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Computing Radial Anomaly in Kepler's Equation via Weierstrass Elliptic Function

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Abstract

The Weierstrass elliptic function is presented in the light of the astrodynamical equation. We synchronise the Weierstrass elliptic function with the elliptic curve which relates the set of Sato weights with the genus (n, s), where $n < s$ and n, s are co-prime, making all equations homogeneous. The duplication formula of the Weierstrass formula, as previously used in Uwamusi [14], is introduced as an indicator of how the function behaves near and far away from the origin of a complex number. The differential equation satisfied by the Weierstrass function is explained, and the invariant discriminant function of the Weierstrass elliptic function is taken as an important tool. A method for speeding up the computation process in the radial anomaly in Kepler's equation, which provides the time of root passage between a pseudo-time and a stable variable time, is introduced via the Chebyshev–Halley iteration formula of third order in the light of the Weierstrass elliptic function. This thus provides in-depth information regarding the radius of peri-center passage and the real root closest to the exact solution. Also in the paper is a computation of the Schwarzian derivative for the Weierstrass elliptic function. Numerical examples are demonstrated with these methods, and the results obtained are quite impressive.

1 Introduction

The Weierstrass elliptic function has become a prominent tool in applied mathematics and physics, particularly in relativity and astrodynamics. Recent applications include computing constant radial acceleration, analyzing the Stark problem, and solving the two fixed-center problem. The historical development of astrodynamical studies, such as the flywheel problem, traces back to Stark's introduction in particle physics (1914), as discussed by Izzo and Biscani [7].

According to Izzo and Biscani [7], three main problems in astrodynamics rely on the Weierstrass elliptic function:

(i) the constant radial acceleration problem,

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(ii) the Stark problem, and

(iii) the two-point Euler three-body problem.

Central to these is the eccentric anomaly E, used to locate an orbital position after a given elapsed time.

This paper aims to formulate an astrodynamical equation for constant radial acceleration of a point mass in a central gravitational field with additional radial acceleration, using the Weierstrass elliptic function. Numerical computations employ a third-order Chebyshev–Halley method implemented in Python/C++ for MATLAB.

1.1 The preliminaries

The Weierstrass elliptic function is a complex function $\wp(z)$ that is meromorphic with a second-order pole at each lattice point.

Let $\Omega = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$ be a lattice in $\mathbb{C}$. The function $\wp(z)$ is defined in the form:

$$\wp_\Omega(z) = \frac{1}{z^2} + \sum_{\omega \in \Omega \setminus \{0\}}^{\infty} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right). \tag{1.1}$$

Thus, its periodicity with respect to the lattice is of second order for $\wp$ and $\wp'$.

The function $\wp(z)$ has periods $2\omega_1$ and $2\omega_2$, where the set of complex numbers 0, $2\omega_1$, $2\omega_2$, and $2\omega_1 + 2\omega_2$ defines a parallelogram in the Weierstrass elliptic function.

The two different fundamental periods given below are

$$\Omega_1 = a\omega_1 + b\omega_2, \tag{1.2}$$

$$\Omega_2 = c\omega_1 + d\omega_2. \tag{1.3}$$

Forming the area of the parallelogram for the two complex numbers given by

$$A = |\mathrm{Im}(z_1 \overline{z}_2)|,$$

and noting that:

$$\mathrm{Im}(\omega_1\omega_2) = (ad - bc)\,\mathrm{Im}(\omega_1 \overline{\omega}_2),$$

with additional information such that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \pm 1,$$

this describes the modular form of the Weierstrass elliptic function for which

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

Equation (1.1) is then rewritten in the form

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\in\Omega\setminus\{0,0\}}^{\infty} \left\{ \frac{1}{(z+2m\omega_1+2n\omega_2)^2} - \frac{1}{(2m\omega_1+2n\omega_2)^2} \right\}, \tag{1.4}$$

with its derivative for equation (1.4) given as

$$\wp'(z) = -2 \sum_{(m,n)\in\Omega\setminus\{0,0\}}^{\infty} \left\{ \frac{1}{(z+2m\omega_1+2n\omega_2)^3} - \frac{1}{(2m\omega_1+2n\omega_2)^3} \right\}. \tag{1.5}$$

Let the Laurent series for the Weierstrass elliptic function at the point $z = 0$ (see, e.g., Lawden [9]) be defined as

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} \sum_{(m,n)\neq(0,0)} \frac{(k+1)(-1)^k}{(2m\omega_1+2n\omega_2)^{k+2}} z^k = \frac{1}{z^2} + \sum_{k=1}^{\infty} a_k z^k, \tag{1.6}$$

where the coefficients a_k depend only on the lattice.

The invariants g_2, g_3 are defined as

$$g_2 = 60 \sum_{\omega\in\Omega\setminus\{0\}} \omega^{-4}, \tag{1.7}$$

$$g_3 = 140 \sum_{\omega\in\Omega\setminus\{0\}} \omega^{-6}. \tag{1.8}$$

The discriminant function

$$\Delta(z) = g_2^3 - \frac{1}{27} g_3^2 \neq 0, \tag{1.9}$$

plays a fundamental role in characterizing the lattice. The function satisfies

$$\left(\wp'(z)\right)^2 = 4\left(\wp(z)\right)^3 - g_2\wp(z) - g_3 \tag{1.10}$$

which corresponds to an elliptic curve in the (X, Y)-plane with $X = \wp(z)$, $Y = \wp'(z)$.

Additionally, to every (n, s) curve (Elleback [5]) there corresponds the set of Sato weights that makes all the equations homogeneous. These are made possible using the Weierstrass sequence for the (n, s) curves.

1.2 The duplication formula for the Weierstrass elliptic function

The duplication formula in the theory of the Weierstrass elliptic function is a popular approach for solving many complex problems in applied mathematics, physics, and engineering. Basically, the Weierstrass duplication formula uses its derivative and the invariant functions g_2 and g_3 to study the existence behaviour of $\wp(z)$ in cases when the complex variable z is near zero or far away from it. This behavioral pattern was demonstrated in Uwamusi [14]. In the sense of Whittaker and Watson [15], the Weierstrass duplication formula was derived.

Starting from the addition formula for $\wp(z)$ defined as

$$\wp(z+\omega) = -\wp(z) - \wp(\omega) + \left(\frac{1}{2}\frac{\wp'(z)-\wp'(\omega)}{\wp(z)-\wp(\omega)}\right)^2. \tag{1.11}$$

By expressing the second derivative of $\wp(z)$ as

$$\wp''(z) = 6(\wp(z))^2 - \frac{g_2(\Omega)}{2}, \tag{1.12}$$

and further setting

$$\wp(2z) = -2\wp(z) + \left(\frac{1}{2}\frac{\wp''(z)}{\wp'(z)}\right)^2, \tag{1.13}$$

using the substitution $\wp''(z) = 6(\wp(z))^2 - \frac{g_2}{2}$ of equation (1.13) into the given equation below

$$\wp(z) = \frac{1}{z^2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + O(z^6), \tag{1.14}$$

this yields the duplication formula.

$$\wp(2z) = -2\wp(z) + \frac{\left(6\wp^2(z) - \frac{g_2}{2}\right)^2}{4\left(4\wp^3(z) - g_2\wp(z) - g_3\right)}. \tag{1.15}$$

Equation (1.15), when simplified further, leads to the expression

$$\wp(2z) = \frac{1}{4}\wp(z) + \frac{3g_2\wp^2(z) + 9g_3\wp(z) + \frac{g_2^2}{4}}{4\wp'^2(z)}. \tag{1.16}$$

This formula is useful for analyzing behaviour near $z = 0$ and for numerical iterations.

2 Astrodynamical Equation in Weierstrass Form

Consider a polynomial equation of degree four or three

$$P(x) = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4. \tag{2.1}$$

The associated differential equation is

$$\left(\frac{dx}{du}\right)^2 = P(x). \tag{2.2}$$

Under the linear transformation $x = \alpha\wp(\tau) + \beta x = \alpha\wp(\tau) + \beta$, the equation can be mapped to the Weierstrass form (1.10). Comparing coefficients,

$$\sqrt{a_0z^4 + 4a_1z^3 + 6a_2z^2 + 4a_3z + a_4} = \frac{\sqrt{4x^3 - g_2x - g_3}}{(\eta x + \delta)^2}, \tag{2.3}$$

and the integral equation becomes

$$\int R\Big(z, \sqrt{a_0z^4 + 4a_1z^3 + 6a_2z^2 + 4a_3z + a_4}\Big)\, dz = \int R\left(\frac{\alpha x + \beta}{\eta x + \delta}, \frac{\sqrt{4x^3 - g_2x - g_3}}{(\eta x + \delta)^2}\right) \frac{dx}{(\eta x + \delta)^2},$$

leading to the following expression:

$$\int \frac{dz}{\sqrt{a_0z^4 + 4a_1z^3 + 6a_2z^2 + 4a_3z + a_4}} = \int \frac{dx}{\sqrt{4x^3 - g_2x - g_3}}. \tag{2.4}$$

This yields the astrodynamical integral (Brizard [3]) in the form:

$$\tau(x) = \tau_n \pm \int_{x_n}^{x} \frac{1}{\sqrt{a_0s^4 + 4a_1s^3 + 6a_2s^2 + 4a_3s + a_4}}\, ds. \tag{2.5}$$

Equation (2.5) expresses the astrodynamical problem between pseudo-time τ and the state variable x for a polynomial equation of degree four as it relates to equation (2.1) with no repeated root.

The inversion formula corresponding to equation (2.5) is

$$x(\tau) = x_n + \frac{K}{\wp(\tau - \tau_n, g_2, g_3) - H}. \tag{2.6}$$

The x_n which appeared in equation (2.6) is any root of f, and the variables K and H are defined as

$$K = \frac{1}{4} f'(x_0), \qquad H = \frac{1}{24} f''(x_0).$$

The lattice invariants are

$$g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2,$$

$$g_3 = a_0a_2a_4 + 2a_1a_2a_3 - a_2^3 - a_0a_3^2 - a_1^2a_4.$$

We call x_n the time of root passage for a transition period from $x(\tau) = x_n$.

For $\tau = 0$, in equation (2.6) one has that

$$\wp(\tau_n) = K + \frac{H}{x_0 - x_n}, \qquad x_0 = x(0), \tag{2.7}$$

$$x'(\tau) = -\frac{(x(\tau) - x_n)^2}{K}\, \wp'(\tau - \tau_n), \tag{2.8}$$

$$\wp'(\tau_n) = \frac{Kx_0'}{(x_0 - x_n)^2}. \tag{2.9}$$

3 Computational Aspect

In this section we present computed results for the described methods.

3.1 The anomaly

Let a be the semi-major axis of the orbit with center C, which circumscribes the orbit and is tangent to the orbit at periapsis and apoapsis. Let x and y be the respective distances from a point C to D with a units, and from D to S with b units, making an angle E with the horizontal axis. Then

$$x = a\cos E, \qquad y = b\sin E.$$

If b is the semi-minor axis of a circle whose center lies at C and is tangent to the orbit at the point E, taking the values $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, then we solve for $\cos E$ and $\sin E$ to obtain

$$\cos E = \frac{x}{a}, \qquad \sin E = \frac{y}{b}.$$

The eccentricity of the orbit in terms of the semi-major and semi-minor axes is

$$\begin{aligned} e &= \sqrt{1 - \frac{b^2}{a^2}} \\ b^2 &= a^2(1 - e^2) \\ r^2 &= a^2(\cos E - e)^2 + a^2(1 - e^2)\sin^2 E \\ &= a^2(\cos^2 E - 2e\cos E + e^2 + (1 - e^2)\sin^2 E) \\ &= a^2(1 - 2e\cos E + e^2\cos^2 E) \\ &= a^2(1 - e\cos E)^2. \end{aligned}$$

Therefore, taking the square root of both sides, we have

$$r = a(1 - e\cos E).$$

The elapsed time between the initial and terminal points on an orbit is an important tool for computation. Kepler's law relates the degree of closeness between E and pseudo-time. Then the change in the position coordinate is

$$\dot{r} = ae\dot{E}\sin E.$$

Let the true anomaly be v. Then an expression for the change in $\dot{r}$ is

$$\dot{r} = r^2\frac{e\sin v}{\rho}\dot{v},$$

where $\rho = a(1 - e^2)$. By letting $u = GM$, the gravitational parameter of the planet, we obtain

$$\dot{r}e\sqrt{\frac{u}{\rho}} = e\sqrt{\frac{u}{a(1 - e^2)}}\sin v,$$

where

$$a(1 - e\cos E)\dot{E} = \sqrt{\frac{u}{a}},$$

leading to

$$(1 - e\cos E)\dot{E} = \sqrt{\frac{u}{a^3}}.$$

The last equation provides a starting point for deriving a relationship between the eccentric anomaly and time. Considering that $\frac{u}{a^3}$ is approximately constant, we then integrate in the form

$$\int (1 - e\cos E)\, dE = E - e\sin E = \int \sqrt{\frac{u}{a^3}}\, dt = \sqrt{\frac{u}{a^3}}\, t + C.$$

If $E(t_0) = E_0$, then we have

$$E(t_0) - e\sin E(t_0) = E_0 - e\sin E_0 = \sqrt{\frac{u}{a^3}}\, t_0 + C.$$

Solving for C gives

$$C = E_0 - e\sin E_0 - \sqrt{\frac{u}{a^3}}\, t_0.$$

Thus, the final solution is

$$E(t) - e\sin(E) - (E_0 - e\sin E_0) = \sqrt{\frac{u}{a^3}}\, (t - t_0),$$

called Kepler's equation. The quantity

$$M(t) = E(t) - e\sin(E),$$

known as the mean anomaly, can be computed most effectively by a fast iterative zero-finding formula.

3.2 Results

In this section, we concentrate on the computational aspect of Kepler's anomaly and the astrodynamical applications of the Weierstrass elliptic functions. We use the methods of Brent [2], Biscani and Izzo [1], Gutiérrez and Hernández [6], and Tommasini and Olivieri [12] to synchronize the applications of the Weierstrass elliptic function in astrodynamics.

Problem 1. We set out to compute Kepler's anomaly:

$$f(E) = E - e\sin E - M, \qquad E_0 = 0.000125786, \qquad M = 1.0, \qquad e = 0.7.$$

We made use of the Chebyshev–Halley third-order method and the Newton–Brent method, respectively.

Method 1: The Chebyshev method

$$x_{k+1} = x_k - \left(1 + \frac{1}{2}\left(\frac{L(x_k)}{1 - \alpha L(x_k)}\right)\right)\frac{f(x_k)}{f'(x_k)}, \qquad (k = 0, 1, 2, \ldots).$$

where, for instance, $\alpha \in \mathbb{R}$,

$$L(x_k) = \frac{f''(x_k)}{f'(x_k)^2}.$$

The computed results are given in Table 1.

Table 1: Results for the Chebyshev–Halley method

Iteration	x_k	$f(x_k)$	Error estimate
0	1.0	-0.295574	-
1	1.35692	0.01412	0.35692
2	1.34734	0.00003	0.00958
3	1.34730	0.00000	0.00004

Method 2: Newton–Brent Method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \qquad k = 0, 1, 2, 3,$$

$$x_{k+1} = x_k - \frac{f(x_{k-1})f(x_k)}{[f(x_{k-2}) - f(x_{k-1})]\,[f(x_{k-2}) - f(x_k)]}.$$

Table 2: Results for the Newton–Brent Method

Iteration	x_k	$f(x_k)$	Error estimate
0	1.0	-0.295374	-
1	1.35715	0.01405	0.33715
2	1.34730	0.00000	0.00985

Problem 2. In this astrodynamical problem, we take the arbitrary polynomial as a case study:

$$f(x) = x^4 - 3x^2 + 2.$$

The roots are $x = \pm 1, \pm 1.4142$. The lattice invariants are $g_2 = 12$, $g_3 = 8$. The computed results are presented in Table 3.

Table 3: Radial Anomaly (Inverse Weierstrass Time

Root x	Radial Anomaly (Inverse Time)
$-\sqrt{2}$	0.58966
-1	0.58966
1	0.58966
$\sqrt{2}$	0.58966

The computed pericenter root passage can be read from Table 3:

Root closest to 0: $x = -1$, corresponding time = 0.58966.

Problem 3. The polynomial

$$f(x) = x^4 - 2x^3 - x^2 + 2x + 1$$

has both real and complex roots. The roots are

$$x_1 = -0.61803, \qquad x_2 = -0.61803$$

(multiplicity observed numerically).

The lattice invariants are $g_2 = 18$, $g_3 = 10.0$. The computed results are presented in Table 4.

Table 4: Radial Anomaly (Weierstrass inverse time)

Root x	Radial Anomaly (inverse time)
-0.61803	0.58429
-0.61803	0.58432

Table 4 indicates the pericenter root passage and the Weierstrass inverse time:

Pericenter root $x = -0.61803$ and the Weierstrass inverse time = 0.58432.

3.3 Results for Schwarzian derivative

In Koss [8], the Schwarzian derivative for the Weierstrass elliptic function was described without computing the actual value. We therefore computed the Schwarzian derivative, given that z is not a critical point or a pole of a meromorphic function f:

$$S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

Considering

$$\wp(z) = \frac{1}{z^2} + \dots$$

around $z = 1.0$, and using $g_2 = 60$ and $g_3 = 140$ with standard normalization, the results for the Schwarzian derivative of the Weierstrass elliptic function are presented in Table 5.

Table 5: Schwarzian Derivative

z	$\wp(z)$	$\wp'(z)$	$\wp''(z)$	Schwarzian $S(f)(z)$
1.0	1.2337	-2.4563	6.9841	-7.8214
0.5	4.1235	-12.875	66.358	-31.242

4 Discussion

The numerical results computed from sample Problems 1–4 are displayed in Tables 1–4. As a prelude to our computations, Problem 1 consists of an iterative technique for computing stationary roots of a single-variable polynomial equation. This made use of the Chebyshev–Halley third-order method and the Newton–Brent third-order method. The second and third problems compute the Kepler radial anomaly using the Weierstrass elliptic function. The pericenter root describing the root passage to the real root is calculated from Equation (2.6) with the help of the Chebyshev–Halley method, and then the inverse Weierstrass time is obtained. The described Problems 3 and 4 are dynamical problems used in astrodynamics, that is, a flywheel problem. These results are shown in Tables 3 and 4. Table 5 illustrates the computed values for the Schwarzian derivative pertaining to the Weierstrass function. This section describes the Julia set in the Weierstrass elliptic function. The importance of the Schwarzian derivative can be found in the characterization of conformal mappings, its link to differential equations, the analysis of complex dynamics and Julia sets, its relation to modular forms, and in the calculation of arc length. All problems enumerated above were calculated and programmed in Python/C++ for MATLAB routines.

5 Conclusion

The need to compute the radial anomaly in Kepler's equation using the Weierstrass elliptic function was the motivation for this study. After reviewing the preliminaries of the Weierstrass elliptic function, where the use of the invariant discriminant function is an important tool for the analysis of roots leading to the discussion of the duplication formula, we then presented the (n, s) curve as the algebraic equation

$$y^n = x^s + a_{s-1}x^{s-1} + \cdots + a_1 x + a_0,$$

where $n < s$ and n, s are coprime, with genus

$$g = \frac{1}{2}(n-1)(s-1),$$

in the Weierstrass elliptic function framework. We transformed a polynomial equation into the inverse Weierstrass phase function, from which an astrodynamical equation was obtained.

We computed the inverse Weierstrass time for the radial anomaly, and the results obtained are sufficiently accurate for the solution. The numerical toolboxes used were the Chebyshev–Halley and Newton–Brent methods. Also computed is the Schwarzian derivative for the Weierstrass elliptic function. The computed results are sufficiently accurate for the desired precision.

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