

## Some Results on Subclasses of Multivalent and Meromorphic Functions Defined by Multiplier Transformations

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### Abstract

The objective of the present work is to investigate a specific family of uniformly meromorphic of multivalent functions defined in  $\mathbb{U}^*$  associated with multiplier transformations. We get some results for this class, like, coefficient estimates, distortion theorem, closure theorem and radii of starlike.

### 1. Introduction

Assume that  $\mathcal{U}_\epsilon$  be the family of functions that take the following type:

$$\mathbb{J}(\eta) = a_0 \eta^{-\epsilon} + \sum_{n=1-\epsilon}^{\infty} a_n \eta^n \quad a_0 > 0, \quad a_n \geq 0, \quad \epsilon \in N = \{1, 2, \dots\}, \quad (1)$$

that are multivalent meromorphic in  $\mathbb{U}^* = \{\eta \in \mathbb{C}, 0 < |\eta| < 1\} = U/\{0\}$ .

For  $n \in \mathbb{Z}, \ell > 0, \mathfrak{X} \geq 0$  and  $\mathbb{J} \in \mathcal{U}_\epsilon$ , the multiplier transformations  $\mathcal{Q}_\epsilon^m(\ell, \mathfrak{X}) : \mathcal{U}_\epsilon \rightarrow \mathcal{U}_\epsilon$  is defined by (see [4])

$$\mathcal{Q}_\epsilon^m(\ell, \mathfrak{X})\mathbb{J}(\eta) = a_0 \eta^{-\epsilon} + \sum_{n=1-\epsilon}^{\infty} \left( \frac{(\ell + \mathfrak{X}(n + \epsilon))}{\ell} \right)^m a_n \eta^n. \quad (2)$$

A function  $\mathbb{J} \in \mathcal{U}_\epsilon$  is named multivalent meromorphic starlike function of order  $\alpha$  whenever it satisfies the condition

$$-Re \left\{ \frac{\eta \mathbb{J}'(\eta)}{\mathbb{J}(\eta)} \right\} > \alpha, \quad (\eta \in \mathbb{U}^*; \quad 0 \leq \alpha < \epsilon), \quad (3)$$

and is named multivalent meromorphic convex functions of order  $\alpha$  whenever it satisfies the condition

$$-Re \left\{ \frac{\eta \mathbb{J}''(\eta)}{\mathbb{J}'(\eta)} + 1 \right\} > \alpha, \quad (\eta \in \mathbb{U}^*; \quad 0 \leq \alpha < \epsilon). \quad (4)$$

We define  $G(x, \ell, \mathfrak{X}, m, \epsilon)$  as the family of functions in  $\mathcal{U}_\epsilon$  that fulfill the condition stated below:

$$Re \left\{ - \left( \frac{\eta (\mathcal{Q}_\epsilon^m(\ell, \mathfrak{X})\mathbb{J}(\eta))''}{(1 + \epsilon)(\mathcal{Q}_\epsilon^m(\ell, \mathfrak{X})\mathbb{J}(\eta))'} + x \right) \right\} \geq \left| \frac{\eta (\mathcal{Q}_\epsilon^m(\ell, \mathfrak{X})\mathbb{J}(\eta))''}{(1 + \epsilon)(\mathcal{Q}_\epsilon^m(\ell, \mathfrak{X})\mathbb{J}(\eta))'} + x + 2 \right|, \quad (5)$$

where  $0 < x < 1, n \in \mathbb{Z}, \ell > 0, \mathfrak{X} \geq 0$ .

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A number of studies have lately focused on analyzing subclasses of meromorphic functions, as documented in references [1,2,3,5,6,7,8,9,10].

## 2. Coefficient Estimates

**Theorem 1.** Let  $\mathbb{J} \in \mathcal{U}_e$ . Then  $\mathbb{J} \in G(x, \ell, \mathfrak{X}, m, e)$ , iff

$$\sum_{n=1-e}^{\infty} n[(e+1)(x+1) + (n-1)] \left( \frac{(\ell + \mathfrak{X}(n+e))}{\ell} \right)^m q_n \leq q_0 e(e+1)x. \quad (6)$$

Sharpness of the result is attained for the function  $\mathbb{J}$  described as

$$\mathbb{J}(\eta) = q_0 \eta^{-e} + \frac{q_0 e(e+1)x}{n[(e+1)(x+1) + (n-1)] \left( \frac{(\ell + \mathfrak{X}(n+e))}{\ell} \right)^m} \eta^{1-e}, \quad n \geq 1-e. \quad (7)$$

**Proof.** Suppose that  $\mathbb{J} \in G(x, \ell, \mathfrak{X}, m, e)$ . Then

$$\operatorname{Re} \left\{ - \left( \frac{\eta (\mathcal{Q}_e^m(\ell, \mathfrak{X})\mathbb{J}(\eta))''}{(e+1)(\mathcal{Q}_e^m(\ell, \mathfrak{X})\mathbb{J}(\eta))'} + x \right) \right\} \geq \left| \frac{\eta (\mathcal{Q}_e^m(\ell, \mathfrak{X})\mathbb{J}(\eta))''}{(e+1)(\mathcal{Q}_e^m(\ell, \mathfrak{X})\mathbb{J}(\eta))'} + x + 2 \right|.$$

But

$$\begin{aligned} \operatorname{Re} \left\{ \frac{\eta (\mathcal{Q}_e^m(\ell, \mathfrak{X})\mathbb{J}(\eta))''}{(e+1)(\mathcal{Q}_e^m(\ell, \mathfrak{X})\mathbb{J}(\eta))'} + x \right\} &\leq \left| \frac{\eta (\mathcal{Q}_e^m(\ell, \mathfrak{X})\mathbb{J}(\eta))''}{(e+1)(\mathcal{Q}_e^m(\ell, \mathfrak{X})\mathbb{J}(\eta))'} + x + 2 \right| \\ &\leq \operatorname{Re} \left\{ - \left( \frac{\eta (\mathcal{Q}_e^m(\ell, \mathfrak{X})\mathbb{J}(\eta))''}{(1+e)(\mathcal{Q}_e^m(\ell, \mathfrak{X})\mathbb{J}(\eta))'} + x \right) \right\}, \end{aligned}$$

that is

$$\operatorname{Re} \left\{ \frac{\eta (\mathcal{Q}_e^m(\ell, \mathfrak{X})\mathbb{J}(\eta))''}{(1+e)(\mathcal{Q}_e^m(\ell, \mathfrak{X})\mathbb{J}(\eta))'} + 2(x+1) \right\} \leq 0,$$

by (2) we get

$$\begin{aligned} &\sum_{n=1-e}^{\infty} n[(e+1)(x+1) + (n-1)] \left( \frac{(\ell + \mathfrak{X}(n+e))}{\ell} \right)^m q_n \leq q_0 e(e+1)x \\ \operatorname{Re} \left\{ \frac{-2e(e+1)xq_0\eta^{-e-1} + \sum_{n=1-e}^{\infty} 2n[(n-1) + (e+1)(x+1)] \left( \frac{(\ell + \mathfrak{X}(n+e))}{\ell} \right)^m q_n \eta^{n-1}}{-eq_0(e+1)\eta^{-e-1} + \sum_{n=1-e}^{\infty} n(e+1) \left( \frac{(\ell + \mathfrak{X}(n+e))}{\ell} \right)^m q_n \eta^{n-1}} \right\} &\leq 0. \end{aligned}$$

Letting  $\eta$  to take real values as  $\eta \rightarrow 1^-$ , we get

$$\frac{-2e(e+1)xq_0 + \sum_{n=1-e}^{\infty} 2n[(n-1) + (e+1)(x+1)] \left( \frac{(\ell + \mathfrak{X}(n+e))}{\ell} \right)^m q_n}{-eq_0(e+1) + \sum_{n=1-e}^{\infty} n(e+1) \left( \frac{(\ell + \mathfrak{X}(n+e))}{\ell} \right)^m q_n} \leq 0,$$

that is

$$-2e(e + 1)xq_0 + \sum_{n=1-e}^{\infty} 2n[(n - 1) + (e + 1)(x + 1)] \left(\frac{\ell + \mathfrak{X}(n + e)}{\ell}\right)^m q_n \leq 0,$$

which is equivalent to (6).

On the other hand, if we assume that inequality (6) holds, we deduce that

$$\begin{aligned} \left| \frac{\eta (\mathcal{Q}_e^m(\ell, \mathfrak{X})\mathbb{J}(\eta))''}{(e + 1)(\mathcal{Q}_e^m(\ell, \mathfrak{X})\mathbb{J}(\eta))'} + x + 2 \right| &= \left| \frac{-2e(e + 1)(x + 2)q_0 \eta^{-e-1} + \sum_{n=1-e}^{\infty} n \left[ 2(n - 1) + (x + 2)(e + 1) \left(\frac{\ell + \mathfrak{X}(n + e)}{\ell}\right)^m \right] q_n \eta^{n-1}}{-e q_0 (e + 1) \eta^{-e-1} + \sum_{n=1-e}^{\infty} n(e + 1) \left(\frac{\ell + \mathfrak{X}(n + e)}{\ell}\right)^m q_n \eta^{n-1}} \right| \\ &\leq \frac{-2e(e + 1)(x + 2)q_0 - \sum_{n=1-e}^{\infty} n \left[ 2(n - 1) + (x + 2)(e + 1) \left(\frac{\ell + \mathfrak{X}(n + e)}{\ell}\right)^m \right] q_n \eta^{n+e}}{-e q_0 (e + 1) + \sum_{n=1-e}^{\infty} n(e + 1) \left(\frac{\ell + \mathfrak{X}(n + e)}{\ell}\right)^m q_n \eta^{n+e}}. \end{aligned}$$

Taking the limit as  $\eta \rightarrow 1^-$  along the real axis yields the required inequality (6). Therefore, the proof of Theorem 1 is established.

**Corollary 1.** If  $\mathbb{J} \in G(x, \ell, \mathfrak{X}, m, e)$ , then

$$q_n \leq \frac{e(e + 1)xq_0}{n[(n - 1) + (e + 1)(x + 1)] \left(\frac{\ell + \mathfrak{X}(n + e)}{\ell}\right)^m} \quad (n, e \in N = \{1, 2, \dots\}). \tag{8}$$

**Theorem 2.** Let  $\mathbb{J} \in G(x, \ell, \mathfrak{X}, m, e)$ . Then

$$\frac{q_0}{r^e} - \frac{e(e + 1)x}{(1 - e)[-e + (e + 1)(x + 1)] \left(\frac{\ell + \mathfrak{X}}{\ell}\right)^m} r^{1-e} \leq |\mathbb{J}(\eta)| \leq \frac{q_0}{r^e} + \frac{e(e + 1)x}{(1 - e)[-e + (e + 1)(x + 1)] \left(\frac{\ell + \mathfrak{X}}{\ell}\right)^m} r^{1-e}.$$

**Proof.** Since  $\mathbb{J} \in G(x, \ell, \mathfrak{X}, m, e)$ , we get

$$\begin{aligned} (1 - e)[-e + (e + 1)(x + 1)] \left(\frac{\ell + \mathfrak{X}}{\ell}\right)^m \sum_{n=1-e}^{\infty} q_n \\ \leq \sum_{n=1-e}^{\infty} n[(n - 1) + (e + 1)(x + 1)] \left(\frac{\ell + \mathfrak{X}(n + e)}{\ell}\right)^m q_n \leq e(e + 1)x. \end{aligned}$$

We have

$$|\mathbb{J}(\eta)| = \frac{q_0}{|\eta|^e} + \sum_{n=1-e}^{\infty} q_n |\eta|^n \leq \frac{q_0}{|\eta|^e} + |\eta|^{1-e} \sum_{n=1-e}^{\infty} q_n \leq \frac{q_0}{r^e} + \frac{e(e + 1)x}{(1 - e)[-e + (e + 1)(x + 1)] \left(\frac{\ell + \mathfrak{X}}{\ell}\right)^m} r^{1-e}.$$

Also, similarly, we have

$$|\mathbb{J}(\eta)| \geq \frac{q_0}{r^e} - \frac{e(e + 1)x}{(1 - e)[-e + (e + 1)(x + 1)] \left(\frac{\ell + \mathfrak{X}}{\ell}\right)^m} r^{1-e}.$$

Sharpness of the result is attained for the function  $\mathbb{J}$  described as

$$\mathbb{J}(\eta) = q_0 \eta^{-e} + \frac{q_0 e(e+1)x}{(1-e)[-e+(e+1)(x+1)] \left(\frac{\ell+\mathfrak{X}}{\ell}\right)^m} \eta^{1-e}, \quad e \in N.$$

Now, let us consider the functions  $\mathbb{J}_j(\eta)$ , defined for  $j = 1, 2, \dots, q$ , by

$$\mathbb{J}_j(\eta) = q_{0,j} \eta^{-e} + \sum_{n=1-e}^{\infty} q_{n,j} \eta^n \quad (q_{0,j} > 0, q_{n,j} \geq 0, e \in N = \{1, 2, \dots\}). \quad (9)$$

**Theorem 3.** Assume that the functions  $\mathbb{J}_j(\eta)$  defined by (9) lies in the family  $G(x, \ell, \mathfrak{X}, m, e)$  for  $j = 1, 2, \dots, q$ . Then the function  $h(\eta)$  defined as

$$h(\eta) = q_{0,j} \eta^{-e} + \sum_{n=1-e}^{\infty} c_n \eta^n$$

is also in the family  $G(x, \ell, \mathfrak{X}, m, e)$ , where

$$c_n = \frac{1}{q} \sum_{j=1}^q q_{n,j}.$$

**Proof.** Since  $\mathbb{J}_j(\eta) \in G(x, \ell, \mathfrak{X}, m, e)$  ( $j = 1, 2, \dots, q$ ) and by Theorem 1, we obtain

$$\sum_{n=1-e}^{\infty} n[(n-1) + (e+1)(x+1)] \left(\frac{(\ell + \mathfrak{X}(n+e))}{\ell}\right)^m q_{n,j} \leq e(e+1)x.$$

Hence

$$\begin{aligned} & \sum_{n=1-e}^{\infty} n[(n-1) + (e+1)(x+1)] \left(\frac{(\ell + \mathfrak{X}(n+e))}{\ell}\right)^m c_n \\ &= \sum_{n=1-e}^{\infty} n[(n-1) + (e+1)(x+1)] \left(\frac{(\ell + \mathfrak{X}(n+e))}{\ell}\right)^m \left(\frac{1}{q} \sum_{j=1}^q q_{n,j}\right) \\ &= \frac{1}{q} \sum_{j=1}^q \sum_{n=1-e}^{\infty} n[(n-1) + (e+1)(x+1)] \left(\frac{(\ell + \mathfrak{X}(n+e))}{\ell}\right)^m q_{n,j} \leq e(e+1)x. \end{aligned}$$

This shows that  $h(\eta) \in G(x, \ell, \mathfrak{X}, m, e)$ .

**Theorem 4.** Let  $\mathbb{J}$  be an element of  $G(x, \ell, \mathfrak{X}, m, e)$ . Then  $\mathbb{J}$  is multivalent meromorphic starlike of order  $\varphi$  ( $0 \leq \varphi < 1$ ) in the region  $|\eta| < R_1$  such that

$$R_1 = \inf_n \left\{ \frac{n(e-\varphi)[(e+1)(x+1) + (n-1)] \left(\frac{(\ell + \mathfrak{X}(n+e))}{\ell}\right)^m}{e(e+1)x(n+\varphi)} \right\}^{\frac{1}{n-e}}.$$

The obtained result is exact for the function  $\mathbb{J}$  defined in (7).

**Proof.** It is enough to demonstrate that

$$\left| \frac{\eta \mathbb{J}'(\eta)}{\mathbb{J}(\eta)} + e \right| \leq e - \varphi \quad \text{for } |\eta| < R_1. \tag{10}$$

But

$$\left| \frac{\eta \mathbb{J}'(\eta)}{\mathbb{J}(\eta)} + e \right| = \left| \frac{\eta \mathbb{J}'(\eta) + e \mathbb{J}(\eta)}{\mathbb{J}(\eta)} \right| \leq \frac{\sum_{n=1-e}^{\infty} (n+e) q_n |\eta|^{n-e}}{1 + \sum_{n=1-e}^{\infty} q_n |\eta|^{n-e}}.$$

Thus (10) will be satisfied if

$$\frac{\sum_{n=1-e}^{\infty} (n+e) q_n |\eta|^{n-e}}{q_0 + \sum_{n=1-e}^{\infty} q_n |\eta|^{n-e}} \leq e - \varphi,$$

or if

$$\sum_{n=1-e}^{\infty} \frac{n + \varphi}{q_0(e - \varphi)} q_n |\eta|^{n-e} \leq 1. \tag{11}$$

Since  $\mathbb{J} \in G(x, \ell, \mathfrak{X}, m, e)$ , we have

$$\sum_{n=1-e}^{\infty} \frac{n[(e+1)(x+1) + (n-1)] \left(\frac{\ell + \mathfrak{X}(n+e)}{\ell}\right)^m}{q_0 e(e+1)x} q_n \leq 1.$$

Hence (11) will be true if

$$\frac{n + \varphi}{q_0(e - \varphi)} |\eta|^{n-e} \leq \frac{n[(e+1)(x+1) + (n-1)] \left(\frac{\ell + \mathfrak{X}(n+e)}{\ell}\right)^m}{q_0 e(e+1)x},$$

or equivalently

$$|\eta| \leq \left\{ \frac{n(e - \varphi)[(e+1)(x+1) + (n-1)] \left(\frac{\ell + \mathfrak{X}(n+e)}{\ell}\right)^m}{e(e+1)x(n + \varphi)} \right\}^{\frac{1}{n-e}} \quad (n \geq 1 - e),$$

which is an immediate consequence of the result.

**Theorem 5.** *The family  $G(x, \ell, \mathfrak{X}, m, e)$  is a convex set.*

**Proof.** Consider  $\mathbb{J}_1$  and  $\mathbb{J}_2$  as arbitrary elements of  $G(x, \ell, \mathfrak{X}, m, e)$ . Then for each  $t$  ( $0 \leq t \leq 1$ ), we show that  $(1-t)\mathbb{J}_1 + t\mathbb{J}_2 \in G(x, \ell, \mathfrak{X}, m, e)$ . Thus, we have

$$(1-t)\mathbb{J}_1 + t\mathbb{J}_2 = q_0 \eta^{-e} + \sum_{n=1-e}^{\infty} [(1-t)q_n + tq_n] \eta^n.$$

Hence

$$\begin{aligned} & \sum_{n=1-e}^{\infty} n[(e+1)(x+1) + (n-1)] \left(\frac{\ell + \mathfrak{X}(n+e)}{\ell}\right)^m [(1-t)q_n + tq_n] \\ &= (1-t) \sum_{n=1-e}^{\infty} n[(e+1)(x+1) + (n-1)] \left(\frac{\ell + \mathfrak{X}(n+e)}{\ell}\right)^m q_n \end{aligned}$$

$$+ t \sum_{n=1-e}^{\infty} n[(e+1)(x+1) + (n-1)] \left( \frac{(\ell + \aleph(n+e))}{\ell} \right)^m p_n$$

$$\leq (1-t)q_0e(e+1)x + tq_0e(e+1)x = q_0e(e+1)x.$$

Hence, the proof is established.

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