Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 16, Number 2, March 2026, Pages 293-298
https://doi.org/10.34198/ejms.16226.22.293298

Exponential Type Contraction Mapping Theorems for the Banach, Kannan, Chatterjea, Reich, and Hardy-Rogers Operators in Metric Spaces with Application

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Abstract

In this paper, we introduce the notion of an exponential type contraction operator, and prove the Banach, Kannan, Reich, Chatterjea, and Hardy-Rogers fixed point theorem for such operators in the setting of metric spaces. Finally, we apply the exponential Banach contraction mapping theorem to the Fredholm integral equation.

1 Introduction and Preliminaries

Theorem 1.1. *[1] Let (X, d) be a metric space, and suppose $T : X \mapsto X$ is a mapping satisfying the following contractive condition:*

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$ and $k \in [0, 1)$. If (X, d) is complete, then T has a unique fixed point.

Theorem 1.2. *[2] Let (X, d) be a complete metric space, and let $T : X \mapsto X$ be a mapping such that there exists $K < \frac{1}{2}$ satisfying*

$$d(Tx, Ty) \leq K[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$. Then T has a unique fixed point $v \in X$, and for any $x \in X$, the sequence of iterates $\{T^n x\}$ converges to v, and

$$d(T^{n+1}x, v) \leq K \cdot \left(\frac{K}{1-K}\right)^n d(x, Tx), \quad n = 0, 1, 2, \cdots.$$

Theorem 1.3. *[3] Let (X, d) be a metric space. Suppose $T : X \mapsto X$ is a mapping satisfying*

$$d(Tx, Ty) \leq \alpha(d(x, Ty) + d(y, Tx))$$

for all $x, y \in X$ and $\alpha \in [0, \frac{1}{2})$. If (X, d) is complete, then T has a unique fixed point.

Received: December 2, 2025; Accepted: February 18, 2026; Published online: March 6, 2026

2020 Mathematics Subject Classification: 47H10, 54H25.

Keywords and phrases: Banach contraction, Kannan contraction, Reich contraction, Chatterjea contraction, Hardy-Rogers contraction, Fredholm integral equation.

Theorem 1.4. *[4] Let (X, d) be a complete metric space, and let $f : X \mapsto X$ be a Reich type single valued (a, b, c)-contraction, that is, there exists nonnegative numbers a, b, c with $a + b + c < 1$ such that*

$$d(f(x), f(y)) \leq ad(x, y) + bd(x, f(x)) + cd(y, f(y))$$

for each $x, y \in X$. Then f has a unique fixed point.

Theorem 1.5. *[5] Let (X, d) be a complete metric space, and let $T : X \mapsto X$ be a mapping satisfying*

$$d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty) + ed(x, Ty) + fd(y, Tx)$$

for all $x, y \in X$ where $a, b, c, e, f \geq 0$ and $a + b + c + e + f < 1$. Then T has a unique fixed point.

2 Main Result

Theorem 2.1. *(Exponential Hardy-Rogers Type Contraction Mapping Theorem) Let (X, d) be a complete metric space, and let $T : X \mapsto X$ be a mapping satisfying*

$$e^{d(Tx,Ty)} \leq ae^{d(x,y)} + be^{d(x,Tx)} + ce^{d(y,Ty)} + de^{\frac{1}{2}d(x,Ty)} + fe^{d(y,Tx)}$$

for all $x, y \in X$, where $a, b, c, d, f \geq 0$ and $a + b + c + d + f < 1$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be chosen arbitrarily. Define the sequence $\{x_n\}_{n=0}^{\infty}$ recursively by $x_{n+1} = Tx_n$ for all n and also define

$$P_n = e^{d(x_n,x_{n+1})}.$$

Letting $x = x_{n-1}$ and $y = x_n$ in the contractive definition of the theorem, we have

$$\begin{aligned}
e^{d(x_n,x_{n+1})} &\leq ae^{d(x_{n-1},x_n)} + be^{d(x_{n-1},x_n)} + ce^{d(x_n,x_{n+1})} + de^{\frac{1}{2}d(x_{n-1},x_{n+1})} + fe^{d(x_n,x_n)} \\
&\leq ae^{d(x_{n-1},x_n)} + be^{d(x_{n-1},x_n)} + ce^{d(x_n,x_{n+1})} + de^{\frac{1}{2}(d(x_{n-1},x_n)+d(x_n,x_{n+1}))} + f \\
&\leq ae^{d(x_{n-1},x_n)} + be^{d(x_{n-1},x_n)} + ce^{d(x_n,x_{n+1})} + de^{\frac{1}{2}(2d(x_{n-1},x_n))} + f \\
&\leq (a + b + d)e^{d(x_{n-1},x_n)} + (c + f)e^{d(x_n,x_{n+1})}.
\end{aligned}$$

From the above inequality, we deduce that

$$P_n \leq \frac{a + b + d}{1 - (c + f)} P_{n-1} = \gamma P_n,$$

where $\gamma = \frac{a+b+d}{1-(c+f)} < 1$, since $a + b + c + d + f < 1$. It follows by induction that

$$d(x_n, x_{n+1}) < P_n \leq \gamma^n P_0.$$

Thus, making use of the above inequality and the triangle inequality, we obtain for all $n \geq 0$ and $m \geq 1$ that

$$\begin{aligned} d(x_n, x_{n+m}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+m-1}, x_{n+m}) \\ &\leq P_0(\gamma^n + \gamma^{n+1} + \cdots + \gamma^{n+m-1}) \\ &= P_0\gamma^n \frac{1-\gamma^m}{1-\gamma} \\ &\leq P_0 \frac{\gamma^n}{1-\gamma} \to 0 \text{ as } n, m \to \infty. \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d), ensuring its convergence to a point $x^* \in X$. To confirm that x^* is a fixed point, we substitute $x = x^*$ and $y = x_n$ in the contractive definition of the theorem, then we have

$$e^{d(Tx^*, x_{n+1})} \leq ae^{d(x^*, x_n)} + be^{d(x^*, Tx^*)} + ce^{d(x_n, x_{n+1})} + de^{\frac{1}{2}d(x^*, x_{n+1})} + fe^{d(x_n, Tx^*)}.$$

Passing to the limit as $n \to \infty$ in the above inequality, we deduce that

$$\begin{aligned} e^{d(Tx^*, x^*)} &\leq a + be^{d(x^*, Tx^*)} + c + d + fe^{d(x^*, Tx^*)} \\ &= (a + c + d) + (b + f)e^{d(x^*, Tx^*)}. \end{aligned}$$

From the above, we deduce that

$$e^{d(Tx^*, x^*)} \leq \frac{a + c + d}{1 - (b + f)} < 1, \text{ (since } a + b + c + d + f < 1)$$

which implies that $e^{d(Tx^*, x^*)} \leq 1$. It follows that $d(Tx^*, x^*) = 0$, that is, $x^* = Tx^*$, and so x^* is a fixed point of T. For uniqueness, assume that $Tx = x$ and $Ty = y$, with $x \neq y$. Then, from the contractive condition of the theorem, we obtain

$$\begin{aligned} e^{d(x,y)} &= e^{d(Tx,Ty)} \\ &\leq ae^{d(x,y)} + be^{d(x,x)} + ce^{d(y,y)} + de^{\frac{1}{2}d(x,y)} + fe^{d(y,x)} \\ &\leq ae^{d(x,y)} + b + c + de^{d(x,y)} + fe^{d(x,y)} \\ &= (b + c) + (a + d + f)e^{d(x,y)}. \end{aligned}$$

From the above, we obtain

$$e^{d(x,y)} \leq \frac{b + c}{1 - (a + d + f)} < 1 \text{ (since } a + b + c + d + f < 1).$$

It follows that $e^{d(x,y)} \leq 1$ which implies $d(x, y) = 0$, that is, $x = y$, and so the fixed point is unique. This completes the proof. □

If $b = c = d = f = 0$ in the above theorem, then we obtain the following

Corollary 2.2. *(Exponential Banach Type Contraction Mapping Theorem) Let (X,d) be a complete metric space, and let $T : X \mapsto X$ be a mapping satisfying*

$$e^{d(Tx,Ty)} \leq ae^{d(x,y)}$$

for all $x, y \in X$, where $a \in [0,1)$. Then T has a unique fixed point.

If $a = d = f = 0$ in the above theorem, then we obtain the following

Corollary 2.3. *(Exponential Kannan Type Contraction Mapping Theorem) Let (X,d) be a complete metric space. and let $T : X \mapsto X$ be a mapping satisfying*

$$e^{d(Tx,Ty)} \leq be^{d(x,Tx)} + ce^{d(y,Ty)}$$

for all $x, y \in X$, where $b, c \geq 0$ and $b + c < 1$. Then T has a unique fixed point in X.

If $d = f = 0$ in the above theorem, then we obtain the following

Corollary 2.4. *(Exponential Reich Type Contraction Mapping Theorem) Let (X,d) be a complete metric space, and let $T : X \mapsto X$ be a mapping satisfying*

$$e^{d(Tx,Ty)} \leq ae^{d(x,y)} + be^{d(x,Tx)} + ce^{d(y,Ty)}$$

for all $x, y \in X$, where $a, b, c \geq 0$ and $a + b + c < 1$. Then T has a unique fixed point.

If $a = b = c = 0$ in the above theorem, then we obtain the following

Corollary 2.5. *(Exponential Chatterjea Type Contraction Mapping Theorem) Let (X,d) be a complete metric space, and let $T : X \mapsto X$ be a mapping satisfying*

$$e^{d(Tx,Ty)} \leq de^{\frac{1}{2}d(x,Ty)} + fe^{d(y,Tx)}$$

for all $x, y \in X$, where $d, f \geq 0$ and $d + f < 1$. Then T has a unique fixed point.

3 Application

We apply our result to establish an existence theorem for non-linear Fredholm integral equation. Let $Y = C[0,1]$ be a set of all real continuous functions on $[0,1]$ equipped with the metric $p(u,v) = |u - v| = \max_{t\in[0,1]} |u(t) - v(t)|$, for all $u, v \in C[0,1]$. Then (Y,p) is a complete metric space. Now we consider the non-linear Fredholm integral equation

$$u(t) = v(t) + \int_0^1 K(t,s,u(s))ds, \tag{3.1}$$

where $t, s \in [0,1]$. Assume that $K : [0,1] \times [0,1] \times Y \mapsto \mathbb{R}$ and $v : [0,1] \mapsto \mathbb{R}$ are continuous, where $v(t)$ is a given function in Y.

Theorem 3.1. *Suppose* (Y, p) *be a metric space equipped with the metric* $p(u, v) = |u - v| = \max_{t\in[0,1]} |u(t) - v(t)|$ *for all* $u, v \in Y$*, and* $F : Y \mapsto Y$ *be an operator on* Y *defined by*

$$Fu(t) = v(t) + \int_0^1 K(t, s, u(s))ds. \tag{3.2}$$

If there exists $\mu \in [0, 1)$ *such that for all* $u, v \in Y$*,* $s, t \in [0, 1]$ *satisfying the following inequality*

$$e^{|K(t,s,u(s))-K(t,s,v(s))|} \leq \mu M(u(s), v(s)),$$

where

$$M(u(s), v(s)) = e^{|u(s)-v(s)|}.$$

Then the integral equation (3.2) has a unique solution in Y*.*

Proof. From (3.1) and (3.2), we obtain

$$\begin{aligned} e^{|Fu(t)-Fv(t)|} &= e^{\left|\int_0^1 K(t,s,u(s))ds - \int_0^1 K(t,s,v(s))ds\right|} \\ &\leq e^{\int_0^1 |K(t,s,u(s))-K(t,s,v(s))|ds} \\ &\leq \int_0^1 e^{|K(t,s,u(s))-K(t,s,v(s))|} ds \\ &\leq \mu \int_0^1 e^{|u(s)-v(s)|} ds. \end{aligned}$$

Taking the maximum on both sides for all $t \in [0, 1]$, we obtain

$$\begin{aligned} e^{p(Fu,Fv)} &= e^{\max_{t\in[0,1]} |Fu(t)-Fv(t)|} \\ &\leq \mu \max_{t\in[0,1]} \int_0^1 e^{|u(s)-v(s)|} ds \\ &= \mu e^{p(u,v)}. \end{aligned}$$

Since $Y = C[0, 1]$ is complete metric space, all the conditions of Corollary 2.2 are satisfied. Hence, the integral equation (3.2) has a unique solution in Y. □

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