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An Analysis of Certain Properties of a Subclass of p-Valent Functions Determined by a Generalized Derivative Operator

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Abstract

This study investigates a specific subclass of multivalent functions defined via the application of a generalized derivative operator. Various associated properties are examined, including coefficient inequalities, growth and distortion estimates, the characterization of extreme points, and the determination of radii of close-to-convexity, starlikeness, and convexity for these subclasses.

1. Introduction

Let $P(p)$ represent the class of functions characterized by the following form:

$$f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j, \quad \left(z \in U, a_j \geq 0, p \in \mathbb{N} = \{1,2,3,\dots\}\right). \tag{1.1}$$

This class includes all analytic and p-valent functions in the open unit disk $D = \{z \in \mathbb{C}: |z| < 1\}$. Let $E(p)$ denote a specific subclass of $P(p)$.

The Hadamard product (also known as convolution) $f * g$ of two analytic functions, f, as defined by (1.1), and $g(z)$, given by:

$$g(z) = z^p + \sum_{j=p+1}^{\infty} b_j z^j \tag{1.2}$$

is defined as

$$f(z) * g(z) = (f * g)(z) = z^p + \sum_{j=p+1}^{\infty} a_j b_j z^j, \quad (z \in U, p \in \mathbb{N}). \tag{1.3}$$

A function f belonging to the class $P(p)$, is said to be multivalent starlike of order ∂, multivalent convex of order ∂, and multivalent close-to-convex of order ∂ under the following conditions: $(p \in \mathbb{N}, 0 \leq \partial < p, z \in D)$, where each characterization holds according to its respective mathematical definition:

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$$Re\left\{\frac{z\,f'(z)}{f(z)}\right\} > \partial, \qquad Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \partial \quad \text{and} \quad Re\left\{\frac{f'(z)}{f^{p-1}}\right\} > \partial.$$

Elhaddad and Darus [3] proposed the following operator:

For a function f belonging to the class $P(p)$, the following relation holds:

$$\tilde{O}^m_{\lambda,p}(v,e,a_1,b_1)f(z) = z^p + \sum_{j=p+1}^{\infty}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Gamma(e)}{\Gamma(v(j-p)+e)}\left(\frac{\prod_{n=1}^{s}(a_n)_{j-p}}{\prod_{i=1}^{r}(b_i)_{j-p}}\right)\frac{a_j z^j}{(j-p)!}, \quad (1.4)$$

where $a_n \in \mathbb{C}, b_i \in \mathbb{C}-\{0,-1,-2,\dots\}$ $(i=1,\dots,r, n=1,\dots,s)$, and $s \le r+1$.

For simplicity, we rewrite the above expression as

$$\tilde{O}^m_{\lambda,p}(v,e,a_1,b_1)f(z) = z^p + \sum_{j=p+1}^{\infty}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!}a_j z^j, \quad (1.5)$$

where $m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, $\lambda \ge 0$ and $\Upsilon_{(j-p,v,e)}(a_s,b_r)$, is defined by

$$\Upsilon_{(j-p,v,e)}(a_s,b_r) = \frac{\Gamma(e)}{\Gamma(v(j-p)+e)}\left(\frac{\prod_{n=1}^{s}(a_n)_{j-p}}{\prod_{i=1}^{r}(b_i)_{j-p}}\right). \quad (1.6)$$

For further details regarding this operator, see [2].

By employing the operator defined in Equation (1.5), we introduce the following class of analytic and multivalent functions.

Definition 1.1. A function $f \in P(p)$ is said to belong to the class $P(p,\gamma,\lambda)$ if

$$\left|\frac{(2-p)z[\tilde{O}^m_{\lambda,p}(v,e,a_1,b_1)f(z)]'' + z^2[\tilde{O}^m_{\lambda,p}(v,e,a_1,b_1)f(z)]'''}{(3\lambda-\gamma)z\left[\tilde{O}^m_{\lambda,p}(v,e,a_1,b_1)f(z)\right]'' + \lambda z^2[\tilde{O}^m_{\lambda,p}(v,e,a_1,b_1)f(z)]'''}\right| < 1, \quad (1.7)$$

where $\left(p \ge 1, \frac{1}{2} \le \gamma < 1, 0 \le \lambda \le \frac{1}{2}\right)$, $a_1 \in \mathbb{C}$, $b_1 \in \mathbb{C}\backslash\{0,-1,-2,\dots\}$, $|z|<1$, $v,e \in \mathbb{C}$, $Re(v) > 0$, $Re(e) > 0$, $m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$.

Numerous authors have examined various classes of analytic and multivalent functions, including their coefficients estimates, as documented in Refs. [1], [2], [4], [5], [6], [7], [8], [9], and [10]. In this study, we focus on analyzing and investigating the class $P(p,\gamma,\lambda)$ of analytic and multivalent functions. Additionally, several properties, such as coefficient bounds, growth and distortion theorems, inclusion properties, and extreme points for functions in this class, are established.

2. Geometric Properties for $P(p,\gamma,\lambda)$

In this section, we present theorems along with their proofs to explore certain geometric properties associated with the class $P(p,\gamma,\lambda)$.

Theorem 2.1. *A function f, as defined in equation (1.1), belongs to the class $P(p,\gamma,\lambda)$ if and only if it satisfies the following condition:*

$$\sum_{j=p+1}^{\infty}[p-j-\gamma+\lambda(j+1)]j(j-1)\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!}|a_j| \le [\lambda(1+p)-\gamma(p-1)], \quad (1.8)$$

where $\left(p \geq 1, \frac{1}{2} \leq \gamma < 1, 0 < \lambda \leq \frac{1}{2}\right)$ *and* $\Upsilon_{(j-p,v,e)}(a_s, b_r)$ *is given by* (1.6).

The result is sharp for the function:

$$f(z) = z^p + \frac{[\lambda(1+p)-\gamma]p(p-1)(j-p)!}{[p-j-\gamma+\lambda(j+1)]j(j-1)\Upsilon_{(j-p,v,e)}(a_s,b_r)\left[\frac{p+(j-p)\lambda}{p}\right]^m} z^j. \tag{1.9}$$

Proof. Suppose that $f \in P(p,\gamma,\lambda)$. Then, by (1.7), we have:

$$\left|\frac{(2-p)z\left[\tilde{O}^m_{\lambda,p}(v,e,a_1,b_1)f(z)\right]'' + z^2\left[\tilde{O}^m_{\lambda,p}(v,e,a_1,b_1)f(z)\right]'''}{(3\lambda-\gamma)z\left[\tilde{O}^m_{\lambda,p}(v,e,a_1,b_1)f(z)\right]'' + \lambda z^2\left[\tilde{O}^m_{\lambda,p}(v,e,a_1,b_1)f(z)\right]'''}\right| < 1$$

$$= \left|(3\lambda-\gamma)z\left[\tilde{O}^m_{\lambda,p}(v,e,a_1,b_1)f(z)\right]'' + \lambda z^2\left[\tilde{O}^m_{\lambda,p}(v,e,a_1,b_1)f(z)\right]'''\right|$$

$$-\left|(2-p)z\left[\tilde{O}^m_{\lambda,p}(v,e,a_1,b_1)f(z)\right]'' + z^2\left[\tilde{O}^m_{\lambda,p}(v,e,a_1,b_1)f(z)\right]'''\right|.$$

$$= \left|(3\lambda-\gamma-2+p)z\left[\tilde{O}^m_{\lambda,p}(v,e,a_1,b_1)f(z)\right]''\right| + \left|(\lambda-1)z^2\left[\tilde{O}^m_{\lambda,p}(v,e,a_1,b_1)f(z)\right]'''\right|$$

$$= \left|(3\lambda-\gamma-2+p)z\left[p(p-1)z^{p-2} + j(j-1)\sum_{j=p+1}^{\infty}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!} a_j z^{j-2}\right]\right|$$

$$+\left|(\lambda-1)z^2\left[p(p-1)(p-2)z^{p-3} + j(j-1)(j-2)\sum_{j=p+1}^{\infty}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!} a_j z^{j-3}\right]\right|$$

$$= \left|(3\lambda-\gamma-2+p)p(p-1)z^{p-1} + (3\lambda-\gamma-2+p)j(j-1)\sum_{j=p+1}^{\infty}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!} a_j z^{j-1}\right|$$

$$+\left|(\lambda-1)p(p-1)(p-2)z^{p-1} + (\lambda-1)j(j-1)(j-2)\sum_{j=p+1}^{\infty}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!} a_j z^{j-1}\right|$$

$$= \left|[\lambda(1+p)-\gamma]p(p-1)z^{p-1} + \sum_{j=p+1}^{\infty}[p-j-\gamma+\lambda(j+1)]j(j-1)\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!} a_j z^{j-1}\right|$$

$$\leq [\lambda(1+p)-\gamma]p(p-1)|z^{p-1}| + \sum_{j=p+1}^{\infty}[p-j-\gamma+\lambda(j+1)]j(j-1)\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!} |a_j||z^{j-1}|$$

$$\leq |z^{p-1}| + \sum_{j=p+1}^{\infty}\frac{[p-j-\gamma+\lambda(j+1)]j(j-1)}{[\lambda(1+p)-\gamma]p(p-1)}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!} |a_j||z^{j-1}| \leq 1$$

$\lambda \geq 0$, $j > p, \frac{p+(j-p)\lambda}{p} > 0, (j-p)! > 0$ and $\Upsilon_{(j-p,v,e)}(a_s, b_r)$ is given by (1.6).

$$\sum_{j=p+1}^{\infty} [p - j - \gamma + \lambda(j+1)] j(j-1) \left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s, b_r)}{(j-p)!} |a_j| \leq [\lambda(1+p) - \gamma] p(p-1).$$

Conversely, assume that condition (1.8) holds for $|z| = s$, where $s < 1$, then

$$\left|(3\lambda-\gamma)z\left[\tilde{O}_{\lambda,p}^m(v,e,a_1,b_1)f(z)\right]'' + \lambda z^2\left[\tilde{O}_{\lambda,p}^m(v,e,a_1,b_1)f(z)\right]'''\right|$$

$$-\left|(2-p)z\left[\tilde{O}_{\lambda,p}^m(v,e,a_1,b_1)f(z)\right]'' + z^2\left[\tilde{O}_{\lambda,p}^m(v,e,a_1,b_1)f(z)\right]'''\right|$$

$$= \left|(3\lambda-\gamma-2+p)z\left[\tilde{O}_{\lambda,p}^m(v,e,a_1,b_1)t(z)\right]''\right| + \left|(\lambda-1)z^2\left[\tilde{O}_{\lambda,p}^m(v,e,a_1,b_1)t(z)\right]'''\right|$$

$$= \left|(3\lambda-\gamma-2+p)z\left[p(p-1)z^{p-2} + j(j-1)\sum_{j=p+1}^{\infty}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(d-p)!} a_j z^{j-2}\right]\right|$$

$$+ \left|(\lambda-1)z^2\left[p(p-1)(p-2)z^{p-3} + j(j-1)(j-2)\sum_{j=p+1}^{\infty}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!} a_j z^{j-3}\right]\right|$$

$$= \left|(3\lambda-\gamma-2+p)p(p-1)z^{p-1} + (3\lambda-\gamma-2+p)j(j-1)\sum_{j=p+1}^{\infty}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!} a_j z^{j-1}\right|$$

$$+ \left|(\lambda-1)p(p-1)(p-2)z^{p-1} + (\lambda-1)j(j-1)(j-2)\sum_{j=p+1}^{\infty}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!} a_j z^{j-1}\right|$$

$$= \left|[\lambda(1+p)-\gamma]p(p-1)z^{j-1} + \sum_{j=p+1}^{\infty}[p-j-\gamma+\lambda(j+1)]j(j-1)\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!} a_j z^{j-1}\right|$$

$$\leq [\lambda(1+p)-\gamma]p(p-1)\left|z^{j-1}\right| + \sum_{j=p+1}^{\infty}[p-j-\gamma+\lambda(j+1)]j(j-1)\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!} |a_j|\left|z^{j-1}\right|$$

$$\leq |z^{p-1}| + \sum_{j=p+1}^{\infty}\frac{[p-j-\gamma+\lambda(j+1)]j(j-1)}{[\lambda(1+p)-\gamma]p(p-1)}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!} |a_j|\left|z^{j-1}\right|$$

$$\sum_{j=p+1}^{\infty}[p-j-\gamma+\lambda(j+1)]j(j-1)\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!} |a_j| \leq [\lambda(1+p)-\gamma]p(p-1),$$

where $|a_j|$ is given by (1.8). So, we have:

$$\sum_{j=p+1}^{\infty}[p-j-\gamma+\lambda(j+1)]j(j-1)\left[\frac{p+(j-p)\lambda}{p}\right]^m\frac{\Upsilon_{(j-p,\upsilon,e)}(a_s,b_r)}{(j-p)!}|a_j|-[\lambda(1+p)-\gamma]p(p-1)\leq 0.$$

Thus, $f \in P(p,\gamma,\lambda)$, and the theorem is thereby proven. □

Corollary 2.2. *Let* $f \in P(p,\gamma,\lambda)$. *Then,*

$$a_j \leq \frac{[\lambda(1+p)-\gamma]p(p-1)(j-p)!}{[p-j-\gamma+\lambda(j+1)]j(j-1)\Upsilon_{(j-p,\upsilon,e)}(a_s,b_r)\left[\frac{p+(j-p)\lambda}{p}\right]^m}, \tag{1.10}$$

where, $(j=p+1,p+2,\dots)\left(p\geq 1,\frac{1}{2}\leq\gamma<1,0<\lambda\leq\frac{1}{2}\right)$.

3. Growth and Distortion Theorems

The bounds for $|f(z)|$ and $|f'(z)|$ will be established through the following theorems, which specifically pertain to multivalent functions $f(z)$ expressed in the form.

$$f(z)=z^p+\frac{[\lambda(1+p)-\gamma]p(p-1)}{[\lambda(p+2)-(1+\gamma)]p(1+p)\Upsilon_{(1,\upsilon,e)}(a_s,b_r)\left[\frac{p+\lambda}{p}\right]^m}z^{p+1}.$$

Theorem 3.1. *If the function* f *belongs to the class* $P(p,\gamma,\lambda)$, *as defined in Equation* 1.2, *then for* $|z|=s<1$,

$$s^p-s^{p+1}\frac{[\lambda(1+p)-\gamma]p(p-1)}{[\lambda(p+2)-(1+\gamma)]p(1+p)\Upsilon_{(1,\upsilon,e)}(a_s,b_r)\left[\frac{p+\lambda}{p}\right]^m}\leq|f(z)|$$

$$\leq s^p+s^{p+1}\frac{[\lambda(1+p)-\gamma]p(p-1)}{[\lambda(p+2)-(1+\gamma)]p(1+p)\Upsilon_{(1,\upsilon,e)}(a_s,b_r)\left[\frac{p+\lambda}{p}\right]^m}.$$

Proof. By considering Theorem 2.1, we obtain

$$\sum_{j=p+1}^{\infty}\frac{[p-j-\gamma+\lambda(j+1)]j(j-1)}{[\lambda(1+p)-\gamma]p(p-1)}\left[\frac{p+(j-p)\lambda}{p}\right]^m\frac{\Upsilon_{(j-p,\upsilon,e)}(a_s,b_r)}{(j-p)!}|a_j|\leq 1.$$

Using the specific characteristics of analytic p-valent functions, we also have

$$\frac{[\gamma-\lambda(j+1)]j(j-1)}{[\lambda(1+p)-\gamma]p(p-1)}\left[\frac{p+\lambda}{p}\right]^m\Upsilon_{(1,\upsilon,e)}(a_s,b_r)\sum_{j=p+1}^{\infty}|a_j|$$

$$\leq\sum_{j=p+1}^{\infty}\frac{[p-j-\gamma+\lambda(j+1)]j(j-1)}{[\lambda(1+p)-\gamma]p(p-1)}\left[\frac{p+(j-p)\lambda}{p}\right]^m\frac{\Upsilon_{(j-p,\upsilon,e)}(a_s,b_r)}{(j-p)!}|a_j|\leq 1.$$

Hence, we have

$$\sum_{j=p+1}^{\infty}|a_j|\leq\frac{[\lambda(1+p)-\gamma]p(p-1)}{[\lambda(p+2)-(1+\gamma)]p(1+p)\Upsilon_{(1,\upsilon,e)}(a_s,b_r)\left[\frac{p+\lambda}{p}\right]^m}.$$

From (1.1), we have

$$|f(z)| = \left| z^p + \sum_{j=p+1}^{\infty} a_j z^j \right| \le |z^p| + |z^{p+1}| \sum_{j=p+1}^{\infty} |a_j| \le s^p + s^{p+1} \sum_{j=p+1}^{\infty} |a_j|$$

$$\le s^p + \frac{[\lambda(1+p)-\gamma]p(p-1)}{[\lambda(p+2)-(1+\gamma)]p(1+p)\Upsilon_{(1,v,e)}(a_s,b_r)\left[\frac{p+\lambda}{p}\right]^m} s^{p+1}.$$

Similarly, the opposing argument can be demonstrated as follows:

$$|f(z)| = \left| z^p + \sum_{j=p+1}^{\infty} a_j z^j \right| \ge |z^p| - |z^{p+1}| \sum_{j=p+1}^{\infty} |a_j| \ge s^p - s^{p+1} \sum_{j=p+1}^{\infty} |a_j|$$

$$\ge s^p - \frac{[\lambda(1+p)-\gamma]p(p-1)}{[\lambda(p+2)-(1+\gamma)]p(1+p)\Upsilon_{(1,v,e)}(a_s,b_r)\left[\frac{p+\lambda}{p}\right]^m} s^{p+1}.$$

This concludes the proof. □

Theorem 3.2. *If* $f \in L_{v,e}^m(a_s, b_r, \lambda; j, p)$*, then for* $|z| = s, s < 1$*, we have*

$$ps^{p-1} - \frac{[\lambda(1+p)-\gamma]p(p-1)}{[\lambda(p+2)-(1+\gamma)]p(1+p)\Upsilon_{(1,v,e)}(a_s,b_r)\left[\frac{p+\lambda}{p}\right]^m} s^p \le |f'(z)|$$

$$\le ps^{p-1} + \frac{[\lambda(1+p)-\gamma]p(p-1)}{[\lambda(p+2)-(1+\gamma)]p(1+p)\Upsilon_{(1,v,e)}(a_s,b_r)\left[\frac{p+\lambda}{p}\right]^m} s^p.$$

Proof. Let $f \in L_{v,e}^m(a_s, b_r, \lambda; j, p)$. Then, from (1.8), we have

$$\sum_{j=p+1}^{\infty} |a_j| \le \frac{[\lambda(1+p)-\gamma]p(p-1)}{[\lambda(p+2)-(1+\gamma)]p(1+p)\Upsilon_{(1,v,e)}(a_s,b_r)\left[\frac{p+\lambda}{p}\right]^m}.$$

Also, from (1.1), we have

$$|f'(z)| = \left| pz^{p-1} + \sum_{j=p+1}^{\infty} j a_j z^{j-1} \right| \le ps^{p-1} + (p+1)s^p \sum_{j=p+1}^{\infty} |a_j|$$

$$\le ps^{p-1} + \frac{[\lambda(1+p)-\gamma]p(p-1)}{[\lambda(p+2)-(1+\gamma)]p(1+p)\Upsilon_{(1,v,e)}(a_s,b_r)\left[\frac{p+\lambda}{p}\right]^m} s^p.$$

Similarly, by reversing the inequality, we have

$$|f'(z)| = \left| pz^{p-1} + \sum_{j=p+1}^{\infty} j a_j z^{j-1} \right| \ge ps^{p-1} - (p+1)s^p \sum_{j=p+1}^{\infty} |a_j|$$

$$\ge ps^{p-1} - \frac{[\lambda(1+p)-\gamma]p(p-1)}{[\lambda(p+2)-(1+\gamma)]p(1+p)\Upsilon_{(1,v,e)}(a_s,b_r)\left[\frac{p+\lambda}{p}\right]^m} s^p.$$

This completes the proof. □

4. Radii of Starlikeness, Convexity and Close-to-Convexity

In this section, the subsequent theorems are reformulated in terms of the radii of starlikeness, convexity, and close-to-convexity.

Theorem 4.1. *If the function $f(z)$, defined by* (1.2), *belongs to the class $P(p,\gamma,\lambda)$, then it is multivalent starlike of order $\partial (0 \le \partial < p)$ in the open disk $|z| \le s_1$, such that*

$$s_1(p,\gamma,\lambda,\partial) = inf_j \left[\sum_{j=p+1}^{\infty} \frac{(p-\partial)[p-j-\gamma+\lambda(j+1)]j(j-1)}{(j-\partial)[\lambda(1+p)-\gamma]p(p-1)} \left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!} \right]^{\frac{1}{j-p}}, \quad (j \ge p+1).$$

The result is sharp for the extremal function $f(z)$ given by (1.9).

Proof. It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - p \right| \le p - \partial, \quad (0 \le \partial < p),$$

for $|z| < s_1(p,\gamma,\lambda,\partial)$. We have

$$\left| \frac{zf'(z)}{f(z)} - p \right| = \left| \frac{z\left[pz^{p-1} + \sum_{j=p+1}^{\infty} j a_j z^{j-1}\right] - p\left[z^p + \sum_{j=p+1}^{\infty} a_j z^j\right]}{z^p + \sum_{j=p+1}^{\infty} a_j z^j} \right|$$

$$= \left| \frac{\left[\sum_{j=p+1}^{\infty} j a_j z^j\right] - p\left[\sum_{j=p+1}^{\infty} a_j z^j\right]}{z^p + \sum_{j=p+1}^{\infty} a_j z^j} \right| \le \frac{\left[\sum_{j=p+1}^{\infty} (j-p)|a_j||z|^{j-p}\right]}{\left[1 - \sum_{j=p+1}^{\infty} |a_j||z|^{j-p}\right]}.$$

Thus,

$$\left| \frac{zf'(z)}{f(z)} - p \right| \le p - \partial,$$

if

$$\sum_{j=p+1}^{\infty} \frac{(j-\partial) a_j |z|^{j-p}}{(p-\partial)} \le 1.$$

Therefore, by Corollary 2.2, the above inequality holds if

$$\frac{(j-\partial)|z|^{j-p}}{(p-\partial)} \le \frac{[p-j-\gamma+\lambda(j+1)]j(j-1)}{[\lambda(1+p)-\gamma]p(p-1)} \left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!}.$$

Equivalently, if

$$|z| \le \left[\frac{(p-\partial)[p-j-\gamma+\lambda(j+1)]j(j-1)}{(j-\partial)[\lambda(1+p)-\gamma]p(p-1)} \left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!} \right]^{\frac{1}{j-p}}. \quad (1.11)$$

Hence, the theorem follows easily from (1.11). □

Theorem 4.2. *If the function $f(z)$, defined by* (1.2), *belongs to the class $P(p,\gamma,\lambda)$, then $f(z)$ is multivalent convex of order ∂ $(0 \le \partial < p)$ in the open disk $|z| < s_2$, where*

$$s_2(p,\gamma,\lambda,\partial) = inf_j \left[\frac{(p-\partial)[p-j-\gamma+\lambda(j+1)](j-1)}{(j-\partial)[\lambda(1+p)-\gamma]p(p-1)} \left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!} \right]^{\frac{1}{j-p}}, \quad (j \ge p+1).$$

The result is sharp for the extremal function $f(z)$ *given by* (1.9).

Proof. It is sufficient to show that

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right| \le p-\partial, \quad (0 \le \partial < p),$$

for $|z| < s_2(p,\gamma,\lambda,\partial)$. We have

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right| = \left|1+\frac{z\left[p(p-1)z^{p-2}+\sum_{j=p+1}^{\infty} j(j-1)a_j z^{j-2}\right]-p\left[pz^{p-1}+\sum_{j=p+1}^{\infty} j a_j z^{j-1}\right]}{\left[pz^{p-1}+\sum_{j=p+1}^{\infty} j a_j z^{j-1}\right]}\right|$$

$$= \left|\frac{\left[\sum_{j=p+1}^{\infty} j^2 a_j z^{j-1}\right]-\left[pj\sum_{j=p+1}^{\infty} a_j z^{j-1}\right]}{\left[pz^{p-1}+\sum_{j=p+1}^{\infty} j a_j z^{j-1}\right]}\right|.$$

Then

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right| \le \frac{\left[\sum_{j=p+1}^{\infty} j(j-p)a_j|z|^{j-p}\right]}{\left[1-\sum_{j=p+1}^{\infty} j a_j|z|^{j-p}\right]}.$$

Thus,

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right| \le p-\partial,$$

if

$$\sum_{j=p+1}^{\infty} \frac{j(j-\partial)a_j|z|^{j-p}}{(p-\partial)} \le 1.$$

Therefore, by Corollary 1.9, the above inequality holds if

$$\frac{j(j-\partial)|z|^{j-p}}{(p-\partial)} \le \frac{[p-j-\gamma+\lambda(j+1)]j(j-1)}{[\lambda(1+p)-\gamma]p(p-1)}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!}.$$

Equivalently, if

$$|z| \le \left[\frac{(p-\partial)[p-j-\gamma+\lambda(j+1)](j-1)}{(j-\partial)[\lambda(1+p)-\gamma]p(p-1)}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!}\right]^{\frac{1}{j-p}}. \tag{1.12}$$

Hence, the theorem follows easily from (1.12). □

Theorem 4.3. *Let the function* $f(z)$ *defined by* (1.2) *be in the class* $P(p,\gamma,\lambda)$. *Then* $f(z)$ *is multivalent close-to-convex of order* ∂ $(0 \le \partial < p)$ *in the open disk* $|z| < s_3$, *where*

$$s_3(p,\gamma,\lambda,\partial) = inf_j\left[\frac{(p-\partial)[p-j-\gamma+\lambda(j+1)](j-1)}{[\lambda(1+p)-\gamma]p(p-1)}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!}\right]^{\frac{1}{j-p}}, \quad (j \ge p+1).$$

The result is sharp for the external function $f(z)$ *given by* (1.9).

Proof. It is sufficient to show that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right| \le p-\partial, \quad (0 \le \partial < p),$$

for $|z| < s_3(p,\gamma,\lambda,\partial)$. We have

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| = \left|\frac{pz^{p-1} + \sum_{j=p+1}^{\infty} ja_j z^{j-1} - pz^{p-1}}{z^{p-1}}\right| = \left|\sum_{j=p+1}^{\infty} ja_j z^{j-p}\right| \le \sum_{j=p+1}^{\infty} ja_j |z|^{j-p}.$$

Thus

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \le p - \partial$$

if

$$\sum_{j=p+1}^{\infty} \frac{ja_j|z|^{j-p}}{(p-\partial)} \le 1.$$

Therefore, by Corollary 2.2, the above inequality holds if

$$\frac{j|z|^{j-p}}{(p-\partial)} \le \frac{[p-j-\gamma+\lambda(j+1)]j(j-1)}{[\lambda(1+p)-\gamma]p(p-1)}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!}.$$

Equivalently, if

$$|z| \le \left[\frac{(p-\partial)[p-j-\gamma+\lambda(j+1)](j-1)}{[\lambda(1+p)-\gamma]p(p-1)}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!}\right]^{\frac{1}{j-p}}. \qquad (1.13)$$

Hence, the theorem follows easily from (1.13). □

5. Extreme Points

The theorem below addresses the extreme points of the class $P(p,\gamma,\lambda)$.

Theorem 5.1. *Let* $f_p(z) = z^p$ *and*

$$f_j(z) = z^p + \sum_{j=p+1}^{\infty} \frac{[p-j-\gamma+\lambda(j+1)]j(j-1)}{[\lambda(1+p)-\gamma]p(p-1)}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!} z^j$$

where $\left(j \ge p+1, p \ge 1, \frac{1}{2} \le \gamma < 1, 0 < \lambda \le \frac{1}{2}\right)$.

Then the function $f(z)$ *belongs to the class* $P(p,\gamma,\lambda)$ *if and only if it can be written as:*

$$f(z) = \mathcal{L}_p z^p + \sum_{j=p+1}^{\infty} \mathcal{L}_j f_j(z), \qquad (1.14)$$

such that

$$(\mathcal{L}_p \ge 0, \mathcal{L}_j \ge 0, j \ge p+1) \text{ and } \mathcal{L}_p + \sum_{j=p+1}^{\infty} \mathcal{L}_j = 1.$$

Proof. Suppose that $f(z)$ thus defined in (1.14), then

$$f(z)=\mathcal{L}_p z^p+\sum_{j=p+1}^{\infty}\mathcal{L}_j\left[z^p+\frac{[\lambda(1+p)-\gamma]p(p-1)}{[p-j-\gamma+\lambda(j+1)]j(j-1)}\left[\frac{p}{p+(j-p)\lambda}\right]^m\frac{(j-p)!}{\Upsilon_{(j-p,v,e)}(a_s,b_r)}z^j\right]$$

$$=z^p+\sum_{j=p+1}^{\infty}\left[\frac{[\lambda(1+p)-\gamma]p(p-1)}{[p-j-\gamma+\lambda(j+1)]j(j-1)}\left[\frac{p}{p+(j-p)\lambda}\right]^m\frac{(j-p)!}{\Upsilon_{(j-p,v,e)}(a_s,b_r)}\right]\mathcal{L}_j z^j.$$

Hence

$$\sum_{j=p+1}^{\infty}\left[\frac{[\lambda(1+p)-\gamma]p(p-1)}{[p-j-\gamma+\lambda(j+1)]j(j-1)}\left[\frac{p}{p+(j-p)\lambda}\right]^m\frac{(j-p)!}{\Upsilon_{(j-p,v,e)}(a_s,b_r)}\right]$$
$$\times\left[\frac{[p-j-\gamma+\lambda(j+1)]j(j-1)}{[\lambda(1+p)-\gamma]p(p-1)}\left[\frac{p+(j-p)\lambda}{p}\right]^m\frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!}\right]\mathcal{L}_j$$
$$=\sum_{j=p+1}^{\infty}\mathcal{L}_j=1-\mathcal{L}_p\le 1.$$

Thus $f\in P(p,\gamma,\lambda)$.

Conversely, suppose that $f(z)\in P(p,\gamma,\lambda)$ we may be setting

$$\mathcal{L}_j=\sum_{j=p+1}^{\infty}\left[\frac{[p-j-\gamma+\lambda(j+1)]j(j-1)}{[\lambda(1+p)-\gamma]p(p-1)}\left[\frac{p+(j-p)\lambda}{p}\right]^m\frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!}\right]a_j,$$

where a_j is defined in (1.10). Then

$$f(z)=z^p+\sum_{j=p+1}^{\infty}a_j z^j=z^p+\sum_{j=p+1}^{\infty}\left[\frac{[\lambda(1+p)-\gamma]p(p-1)}{[p-j-\gamma+\lambda(j+1)]j(j-1)}\left[\frac{p}{p+(j-p)\lambda}\right]^m\frac{(j-p)!}{\Upsilon_{(j-p,v,e)}(a_s,b_r)}\right]\mathcal{L}_j z^j$$

$$=z^p+\sum_{j=p+1}^{\infty}\left[f_j(z)-z^p\right]=\sum_{j=p+1}^{\infty}\mathcal{L}_j f_j(z)+(1-\sum_{j=p+1}^{\infty}\mathcal{L}_j)z^p.$$

Thus

$$f(z)=\mathcal{L}_p z^p+\sum_{j=p+1}^{\infty}\mathcal{L}_j f_j(z).$$

This completes the proof of Theorem 2.1. □

6. Convolution Properties

In this section, we present the following theorems, which explain the convolution properties of functions belonging to the class $P(p,\gamma,\lambda)$.

Theorem 6.1. *Let the functions $f_s(z)$ belong to the class $P(p,\gamma,\lambda)$ such that*

$$f_s(z)=z^p+\sum_{j=p+1}^{\infty}a_{j,s}z^j,\qquad (a_{j,s}\ge 0, s=1,2). \tag{1.15}$$

Then $(f_1*f_2)\in P(p,\gamma,k)$, where

$$k \geq \frac{\begin{array}{c} p(p-1)[p-j-\gamma][(1+\lambda p)-\gamma]^2 p^m (j-p)! \\ +j(j-1)\gamma[p-j-\gamma+(\lambda j+1)]^2[p+(j-p)\lambda]^m \Upsilon_{(j-p,v,e)}(a_s,b_r) \end{array}}{\begin{array}{c} (1+p)j(j-1)[p-j-\gamma+(\lambda j+1)]^2[p+(j-p)\lambda]^m \Upsilon_{(j-p,v,e)}(a_s,b_r) \\ -(j+1)p(p-1)[(1+\lambda p)-\gamma]^2 p^m (j-p)! \end{array}}.$$

The result is sharp for the functions f_s $(s = 1,2)$ *given by* (1.9), *where* $k \in \mathbb{C}/\{0\}$.

Proof. We will find the smallest k such that

$$\sum_{j=p+1}^{\infty} \left[\frac{[p-j-\gamma+k(j+1)]j(j-1)}{[k(1+p)-\gamma]p(p-1)} \left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!} \right] a_{j,1}a_{j,2} \leq 1.$$

Since $f_s \in B(p,\gamma,\lambda), (s = 1,2)$,

$$\sum_{j=p+1}^{\infty} \left[\frac{[p-j-\gamma+(\lambda j+1)]j(j-1)}{[(1+\lambda p)-\gamma]p(p-1)} \left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!} \right] a_{j,s} \leq 1, \quad (s = 1,2).$$

By Cauchy-Schwarz inequality, we get

$$\sum_{j=p+1}^{\infty} \left[\frac{[p-j-\gamma+(\lambda j+1)]j(j-1)}{[(1+\lambda p)-\gamma]p(p-1)} \left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!} \right] \sqrt{a_{j,1}a_{j,2}} \leq 1. \quad (1.16)$$

Now, the only thing we need to prove is that:

$$\left[\frac{[p-j-\gamma+k(j+1)]j(j-1)}{[k(1+p)-\gamma]p(p-1)} \left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!} \right] a_{j,1}a_{j,2}$$
$$\leq \left[\frac{[p-j-\gamma+(\lambda j+1)]j(j-1)}{[(1+\lambda p)-\gamma]p(p-1)} \left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!} \right] \sqrt{a_{j,1}a_{j,2}}$$

and equivalently to:

$$\sqrt{a_{j,1}a_{j,2}} \leq \frac{[k(1+p)-\gamma][p-j-\gamma+(\lambda j+1)]}{[p-j-\gamma+k(j+1)][(1+\lambda p)-\gamma]},$$

from (1.16), we have

$$\sqrt{a_{j,1}a_{j,2}} \leq \frac{1}{\frac{[p-j-\gamma+(\lambda j+1)]j(j-1)}{[(1+\lambda p)-\gamma]p(p-1)} \left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!}},$$

this ends well enough to illustrate that

$$\frac{[(1+\lambda p)-\gamma]p(p-1)p^m(j-p)!}{[p-j-\gamma+(\lambda j+1)]j(j-1)[p+(j-p)\lambda]^m \Upsilon_{(j-p,v,e)}(a_s,b_r)} \leq \frac{[k(1+p)-\gamma][p-j-\gamma+(\lambda j+1)]}{[p-j-\gamma+k(j+1)][(1+\lambda p)-\gamma]},$$

then

$$[p-j-\gamma][(1+\lambda p)-\gamma]^2 p(p-1)p^m(j-p)! + k(j+1)[(1+\lambda p)-\gamma]^2 p(p-1)p^m(j-p)!$$
$$\leq (-\gamma)[p-j-\gamma+(\lambda j+1)]^2 j(j-1)[p+(j-p)\lambda]^m \Upsilon_{(j-p,v,e)}(a_s,b_r)$$
$$+ k(1+p)[p-j-\gamma+(\lambda j+1)]^2 j(j-1)[p+(j-p)\lambda]^m \Upsilon_{(j-p,v,e)}(a_s,b_r),$$

$$k \geq \frac{\begin{array}{c} p(p-1)[p-j-\gamma][(1+\lambda p)-\gamma]^2 p^m (j-p)! \\ +j(j-1)\gamma[p-j-\gamma+(\lambda j+1)]^2[p+(j-p)\lambda]^m \Upsilon_{(j-p,v,e)}(a_s,b_r) \end{array}}{\begin{array}{c} (1+p)j(j-1)[p-j-\gamma+(\lambda j+1)]^2[p+(j-p)\lambda]^m \Upsilon_{(j-p,v,e)}(a_s,b_r) \\ -(j+1)p(p-1)[(1+\lambda p)-\gamma]^2 p^m (j-p)! \end{array}}.$$

Thus, the theorem is established. □

Theorem 6.2. *Let the functions* $f_s(z)$ *defined by* (1.15) *in Theorem* 6.1 *belong to the class* $P(p,\gamma,\lambda)$. *Then the function* $f(z) = z^p + \sum_{j=p+1}^{\infty}(a_{j,1}^2 + a_{j,2}^2)z^j$ *also belongs to the class* $P(p,\gamma,\lambda)$, *where* $p(p+1)[1-(\lambda(p+1)+1)+\gamma] - 2p(p-1)[(\lambda p+1)-\gamma] \geq 0$.

Proof. Since $f_1(z) \in B(p,\gamma,\lambda)$, we get

$$\sum_{j=p+1}^{\infty}\left[\frac{[p-j-\gamma+(\lambda j+1)]j(j-1)}{[(1+\lambda p)-\gamma]p(p-1)}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!}\right]^2 a_{j,1}^2\,,$$

where $\left(j \geq p+1, p \geq 1, \frac{1}{2} \leq \gamma < 1, 0 < \lambda \leq \frac{1}{2}\right)$.

$$\sum_{j=p+1}^{\infty}\left[\frac{[p-j-\gamma+(\lambda j+1)]j(j-1)}{[(1+\lambda p)-\gamma]p(p-1)}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!}\right]^2 a_{j,1}^2$$

$$\leq \left(\sum_{j=p+1}^{\infty}\left[\frac{[p-j-\gamma+(\lambda j+1)]j(j-1)}{[(1+\lambda p)-\gamma]p(p-1)}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!}\right] a_{j,1}\right)^2 \leq 1 \qquad (1.17)$$

and

$$\sum_{j=p+1}^{\infty}\left[\frac{[p-j-\gamma+(\lambda j+1)]j(j-1)}{[(1+\lambda p)-\gamma]p(p-1)}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!}\right]^2 a_{j,2}^2$$

$$\leq \left(\sum_{j=p+1}^{\infty}\left[\frac{[p-j-\gamma+(\lambda j+1)]j(j-1)}{[(1+\lambda p)-\gamma]p(p-1)}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!}\right] a_{j,2}\right)^2 \leq 1, \qquad (1.18)$$

combining the inequalities (1.17) and (1.18) gives

$$\sum_{j=p+1}^{\infty}\frac{1}{2}\left[\frac{[p-j-\gamma+(\lambda j+1)]j(j-1)}{[(1+\lambda p)-\gamma]p(p-1)}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!}\right]^2 (a_{j,1}^2 + a_{j,2}^2) \leq 1.$$

According to Theorem 2.1, it is sufficient to show that

$$\sum_{j=p+1}^{\infty}\left[\frac{[p-j-\gamma+(\lambda j+1)]j(j-1)}{[(1+\lambda p)-\gamma]p(p-1)}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!}\right](a_{j,1}^2 + a_{j,2}^2) \leq 1.$$

Thus, if the last inequality is fulfilled, for $(j = p+1, p+2, p+3, \ldots)$

$$\left[\frac{[p-j-\gamma+(\lambda j+1)]j(j-1)}{[(1+\lambda p)-\gamma]p(p-1)}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!}\right]$$
$$\leq \frac{1}{2}\left[\frac{[p-j-\gamma+(\lambda j+1)]j(j-1)}{[(1+\lambda p)-\gamma]p(p-1)}\left[\frac{p+(j-p)\lambda}{p}\right]^m \frac{\Upsilon_{(j-p,v,e)}(a_s,b_r)}{(j-p)!}\right]^2.$$

Or, if

$$[(1+\lambda p)-\gamma]p(p-1)-2j(j-1)[p-j-\gamma+(\lambda j+1)]\geq 0 \tag{1.19}$$

for $j=p+1,p+2,p+3,\ldots,$ the left-hand side of (1.19) is increasing function of j, hence it is satisfied for all j,

$$p(p+1)[\lambda(p+1)-\gamma]-2[(1+\lambda p)-\gamma]p(p-1)\geq 0,$$

which is true by our assumption therefore the proof is complete. □

7. Conclusion

The main aim was to employ the integration of a generalized derivative operator to construct the subclass $P(p,\gamma,\lambda)$ of multivalent functions in the open unit disk. For functions in this subclass, we established various properties such as, coefficient inequalities, growth and distortion estimates, extreme points, radii of close-to-convexity, starlikeness and convexity.

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