\frac{\mathfrak{y} (\mathfrak{L}_e^m(\ell, \mathfrak{X}) \mathbb{J}(\mathfrak{y}))''}{(e+1)(\mathfrak{L}_e^m(\ell, \mathfrak{X}) \mathbb{J}(\mathfrak{y}))'} + x \right) \right\} \geq \left| \frac{\mathfrak{y} (\mathfrak{L}_e^m(\ell, \mathfrak{X}) \mathbb{J}(\mathfrak{y}))''}{(e+1)(\mathfrak{L}_e^m(\ell, \mathfrak{X}) \mathbb{J}(\mathfrak{y}))'} + x + 2 \right|.$$

But

$$\begin{aligned} Re \left\{ \frac{\mathfrak{y} (\mathfrak{L}_e^m(\ell, \mathfrak{X}) \mathbb{J}(\mathfrak{y}))''}{(e+1)(\mathfrak{L}_e^m(\ell, \mathfrak{X}) \mathbb{J}(\mathfrak{y}))'} + x \right\} &\leq \left| \frac{\mathfrak{y} (\mathfrak{L}_e^m(\ell, \mathfrak{X}) \mathbb{J}(\mathfrak{y}))''}{(e+1)(\mathfrak{L}_e^m(\ell, \mathfrak{X}) \mathbb{J}(\mathfrak{y}))'} + x + 2 \right| \\ &\leq Re \left\{ - \left(\frac{\mathfrak{y} (\mathfrak{L}_e^m(\ell, \mathfrak{X}) \mathbb{J}(\mathfrak{y}))''}{(1+e)(\mathfrak{L}_e^m(\ell, \mathfrak{X}) \mathbb{J}(\mathfrak{y}))'} + x \right) \right\}, \end{aligned}$$

that is

$$Re \left\{ \frac{\mathfrak{y} (\mathfrak{L}_e^m(\ell, \mathfrak{X}) \mathbb{J}(\mathfrak{y}))''}{(1+e)(\mathfrak{L}_e^m(\ell, \mathfrak{X}) \mathbb{J}(\mathfrak{y}))'} + 2(x+1) \right\} \leq 0,$$

by (2) we get

$$\begin{aligned} \sum_{n=1-e}^{\infty} n[(e+1)(x+1) + (n-1)] \left(\frac{(\ell + \mathfrak{X}(n+e))}{\ell} \right)^m q_n &\leq q_0 e(e+1)x \\ Re \left\{ \frac{-2e(e+1)xq_0 \mathfrak{y}^{-e-1} + \sum_{n=1-e}^{\infty} 2n[(n-1) + (e+1)(x+1)] \left(\frac{(\ell + \mathfrak{X}(n+e))}{\ell} \right)^m q_n \mathfrak{y}^{n-1}}{-e q_0 (e+1) \mathfrak{y}^{-e-1} + \sum_{n=1-e}^{\infty} n(e+1) \left(\frac{(\ell + \mathfrak{X}(n+e))}{\ell} \right)^m q_n \mathfrak{y}^{n-1}} \right\} &\leq 0. \end{aligned}$$

Letting \mathfrak{y} to take real values as $\mathfrak{y} \rightarrow 1^-$, we get

$$\frac{-2e(e+1)xq_0 + \sum_{n=1-e}^{\infty} 2n[(n-1) + (e+1)(x+1)] \left(\frac{(\ell + \mathfrak{X}(n+e))}{\ell} \right)^m q_n}{-e q_0 (e+1) + \sum_{n=1-e}^{\infty} n(e+1) \left(\frac{(\ell + \mathfrak{X}(n+e))}{\ell} \right)^m q_n} \leq 0,$$

that is

$$-2\epsilon(\epsilon+1)xq_0 + \sum_{n=1-\epsilon}^{\infty} 2n[(n-1) + (\epsilon+1)(x+1)] \left(\frac{(\ell+\mathfrak{X}(n+\epsilon))}{\ell} \right)^m q_n \leq 0,$$

which is equivalent to (6).

On the other hand, if we assume that inequality (6) holds, we deduce that

$$\begin{aligned} \left| \frac{\eta (\mathfrak{L}_\epsilon^m(\ell, \mathfrak{X})\mathbb{J}(\eta))''}{(\epsilon+1)(\mathfrak{L}_\epsilon^m(\ell, \mathfrak{X})\mathbb{J}(\eta))} + x + 2 \right| &= \left| \frac{-2\epsilon(\epsilon+1)(x+2)q_0 \eta^{-\epsilon-1}}{+ \sum_{n=1-\epsilon}^{\infty} n \left[2(n-1) + (x+2)(\epsilon+1) \left(\frac{(\ell+\mathfrak{X}(n+\epsilon))}{\ell} \right)^m \right] q_n \eta^{n-1}} \right. \\ &\quad \left. - \epsilon q_0 (\epsilon+1) \eta^{-\epsilon-1} + \sum_{n=1-\epsilon}^{\infty} n(\epsilon+1) \left(\frac{(\ell+\mathfrak{X}(n+\epsilon))}{\ell} \right)^m q_n \eta^{n-1} \right| \\ &\leq \frac{-2\epsilon(\epsilon+1)(x+2)q_0}{- \sum_{n=1-\epsilon}^{\infty} n \left[2(n-1) + (x+2)(\epsilon+1) \left(\frac{(\ell+\mathfrak{X}(n+\epsilon))}{\ell} \right)^m \right] q_n \eta^{n+\epsilon}} \\ &\quad - \epsilon q_0 (\epsilon+1) + \sum_{n=1-\epsilon}^{\infty} n(\epsilon+1) \left(\frac{(\ell+\mathfrak{X}(n+\epsilon))}{\ell} \right)^m q_n \eta^{n+\epsilon}. \end{aligned}$$

Taking the limit as $\eta \rightarrow 1^-$ along the real axis yields the required inequality (6). Therefore, the proof of Theorem 1 is established.

Corollary 1. If $\mathbb{J} \in G(x, \ell, \mathfrak{X}, m, \epsilon)$, then

$$q_n \leq \frac{\epsilon(\epsilon+1)xq_0}{n[(n-1) + (\epsilon+1)(x+1)] \left(\frac{(\ell+\mathfrak{X}(n+\epsilon))}{\ell} \right)^m} \quad (n, \epsilon \in N = \{1, 2, \dots\}). \quad (8)$$

Theorem 2. Let $\mathbb{J} \in G(x, \ell, \mathfrak{X}, m, \epsilon)$. Then

$$\frac{q_0}{r^\epsilon} - \frac{\epsilon(\epsilon+1)x}{(1-\epsilon)[-e + (\epsilon+1)(x+1)] \left(\frac{\ell+\mathfrak{X}}{\ell} \right)^m} r^{1-\epsilon} \leq |\mathbb{J}(\eta)| \leq \frac{q_0}{r^\epsilon} + \frac{\epsilon(\epsilon+1)x}{(1-\epsilon)[-e + (\epsilon+1)(x+1)] \left(\frac{\ell+\mathfrak{X}}{\ell} \right)^m} r^{1-\epsilon}.$$

Proof. Since $\mathbb{J} \in G(x, \ell, \mathfrak{X}, m, \epsilon)$, we get

$$\begin{aligned} (1-\epsilon)[-e + (\epsilon+1)(x+1)] \left(\frac{\ell+\mathfrak{X}}{\ell} \right)^m \sum_{n=1-\epsilon}^{\infty} q_n &\leq \sum_{n=1-\epsilon}^{\infty} n[(n-1) + (\epsilon+1)(x+1)] \left(\frac{(\ell+\mathfrak{X}(n+\epsilon))}{\ell} \right)^m q_n \leq \epsilon(\epsilon+1)x. \end{aligned}$$

We have

$$|\mathbb{J}(\eta)| = \frac{q_0}{|\eta|^\epsilon} + \sum_{n=1-\epsilon}^{\infty} q_n |\eta|^n \leq \frac{q_0}{|\eta|^\epsilon} + |\eta|^{1-\epsilon} \sum_{n=1-\epsilon}^{\infty} q_n \leq \frac{q_0}{r^\epsilon} + \frac{\epsilon(\epsilon+1)x}{(1-\epsilon)[-e + (\epsilon+1)(x+1)] \left(\frac{\ell+\mathfrak{X}}{\ell} \right)^m} r^{1-\epsilon}.$$

Also, similarly, we have

$$|\mathbb{J}(\eta)| \geq \frac{q_0}{r^\epsilon} - \frac{\epsilon(\epsilon+1)x}{(1-\epsilon)[-e + (\epsilon+1)(x+1)] \left(\frac{\ell+\mathfrak{X}}{\ell} \right)^m} r^{1-\epsilon}.$$

Sharpness of the result is attained for the function \mathbb{J} described as

$$\mathbb{J}(\mathfrak{y}) = q_0 \mathfrak{y}^{-\mathfrak{e}} + \frac{q_0 \mathfrak{e}(\mathfrak{e}+1)x}{(1-\mathfrak{e})(-\mathfrak{e}+(\mathfrak{e}+1)(x+1)) \left(\frac{\ell+\mathfrak{X}}{\ell}\right)^m \mathfrak{y}^{1-\mathfrak{e}}}, \quad \mathfrak{e} \in N.$$

Now, let us consider the functions $\mathbb{J}_j(\mathfrak{y})$, defined for $j = 1, 2, \dots, q$, by

$$\mathbb{J}_j(\mathfrak{y}) = q_{0,j} \mathfrak{y}^{-\mathfrak{e}} + \sum_{n=1-\mathfrak{e}}^{\infty} q_{n,j} \mathfrak{y}^n \quad (q_{0,j} > 0, q_{n,j} \geq 0, \mathfrak{e} \in N = \{1, 2, \dots\}). \quad (9)$$

Theorem 3. Assume that the functions $\mathbb{J}_j(\mathfrak{y})$ defined by (9) lies in the family $G(x, \ell, \mathfrak{X}, m, \mathfrak{e})$ for $j = 1, 2, \dots, q$. Then the function $h(\mathfrak{y})$ defined as

$$h(\mathfrak{y}) = q_{0,j} \mathfrak{y}^{-\mathfrak{e}} + \sum_{n=1-\mathfrak{e}}^{\infty} c_n \mathfrak{y}^n$$

is also in the family $G(x, \ell, \mathfrak{X}, m, \mathfrak{e})$, where

$$c_n = \frac{1}{q} \sum_{j=1}^q q_{n,j}.$$

Proof. Since $\mathbb{J}_j(\mathfrak{y}) \in G(x, \ell, \mathfrak{X}, m, \mathfrak{e})$ ($j = 1, 2, \dots, q$) and by Theorem 1, we obtain

$$\sum_{n=1-\mathfrak{e}}^{\infty} n[(n-1) + (\mathfrak{e}+1)(x+1)] \left(\frac{(\ell+\mathfrak{X}(n+\mathfrak{e}))}{\ell}\right)^m q_{n,j} \leq \mathfrak{e}(\mathfrak{e}+1)x.$$

Hence

$$\begin{aligned} & \sum_{n=1-\mathfrak{e}}^{\infty} n[(n-1) + (\mathfrak{e}+1)(x+1)] \left(\frac{(\ell+\mathfrak{X}(n+\mathfrak{e}))}{\ell}\right)^m c_n \\ &= \sum_{n=1-\mathfrak{e}}^{\infty} n[(n-1) + (\mathfrak{e}+1)(x+1)] \left(\frac{(\ell+\mathfrak{X}(n+\mathfrak{e}))}{\ell}\right)^m \left(\frac{1}{q} \sum_{j=1}^q q_{n,j}\right) \\ &= \frac{1}{q} \sum_{j=1}^q \sum_{n=1-\mathfrak{e}}^{\infty} n[(n-1) + (\mathfrak{e}+1)(x+1)] \left(\frac{(\ell+\mathfrak{X}(n+\mathfrak{e}))}{\ell}\right)^m q_{n,j} \leq \mathfrak{e}(\mathfrak{e}+1)x. \end{aligned}$$

This shows that $h(\mathfrak{y}) \in G(x, \ell, \mathfrak{X}, m, \mathfrak{e})$.

Theorem 4. Let \mathbb{J} be an element of $G(x, \ell, \mathfrak{X}, m, \mathfrak{e})$. Then \mathbb{J} is multivalent meromorphic starlike of order φ ($0 \leq \varphi < 1$) in the region $|\mathfrak{y}| < R_1$ such that

$$R_1 = \inf_n \left\{ \frac{n(\mathfrak{e}-\varphi)[(\mathfrak{e}+1)(x+1) + (n-1)] \left(\frac{(\ell+\mathfrak{X}(n+\mathfrak{e}))}{\ell}\right)^m}{\mathfrak{e}(\mathfrak{e}+1)x(n+\varphi)} \right\}^{\frac{1}{n-\mathfrak{e}}}.$$

The obtained result is exact for the function \mathbb{J} defined in (7).

Proof. It is enough to demonstrate that

$$\left| \frac{\eta J'(\eta)}{J(\eta)} + e \right| \leq e - \varphi \quad \text{for } |\eta| < R_1. \quad (10)$$

But

$$\left| \frac{\eta J'(\eta)}{J(\eta)} + e \right| = \left| \frac{\eta J'(\eta) + e J(\eta)}{J(\eta)} \right| \leq \frac{\sum_{n=1-e}^{\infty} (n+e) q_n |\eta|^{n-e}}{1 + \sum_{n=1-e}^{\infty} q_n |\eta|^{n-e}}.$$

Thus (10) will be satisfied if

$$\frac{\sum_{n=1-e}^{\infty} (n+e) q_n |\eta|^{n-e}}{q_0 + \sum_{n=1-e}^{\infty} q_n |\eta|^{n-e}} \leq e - \varphi,$$

or if

$$\sum_{n=1-e}^{\infty} \frac{n + \varphi}{q_0(e - \varphi)} q_n |\eta|^{n-e} \leq 1. \quad (11)$$

Since $J \in G(x, \ell, \mathfrak{X}, m, e)$, we have

$$\sum_{n=1-e}^{\infty} \frac{n[(e+1)(x+1) + (n-1)] \left(\frac{(\ell+\mathfrak{X}(n+e))}{\ell} \right)^m}{q_0 e (e+1) x} q_n \leq 1.$$

Hence (11) will be true if

$$\frac{n + \varphi}{q_0(e - \varphi)} |\eta|^{n-e} \leq \frac{n[(e+1)(x+1) + (n-1)] \left(\frac{(\ell+\mathfrak{X}(n+e))}{\ell} \right)^m}{q_0 e (e+1) x},$$

or equivalently

$$|\eta| \leq \left\{ \frac{n(e - \varphi)[(e+1)(x+1) + (n-1)] \left(\frac{(\ell+\mathfrak{X}(n+e))}{\ell} \right)^m}{e(e+1)x(n + \varphi)} \right\}^{\frac{1}{n-e}} \quad (n \geq 1 - e),$$

which is an immediate consequence of the result.

Theorem 5. *The family $G(x, \ell, \mathfrak{X}, m, e)$ is a convex set.*

Proof. Consider J_1 and J_2 as arbitrary elements of $G(x, \ell, \mathfrak{X}, m, e)$. Then for each t ($0 \leq t \leq 1$), we show that $(1-t)J_1 + tJ_2 \in G(x, \ell, \mathfrak{X}, m, e)$. Thus, we have

$$(1-t)J_1 + tJ_2 = q_0 \eta^{-e} + \sum_{n=1-e}^{\infty} [(1-t)q_n + tq_n] \eta^n.$$

Hence

$$\begin{aligned} & \sum_{n=1-e}^{\infty} n[(e+1)(x+1) + (n-1)] \left(\frac{(\ell+\mathfrak{X}(n+e))}{\ell} \right)^m [(1-t)q_n + tq_n] \\ &= (1-t) \sum_{n=1-e}^{\infty} n[(e+1)(x+1) + (n-1)] \left(\frac{(\ell+\mathfrak{X}(n+e))}{\ell} \right)^m q_n \end{aligned}$$

$$\begin{aligned}
& + t \sum_{n=1-\epsilon}^{\infty} n[(\epsilon+1)(x+1) + (n-1)] \left(\frac{(\ell + \mathfrak{X}(n+\epsilon))}{\ell} \right)^m p_n \\
& \leq (1-t)q_0\epsilon(\epsilon+1)x + tq_0\epsilon(\epsilon+1)x = q_0\epsilon(\epsilon+1)x.
\end{aligned}$$

Hence, the proof is established.

References

- [1] Aouf, M. K. (2008). Certain subclasses of meromorphically p -valent functions with positive or negative coefficients. *Mathematical and Computer Modelling*, 47(9–10), 997–1008. <https://doi.org/10.1016/j.mcm.2007.04.018>
- [2] Aouf, M. K. (1990). On a class of meromorphic multivalent functions with positive coefficients. *Mathematica Japonica*, 35, 603–608.
- [3] Aouf, M. K., & Darwish, H. E. (2008). A certain subclass of p -valent meromorphically starlike functions with alternating coefficients. *Indian Journal of Pure and Applied Mathematics*, 39(2), 157–166.
- [4] El-Ashwah, R. M. (2009). A note on certain meromorphic p -valent functions. *Applied Mathematics Letters*, 22, 1756–1759. <https://doi.org/10.1016/j.aml.2009.06.026>
- [5] Joshi, S. B., & Srivastava, H. M. (1999). A certain family of meromorphically multivalent functions. *Computers & Mathematics with Applications*, 38(3–4), 201–211. [https://doi.org/10.1016/S0898-1221\(99\)00194-7](https://doi.org/10.1016/S0898-1221(99)00194-7)
- [6] Mogra, M. L. (1990). Meromorphic multivalent functions with positive coefficients. *Mathematica Japonica*, 35(1), 1–11.
- [7] Raina, R. K., & Srivastava, H. M. (2006). A new class of meromorphically multivalent functions with applications to generalized hypergeometric functions. *Mathematical and Computer Modelling*, 43(3–4), 350–356. <https://doi.org/10.1016/j.mcm.2005.09.031>
- [8] Reddy, T. R., Sharma, R. B., & Saroja, K. (2013). A new subclass of meromorphic functions with positive coefficients. *Indian Journal of Pure and Applied Mathematics*, 44(1), 29–46. <https://doi.org/10.1007/s13226-013-0002-2>
- [9] Srivastava, H. M., Suchithra, K., Stephen, B. A., & Sivasubramanian, S. (2006). Inclusion and neighborhood properties of certain subclasses of analytic and multivalent functions of complex order. *Journal of Inequalities in Pure and Applied Mathematics*, 7, Article 191, 1–8.
- [10] Tehranchi, A., & Kulkarni, S. R. (2008). An application of differential subordination for the class of meromorphic multivalent functions with complex order. *Southeast Asian Bulletin of Mathematics*, 32, 379–392.

This is an open access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted, use, distribution and reproduction in any medium, or format for any purpose, even commercially provided the work is properly cited.
