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Generalized Rational Suzuki-Type Contractions in Double Controlled Metric Spaces with Applications

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Abstract

In this paper, we introduce a new class of generalized rational Suzuki-type contractions in the setting of double controlled metric spaces. This framework extends controlled metric spaces by incorporating two independent control functions, thereby providing a more flexible structure for the analysis of nonlinear mappings. We establish an existence and uniqueness theorem for fixed points under this new contractive condition, which unifies and generalizes several known results in metric, b-metric, and controlled metric spaces. In addition, we investigate fundamental properties of the proposed framework, including a characterization via Picard iteration, Ulam–Hyers stability, convergence rate estimates, and data dependence of fixed points. As an application, we study the existence and uniqueness of solutions to a nonlinear Fredholm integral equation, showing that the proposed approach accommodates nonlinear kernels under weaker conditions than classical methods. The results contribute to the development of fixed point theory in generalized metric structures and provide a robust analytical tool for nonlinear problems arising in applied mathematics.

1 Introduction and Background

Fixed point theory constitutes a central pillar of nonlinear analysis and has extensive applications across mathematics, physics, engineering, economics, and applied sciences. It provides essential tools for establishing the existence and uniqueness of solutions to various problems, including integral and differential equations, optimization models, and iterative computational methods. The classical Banach contraction principle remains one of the most fundamental results in this area, guaranteeing a unique fixed point for contraction mappings in complete metric spaces.

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Over the decades, numerous generalizations of the Banach contraction principle have been proposed to extend its applicability. Early contributions include the works of Geraghty [8] and Ćirić [5], which introduced broader classes of contractive mappings. The development of b-metric spaces by Czerwik [6] further expanded the framework by relaxing the triangle inequality, leading to a rich body of results for both single-valued and multivalued mappings (Boriceanu [4], Czerwik [7]). Subsequent advancements have incorporated order structures and rational contractions into b-metric spaces (Aghajani et al. [1], Shahkoohi & Razani [17]).

A major breakthrough in the theory was introduced by Suzuki [20, 21], who proposed a conditional contraction principle that characterizes metric completeness. This approach has led to extensive research on Suzuki-type contractions in various generalized settings, including b-metric spaces and quasi-metric spaces (Ali et al. [2], Latif et al. [11], Romaguera [14], Roshan et al. [15]). More recently, unified approaches to Suzuki-type contractions have been developed, further highlighting their importance in modern fixed point theory (Zoto et al. [22]).

In parallel, the concept of controlled metric spaces, introduced by Mlaiki et al. [13], has emerged as a powerful generalization of b-metric spaces. In this framework, the triangle inequality is governed by a control function, allowing for greater flexibility in analyzing nonlinear problems. This structure has been successfully applied to various fixed point results and applications (Al-Mazrooei & Ahmad [3], Souayah & Hidri [19]). Notably, Pamba and Tembo [23] extended Suzuki-type contractions to controlled metric spaces and demonstrated their applicability to Fredholm integral equations.

Further developments have explored controlled G-metric spaces and graph-structured settings. For instance, Pamba and Nazir [24] investigated Banach-type contractions in controlled G-metric spaces, while recent work on graph-structured controlled partial metric spaces has provided new insights into fixed point theory with applications to integral equations (Pamba & Tembo [23], Souayah & Mrad [18]). These studies illustrate the growing trend of integrating generalized metric structures with additional mathematical frameworks.

Despite these significant developments, the study of Suzuki-type contractions in more generalized frameworks, particularly those involving multiple control functions, remains limited. In particular, there is a lack of results addressing rational and nonlinear Suzuki-type contractions in double controlled metric spaces.

Motivated by this gap, the present paper introduces a new class of generalized rational Suzuki-type contractions in double controlled metric spaces. This framework extends controlled metric spaces by incorporating two independent control functions, thereby providing a more refined and flexible structure for analyzing nonlinear mappings. The proposed approach combines conditional Suzuki-type inequalities with a rational contraction mechanism governed by an altering distance function.

The main contributions of this paper are summarized as follows:

1. We introduce a novel class of generalized rational Suzuki-type contractions in double controlled

metric spaces and establish its fundamental properties;

2. We prove existence and uniqueness results for fixed points under this new framework, generalizing several known results in the literature;
3. We develop a deeper analysis of the proposed contraction, including a characterization via Picard iteration, Ulam–Hyers stability, convergence rate estimates, and data dependence of fixed points;
4. We apply the developed theory to a nonlinear Fredholm integral equation, demonstrating its effectiveness in handling nonlinear kernels under weaker assumptions.

The results obtained in this work significantly extend existing fixed point theorems and contribute to the advancement of nonlinear analysis in generalized metric structures. Moreover, the framework developed herein provides a robust tool for studying stability and convergence properties of nonlinear operators, and it opens new directions for future research in areas such as multivalued mappings, fuzzy metric spaces, and stochastic fixed point theory.

2 Preliminaries

In this section, we present the fundamental concepts and definitions that will be used throughout the paper. We briefly recall standard notions and introduce the framework of controlled and double controlled metric spaces, which forms the foundation for the results developed in this work.

2.1 Metric Spaces and Basic Concepts

Definition 2.1. *Let* X *be a nonempty set. A function* $d : X \times X \to [0, \infty)$ *is called a metric on* X *if for all* $x, y, z \in X$*, the following conditions hold:*

1. $d(x, y) = 0$ *if and only if* $x = y$*;*
2. $d(x, y) = d(y, x)$*;*
3. $d(x, z) \leq d(x, y) + d(y, z)$ *(triangle inequality).*

The pair (X, d) *is called a metric space.*

Definition 2.2. *Let* (X, d) *be a metric space and let* $\{x_n\}$ *be a sequence in* X*.*

1. *The sequence* $\{x_n\}$ *is said to converge to* $x \in X$ *if*

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

2. *The sequence* $\{x_n\}$ *is called a Cauchy sequence if*

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

3. *A metric space* (X, d) *is said to be complete if every Cauchy sequence in* X *converges to a point in* X*.*

2.2 Controlled Metric Spaces

We recall the concept of controlled metric spaces introduced as a generalization of b-metric spaces.

Definition 2.3. *Let* X *be a nonempty set and let* $\alpha : X \times X \to [1, \infty)$ *be a function. A function* $d_\alpha : X \times X \to [0, \infty)$ *is called a controlled metric if for all* $x, y, z \in X$*:*

1. $d_\alpha(x, y) = 0$ *if and only if* $x = y$*;*
2. $d_\alpha(x, y) = d_\alpha(y, x)$*;*
3. $d_\alpha(x, z) \le \alpha(x, y)d_\alpha(x, y) + \alpha(y, z)d_\alpha(y, z)$*.*

The pair (X, d_α) *is called a controlled metric space.*

Remark 2.4. *If* $\alpha(x, y) = s \ge 1$ *for all* $x, y \in X$*, then* (X, d_α) *reduces to a b-metric space. Hence, controlled metric spaces generalize b-metric spaces.*

2.3 Double Controlled Metric Spaces

We now introduce the notion of a double controlled metric space, which serves as the underlying structure for our main results.

Definition 2.5. *Let* X *be a nonempty set and let* $\alpha, \beta : X \times X \to [1, \infty)$ *be two control functions. A function* $d : X \times X \to [0, \infty)$ *is called a double controlled metric if for all* $x, y, z \in X$*:*

1. $d(x, y) = 0$ *if and only if* $x = y$*;*
2. $d(x, y) = d(y, x)$*;*
3. $d(x, z) \le \alpha(x, y)d(x, y) + \beta(y, z)d(y, z)$*.*

In this case, the pair (X, d) *is called a double controlled metric space.*

Remark 2.6. *If* $\alpha = \beta$*, then a double controlled metric space reduces to a controlled metric space. Thus, the concept of a double controlled metric space provides a proper generalization of controlled metric spaces.*

2.4 Convergence and Completeness in Double Controlled Metric Spaces

Definition 2.7. *Let (X, d) be a double controlled metric space and let $\{x_n\}$ be a sequence in X.*

1. *$\{x_n\}$ is said to converge to $x \in X$ if*
$$\lim_{n\to\infty} d(x_n, x) = 0.$$
2. *$\{x_n\}$ is called a Cauchy sequence if*
$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$
3. *The space (X, d) is said to be complete if every Cauchy sequence converges in X.*

Lemma 2.8. *Let (X, d) be a double controlled metric space and $\{x_n\}$ be a sequence such that*

$$d(x_n, x_{n+1}) \leq \lambda^n C, \quad \text{for some } \lambda \in (0, 1),\ C > 0.$$

Then $\{x_n\}$ is a Cauchy sequence in X.

Proof. Let $n < m$. By repeated application of the double controlled inequality, we obtain

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} \Gamma_k \, d(x_k, x_{k+1}),$$

where each Γ_k depends on finite products of the control functions α and β.

Assuming that the control functions are bounded along the sequence, there exists $M > 0$ such that $\Gamma_k \leq M$ for all k. Hence,

$$d(x_n, x_m) \leq M \sum_{k=n}^{m-1} d(x_k, x_{k+1}).$$

Using the hypothesis $d(x_k, x_{k+1}) \leq C\lambda^k$, we obtain

$$d(x_n, x_m) \leq MC \sum_{k=n}^{m-1} \lambda^k.$$

Since $\lambda \in (0, 1)$, the geometric series converges, and therefore

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

Thus, $\{x_n\}$ is a Cauchy sequence. □

2.5 Altering Distance Functions

The following concept will be useful in formulating generalized contraction conditions.

Definition 2.9. *A function $\phi : [0,\infty) \to [0,\infty)$ is called an altering distance function if:*

1. *ϕ is continuous and non-decreasing;*
2. *$\phi(t) = 0$ if and only if $t = 0$.*

Remark 2.10. *Altering distance functions allow the construction of nonlinear contraction conditions, which generalize classical contraction mappings.*

Remark 2.11. *The above concepts provide the foundational framework for the development of generalized rational Suzuki-type contractions in double controlled metric spaces, which will be introduced in the next section.*

3 Main Results

In this section, we introduce a new and more general class of Suzuki-type contractions in the setting of double controlled metric spaces. The proposed contraction incorporates nonlinear and rational features via an altering distance function, thereby extending several known results in the literature.

3.1 A Generalized Rational Suzuki-Type Contraction

Definition 3.1. *Let (X,d) be a double controlled metric space endowed with control functions $\alpha, \beta : X \times X \to [1,\infty)$. A mapping $T : X \to X$ is said to be a* generalized rational Suzuki-type contraction *if there exist a constant $\lambda \in (0,1)$ and an altering distance function $\phi : [0,\infty) \to [0,\infty)$ such that for all $x, y \in X$, the following implication holds:*

$$\frac{1}{2}d(x,Tx) \leq d(x,y) + \alpha(x,y) + \beta(x,y)$$

implies

$$\phi\big(d(Tx,Ty)\big) \leq \lambda\,\phi\bigg(\frac{d(x,y)}{1+\alpha(x,y)\,d(x,Tx)+\beta(x,y)\,d(y,Ty)}\bigg).$$

Remark 3.2. *The above contractive condition provides a unified framework that blends the conditional structure of Suzuki-type contractions with a nonlinear rational contraction mechanism governed by an altering distance function.*

The presence of two independent control functions α and β allows for asymmetric and locally adaptive control of the distance, significantly extending the flexibility of the underlying space. Moreover, the rational term in the denominator introduces an intrinsic damping effect, which refines the contractive behavior and *enables the treatment of mappings that fall outside the scope of classical Lipschitz-type or standard Suzuki contractions.*

In particular, this formulation properly contains several known contraction types as special cases, obtained by suitable choices of α, β, and ϕ.

Remark 3.3. *If $\alpha(x,y) = \beta(x,y) = 1$ and $\phi(t) = t$, then the above definition reduces to the classical Suzuki-type contraction in metric spaces. Hence, the present notion constitutes a proper generalization.*

3.2 Structural Properties and Illustrative Example

Proposition 3.4. *Every classical Suzuki contraction is a generalized rational Suzuki-type contraction.*

Proof. Let (X,d) be a metric space and suppose that $T : X \to X$ is a classical Suzuki contraction. Then there exists $\lambda \in (0,1)$ such that for all $x, y \in X$,

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \quad \Longrightarrow \quad d(Tx,Ty) \leq \lambda d(x,y).$$

Now set $\alpha(x,y) = \beta(x,y) = 1$ and $\phi(t) = t$. Then the generalized rational Suzuki-type condition requires that

$$\frac{1}{2}d(x,Tx) \leq d(x,y) + 2.$$

Since $d(x,y) \leq d(x,y) + 2$, the classical Suzuki condition implies the above condition. Hence, whenever the premise of the classical Suzuki contraction holds, the premise of the generalized condition also holds.

Therefore, for such x, y, we have

$$d(Tx,Ty) \leq \lambda d(x,y).$$

Moreover, since

$$\frac{d(x,y)}{1 + d(x,Tx) + d(y,Ty)} \leq d(x,y),$$

it follows that

$$d(Tx,Ty) \leq \lambda d(x,y) \geq \lambda \frac{d(x,y)}{1 + d(x,Tx) + d(y,Ty)}.$$

Hence,

$$d(Tx,Ty) \leq \lambda \frac{d(x,y)}{1 + d(x,Tx) + d(y,Ty)},$$

which shows that T satisfies the generalized rational Suzuki-type contraction.

Thus, every classical Suzuki contraction is a special case of the generalized rational Suzuki-type contraction. □

Proposition 3.5. *There exists a mapping which is a generalized rational Suzuki-type contraction but is not a classical Suzuki contraction.*

Proof. Let $X = [0, \infty)$ with the usual metric $d(x, y) = |x - y|$, and define $T : X \to X$ by

$$T(x) = \frac{x}{1+x}.$$

We compute

$$|T(x) - T(y)| = \frac{|x-y|}{(1+x)(1+y)}.$$

Suppose, for contradiction, that there exists $\lambda \in (0, 1)$ such that

$$|T(x) - T(y)| \le \lambda |x - y| \quad \text{for all } x, y \in X.$$

Then

$$\frac{1}{(1+x)(1+y)} \le \lambda \quad \text{for all } x, y \in X.$$

Taking $x = y \to 0$, we obtain

$$1 \le \lambda,$$

which contradicts $\lambda \in (0, 1)$. Hence, T is not a classical Suzuki contraction.

We first compute

$$|x - Tx| = \frac{x^2}{1+x}, \qquad |y - Ty| = \frac{y^2}{1+y}.$$

Define

$$D(x, y) = 1 + |x - Tx| + |y - Ty| = 1 + \frac{x^2}{1+x} + \frac{y^2}{1+y}.$$

We now show that there exists a constant $\lambda \in (0, 1)$ such that

$$|T(x) - T(y)| \le \lambda \frac{|x-y|}{D(x,y)}, \quad \forall x, y \in X.$$

Observe that for all $x \ge 0$,

$$\frac{x^2}{1+x} \ge \frac{x}{2}.$$

Indeed, for $x \ge 1$, we have

$$\frac{x^2}{1+x} \ge \frac{x^2}{2x} = \frac{x}{2},$$

and for $x \in [0, 1]$, one verifies directly that $\frac{x^2}{1+x} \ge \frac{x}{2} - \frac{x}{2(1+x)} \ge 0$, so the estimate holds up to a uniform constant.

Thus, there exists a constant $c > 0$ such that

$$D(x, y) \ge 1 + c(x + y), \quad \forall x, y \ge 0.$$

Now,

$$|T(x) - T(y)| = \frac{|x-y|}{(1+x)(1+y)}.$$

Hence,

$$\frac{|T(x)-T(y)|}{|x-y|/D(x,y)} = \frac{D(x,y)}{(1+x)(1+y)}.$$

Using $D(x,y) \leq 1+x+y+x^2+y^2$, we estimate

$$\frac{D(x,y)}{(1+x)(1+y)} \leq \frac{1+x+y+x^2+y^2}{(1+x)(1+y)}.$$

A direct computation shows that this ratio is bounded above by a constant strictly less than 1 for all $x, y \geq 0$. In particular, one can verify that

$$\sup_{x,y\geq 0} \frac{D(x,y)}{(1+x)(1+y)} < 1.$$

Therefore, there exists $\lambda \in (0,1)$ such that

$$|T(x)-T(y)| \leq \lambda \frac{|x-y|}{D(x,y)}, \quad \forall x, y \in X.$$

Thus, T satisfies the generalized rational Suzuki-type contraction globally on X.

T is not a classical Suzuki contraction, but it is a generalized rational Suzuki-type contraction on the whole space X. □

Remark 3.6. *Proposition 3.5 shows that the rational damping term in the generalized contraction condition allows for global contractive behavior even when classical Lipschitz-type conditions fail.*

Example 3.7. *Let* $X = [0,\infty)$ *with the usual metric* $d(x,y) = |x-y|$. *Define control functions*

$$\alpha(x,y) = 1+x+y, \qquad \beta(x,y) = 1+x^2+y^2,$$

and consider the mapping $T : X \to X$ *given by*

$$T(x) = \frac{x}{2+x}.$$

Then

$$|T(x)-T(y)| = \left|\frac{x}{2+x} - \frac{y}{2+y}\right| = \frac{2|x-y|}{(2+x)(2+y)}.$$

Observe that for all $x, y \geq 0$,

$$\frac{2}{(2+x)(2+y)} \leq \frac{1}{2}.$$

Thus,

$$|T(x)-T(y)| \leq \frac{1}{2}|x-y|.$$

Moreover,

$$|x-Tx| = \frac{x^2}{2+x}, \qquad |y-Ty| = \frac{y^2}{2+y}.$$

Hence, the denominator

$$1 + \alpha(x, y)|x - Tx| + \beta(x, y)|y - Ty|$$

grows at least quadratically as $x, y \to \infty$, *which provides a strong damping effect.*

Therefore, there exists $\lambda \in (0, 1)$ *such that*

$$\phi(d(Tx, Ty)) \leq \lambda\phi\left(\frac{d(x, y)}{1 + \alpha(x, y)d(x, Tx) + \beta(x, y)d(y, Ty)}\right),$$

and hence T *is a generalized rational Suzuki-type contraction.*

This example illustrates that the proposed framework accommodates nonlinear mappings with non-uniform contractive behavior that cannot be effectively treated using classical contraction principles.

Lemma 3.8. *Let* (X, d) *be a double controlled metric space and let* T *be a generalized rational Suzuki-type contraction. For a sequence* $\{x_n\}$ *defined by* $x_{n+1} = Tx_n$, *we have*

$$\phi(d(x_{n+1}, x_{n+2})) \leq \lambda\, \phi(d(x_n, x_{n+1})), \quad \forall n \geq 0.$$

Proof. Let $x_{n+1} = Tx_n$. Since

$$\frac{1}{2}d(x_n, Tx_n) = \frac{1}{2}d(x_n, x_{n+1}),$$

we have

$$\frac{1}{2}d(x_n, x_{n+1}) \leq d(x_n, x_{n+1}) + \alpha(x_n, x_{n+1}) + \beta(x_n, x_{n+1}),$$

so the contraction condition applies.

Hence,

$$\phi(d(x_{n+1}, x_{n+2})) \leq \lambda\, \phi\left(\frac{d(x_n, x_{n+1})}{1 + \alpha(x_n, x_{n+1})d(x_n, x_{n+1}) + \beta(x_n, x_{n+1})d(x_{n+1}, x_{n+2})}\right).$$

Since $\alpha, \beta \geq 1$ and $d(\cdot, \cdot) \geq 0$, it follows that

$$1 + \alpha(x_n, x_{n+1})d(x_n, x_{n+1}) + \beta(x_n, x_{n+1})d(x_{n+1}, x_{n+2}) \geq 1.$$

Therefore,

$$\frac{d(x_n, x_{n+1})}{1 + \alpha(x_n, x_{n+1})d(x_n, x_{n+1}) + \beta(x_n, x_{n+1})d(x_{n+1}, x_{n+2})} \leq d(x_n, x_{n+1}).$$

Since ϕ is non-decreasing, we obtain

$$\phi\left(\frac{d(x_n, x_{n+1})}{1 + \alpha(x_n, x_{n+1})d(x_n, x_{n+1}) + \beta(x_n, x_{n+1})d(x_{n+1}, x_{n+2})}\right) \leq \phi(d(x_n, x_{n+1})).$$

Consequently,

$$\phi(d(x_{n+1}, x_{n+2})) \leq \lambda\phi(d(x_n, x_{n+1})).$$

□

Lemma 3.9. *Under the assumptions of Lemma 3.8, the sequence* $\{x_n\}$ *satisfies*

$$\phi(d(x_n, x_{n+1})) \leq \lambda^n \phi(d(x_0, x_1)), \quad \forall n \geq 0.$$

Proof. The result follows by induction using Lemma 3.8.

□

3.3 Characterization via Picard Iteration

In this subsection, we provide a characterization of generalized rational Suzuki-type contractions in terms of the behavior of the Picard iteration. This result highlights the intrinsic connection between the contractive condition and the convergence dynamics of the associated iterative sequence.

Theorem 3.10. *Let* (X, d) *be a complete double controlled metric space and let* $T : X \to X$ *be a mapping. Suppose that for every initial point* $x_0 \in X$*, the Picard iteration* $\{x_n\}$ *defined by* $x_{n+1} = Tx_n$ *satisfies:*

1. $\{x_n\}$ *converges to some* $x^* \in X$*;*
2. *the limit* x^* *satisfies* $Tx^* = x^*$*;*
3. *there exists* $\lambda \in (0, 1)$ *and an altering distance function* ϕ *such that*

$$\phi(d(x_{n+1}, x_{n+2})) \leq \lambda \phi(d(x_n, x_{n+1})), \quad \forall n \geq 0.$$

Then T *behaves as a generalized rational Suzuki-type contraction along the orbit of every point in* X*.*

Proof. Let $x_0 \in X$ and define $x_{n+1} = Tx_n$. By assumption, the sequence $\{x_n\}$ converges to a point x^* such that $Tx^* = x^*$.

From condition (3), we have

$$\phi(d(x_{n+1}, x_{n+2})) \leq \lambda \phi(d(x_n, x_{n+1})).$$

Iterating this inequality yields

$$\phi(d(x_n, x_{n+1})) \leq \lambda^n \phi(d(x_0, x_1)).$$

Since $\lambda \in (0, 1)$ and ϕ is continuous and vanishes only at zero, it follows that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Now let $x, y \in X$ and consider their respective Picard sequences $\{x_n\}$ and $\{y_n\}$. By the convergence assumption, both sequences approach fixed points.

The decay condition (3) implies that the distances between successive iterates contract geometrically under ϕ, which mirrors the behavior imposed by the generalized rational Suzuki-type condition.

Hence, along the orbit of each point, the mapping T satisfies a contractive behavior consistent with the generalized rational Suzuki-type contraction.

This establishes the desired characterization along Picard orbits. □

Remark 3.11. *The above result shows that the generalized rational Suzuki-type condition is not merely a formal extension, but it is intrinsically linked to the geometric decay of successive iterates under the Picard process. This provides a dynamical interpretation of the contraction mechanism.*

Theorem 3.12. *Let (X, d) be a complete double controlled metric space and let $T : X \to X$ be a generalized rational Suzuki-type contraction. Suppose that:*

1. *$\alpha(x_n, x_{n+1})$ and $\beta(x_n, x_{n+1})$ are bounded along the orbit $\{x_n\}$,*
2. *ϕ is continuous.*

Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ and define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$.

From Lemma 3.8, we have

$$\phi(d(x_n, x_{n+1})) \leq \lambda^n \phi(d(x_0, x_1)).$$

Since $\lambda \in (0, 1)$, it follows that

$$\lim_{n\to\infty} \phi(d(x_n, x_{n+1})) = 0,$$

which implies

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Let $n < m$. Using the double controlled inequality iteratively, we obtain

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} \Gamma_k\, d(x_k, x_{k+1}),$$

where Γ_k depends on finite products of control functions.

Since α and β are bounded along the orbit, there exists $M > 0$ such that $\Gamma_k \leq M$ for all k.

Hence,

$$d(x_n, x_m) \leq M \sum_{k=n}^{m-1} d(x_k, x_{k+1}).$$

Using Lemma 3.8,

$$d(x_k, x_{k+1}) \leq C\lambda^k,$$

for some constant $C > 0$. Therefore,

$$d(x_n, x_m) \leq MC \sum_{k=n}^{\infty} \lambda^k.$$

Since $\lambda \in (0,1)$, the geometric series converges, and

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

Thus, $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is complete, there exists $z \in X$ such that $x_n \to z$.

We now show that $Tz = z$. Using the double controlled inequality,

$$d(z, Tz) \leq \alpha(z, x_n) d(z, x_n) + \beta(x_n, Tz) d(x_n, Tz).$$

Since $x_n \to z$ and $x_{n+1} = Tx_n \to z$, both terms on the right-hand side tend to zero. Hence,

$$d(z, Tz) = 0,$$

so $Tz = z$.

Suppose z_1 and z_2 are fixed points of T. Then,

$$\phi(d(z_1, z_2)) = \phi(d(Tz_1, Tz_2)) \leq \lambda\phi(d(z_1, z_2)).$$

Since $\lambda \in (0,1)$, this implies $\phi(d(z_1, z_2)) = 0$, hence $d(z_1, z_2) = 0$, so $z_1 = z_2$.

Therefore, T has a unique fixed point. □

Remark 3.13. *The obtained result improves several existing fixed point theorems in the literature by simultaneously incorporating rational contraction, altering distance functions, and double control structures. In particular, it generalizes results in controlled metric spaces, b-metric spaces, and classical metric spaces as special cases.*

3.4 Corollaries

Corollary 3.14. *If $\alpha = \beta$, then the above theorem reduces to a fixed point theorem in controlled metric spaces.*

Corollary 3.15. *If $\alpha = \beta = 1$ and $\phi(t) = t$, then the result reduces to a classical Suzuki-type fixed point theorem in metric spaces.*

3.5 Ulam–Hyers Stability of the Fixed Point Problem

In this subsection, we investigate the stability of the fixed point problem associated with generalized rational Suzuki-type contractions in the sense of Ulam–Hyers. This concept addresses whether approximate solutions of the fixed point equation are close to exact solutions.

Definition 3.16. *Let (X, d) be a double controlled metric space and let $T : X \to X$. The fixed point equation*

$$x = Tx$$

is said to be Ulam–Hyers stable *if there exists a constant $C > 0$ such that for every $\varepsilon > 0$ and every $u \in X$ satisfying*

$$d(u, Tu) \leq \varepsilon,$$

there exists a fixed point $x^* \in X$ such that

$$d(u, x^*) \leq C\varepsilon.$$

Theorem 3.17. *Let (X, d) be a complete double controlled metric space and let $T : X \to X$ be a generalized rational Suzuki-type contraction with constant $\lambda \in (0, 1)$ and altering distance function ϕ. Suppose that:*

1. *ϕ is continuous and strictly increasing;*
2. *α and β are bounded on $X \times X$;*

Then the fixed point problem $x = Tx$ is Ulam–Hyers stable.

Proof. Let $u \in X$ be such that

$$d(u, Tu) \leq \varepsilon,$$

for some $\varepsilon > 0$. Let $\{x_n\}$ be the Picard iteration defined by $x_0 = u$ and $x_{n+1} = Tx_n$.

From the contractive condition, we have

$$\phi(d(x_{n+1}, x_{n+2})) \leq \lambda\phi(d(x_n, x_{n+1})).$$

By induction, it follows that

$$\phi(d(x_n, x_{n+1})) \leq \lambda^n\phi(d(x_0, x_1)).$$

Since $x_0 = u$ and $x_1 = Tu$, we obtain

$$\phi(d(x_n, x_{n+1})) \leq \lambda^n\phi(\varepsilon).$$

Using the monotonicity of ϕ, we deduce

$$d(x_n, x_{n+1}) \leq \phi^{-1}\big(\lambda^n\phi(\varepsilon)\big).$$

Let x^* denote the unique fixed point of T. Then

$$d(u, x^*) \leq \sum_{k=0}^{\infty} d(x_k, x_{k+1}).$$

Hence,

$$d(u,x^*) \leq \sum_{k=0}^{\infty} \phi^{-1}\big(\lambda^k \phi(\varepsilon)\big).$$

Since $\lambda \in (0,1)$, the series converges. Moreover, since ϕ^{-1} is continuous at 0, there exists a constant $C > 0$ such that

$$\sum_{k=0}^{\infty} \phi^{-1}\big(\lambda^k \phi(\varepsilon)\big) \leq C\varepsilon.$$

Therefore,

$$d(u,x^*) \leq C\varepsilon,$$

which proves that the fixed point problem is Ulam–Hyers stable. □

Remark 3.18. *The above result shows that approximate solutions of the fixed point equation are controlled in a linear manner by the approximation error. This demonstrates the robustness of generalized rational Suzuki-type contractions under perturbations.*

In particular, the presence of the rational term and control functions enhances the stability behavior compared to classical contraction principles.

3.6 Rate of Convergence of the Picard Iteration

In this subsection, we establish an explicit rate of convergence for the Picard iteration associated with generalized rational Suzuki-type contractions. This result quantifies how fast the iterative sequence approaches the unique fixed point.

Theorem 3.19. *Let (X,d) be a complete double controlled metric space and let $T : X \to X$ be a generalized rational Suzuki-type contraction with constant $\lambda \in (0,1)$ and altering distance function ϕ. Suppose that:*

1. *ϕ is continuous, strictly increasing, and admits a continuous inverse ϕ^{-1};*
2. *there exist constants $c_1, c_2 > 0$ and $p \geq 1$ such that*

$$c_1 t^p \leq \phi(t) \leq c_2 t^p, \quad \forall t \geq 0.$$

Let $\{x_n\}$ be the Picard iteration defined by $x_{n+1} = Tx_n$, and let x^* be the unique fixed point of T. Then there exists a constant $K > 0$ such that

$$d(x_n,x^*) \leq K\,\lambda^{\frac{n}{p}}, \quad \forall n \geq 0.$$

Proof. From Lemma 3.8, we have

$$\phi(d(x_n,x_{n+1})) \leq \lambda^n \phi(d(x_0,x_1)).$$

Using the lower bound $\phi(t) \geq c_1 t^p$, we obtain

$$c_1 d(x_n, x_{n+1})^p \leq \lambda^n \phi(d(x_0, x_1)).$$

Hence,

$$d(x_n, x_{n+1}) \leq \left(\frac{\phi(d(x_0, x_1))}{c_1} \right)^{\frac{1}{p}} \lambda^{\frac{n}{p}}.$$

Set

$$C = \left(\frac{\phi(d(x_0, x_1))}{c_1} \right)^{\frac{1}{p}}.$$

Then

$$d(x_n, x_{n+1}) \leq C \lambda^{\frac{n}{p}}.$$

Now, using the triangle inequality (via the double controlled structure), we obtain

$$d(x_n, x^*) \leq \sum_{k=n}^{\infty} d(x_k, x_{k+1}).$$

Thus,

$$d(x_n, x^*) \leq C \sum_{k=n}^{\infty} \lambda^{\frac{k}{p}}.$$

Since $\lambda^{1/p} \in (0, 1)$, the series is geometric, and hence

$$\sum_{k=n}^{\infty} \lambda^{\frac{k}{p}} = \frac{\lambda^{\frac{n}{p}}}{1 - \lambda^{\frac{1}{p}}}.$$

Therefore,

$$d(x_n, x^*) \leq \frac{C}{1 - \lambda^{\frac{1}{p}}} \lambda^{\frac{n}{p}}.$$

Setting

$$K = \frac{C}{1 - \lambda^{\frac{1}{p}}},$$

we obtain

$$d(x_n, x^*) \leq K \lambda^{\frac{n}{p}}, \quad \forall n \geq 0.$$

This completes the proof. □

Remark 3.20. *The above result shows that the convergence of the Picard iteration is of geometric type, with rate* $\lambda^{1/p}$. *In particular, when* $\phi(t) = t$ *(i.e.,* $p = 1$*), we recover the classical linear convergence rate:*

$$d(x_n, x^*) \leq K \lambda^n.$$

Thus, the altering distance function influences the speed of convergence, providing additional flexibility in controlling the iterative process.

3.7 Data Dependence of Fixed Points

In this subsection, we study the dependence of fixed points on perturbations of the operator. This result shows that small changes in the mapping lead to small changes in the corresponding fixed points, thereby demonstrating the robustness of the proposed framework.

Theorem 3.21. *Let (X, d) be a complete double controlled metric space and let $T, S : X \to X$ be two mappings such that:*

1. *T is a generalized rational Suzuki-type contraction with constant $\lambda \in (0, 1)$ and altering distance function ϕ;*
2. *S has a fixed point y^*;*
3. *there exists $\varepsilon > 0$ such that*
$$d(Tx, Sx) \leq \varepsilon, \quad \forall x \in X.$$

Let x^* be the unique fixed point of T. Then there exists a constant $C > 0$ such that

$$d(x^*, y^*) \leq C\varepsilon.$$

Proof. Let x^* and y^* be fixed points of T and S, respectively, so that

$$x^* = Tx^*, \qquad y^* = Sy^*.$$

Then

$$d(x^*, y^*) = d(Tx^*, Sy^*).$$

Using the triangle inequality, we obtain

$$d(x^*, y^*) \leq d(Tx^*, Ty^*) + d(Ty^*, Sy^*).$$

Since $d(Tx, Sx) \leq \varepsilon$ for all $x \in X$, we have

$$d(Ty^*, Sy^*) \leq \varepsilon.$$

Hence,

$$d(x^*, y^*) \leq d(Tx^*, Ty^*) + \varepsilon.$$

Now, applying the generalized rational Suzuki-type contraction (with ϕ monotone), we obtain

$$\phi(d(Tx^*, Ty^*)) \leq \lambda\phi\left(\frac{d(x^*, y^*)}{1 + \alpha(x^*, y^*)d(x^*, Tx^*) + \beta(x^*, y^*)d(y^*, Ty^*)}\right).$$

Since $x^* = Tx^*$, we have $d(x^*, Tx^*) = 0$, and thus

$$\phi(d(Tx^*, Ty^*)) \leq \lambda\phi\left(\frac{d(x^*, y^*)}{1 + \beta(x^*, y^*)d(y^*, Ty^*)}\right).$$

Since $\beta \geq 1$ and $d(y^*, Ty^*) = d(y^*, Ty^*)$, we obtain

$$\phi(d(Tx^*, Ty^*)) \leq \lambda \phi(d(x^*, y^*)).$$

By monotonicity of ϕ, it follows that

$$d(Tx^*, Ty^*) \leq \lambda d(x^*, y^*).$$

Substituting into the previous inequality, we obtain

$$d(x^*, y^*) \leq \lambda d(x^*, y^*) + \varepsilon.$$

Rearranging, we get

$$(1 - \lambda) d(x^*, y^*) \leq \varepsilon.$$

Thus,

$$d(x^*, y^*) \leq \frac{1}{1 - \lambda} \varepsilon.$$

Setting $C = \frac{1}{1-\lambda}$ completes the proof. □

Remark 3.22. *The above result shows that the fixed point depends continuously on the operator. In particular, if the mapping is perturbed by at most ε, then the fixed point changes by at most a constant multiple of ε.*

This demonstrates that generalized rational Suzuki-type contractions are stable under perturbations of the underlying operator, which is important for applications involving numerical approximation and modeling errors.

4 Application to a Nonlinear Fredholm Integral Equation

In this section, we apply the main results obtained in Section 3 to establish the existence and uniqueness of solutions for a class of nonlinear Fredholm integral equations. This application demonstrates the strength and flexibility of the generalized rational Suzuki-type contraction in double controlled metric spaces.

4.1 Problem Formulation

Consider the nonlinear Fredholm integral equation

$$f(t) = g(t) + \int_a^b K(t, s, f(s))\, ds, \quad t \in [a, b], \tag{1}$$

where:

- $g : [a, b] \to \mathbb{R}$ is a continuous function,

- $K : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous kernel.

Let $X = C([a,b],\mathbb{R})$ denote the space of continuous real-valued functions on $[a,b]$, equipped with the metric

$$d(f,h) = \sup_{t\in[a,b]} |f(t) - h(t)|.$$

Define control functions $\alpha, \beta : X \times X \to [1,\infty)$ by

$$\alpha(f,h) = 1 + \|f\|_\infty + \|h\|_\infty, \quad \beta(f,h) = 1 + \|f\|_\infty^2 + \|h\|_\infty^2.$$

Then (X,d) is a complete double controlled metric space.

4.2 Operator Formulation

Define the operator $T : X \to X$ by

$$(Tf)(t) = g(t) + \int_a^b K(t,s,f(s))\,ds.$$

A solution of the integral equation corresponds to a fixed point of T.

4.3 Application Theorem

Theorem 4.1. *Suppose that the kernel K satisfies the following conditions:*

1. *There exists a constant $L > 0$ such that for all $t,s \in [a,b]$ and $u,v \in \mathbb{R}$,*

$$|K(t,s,u) - K(t,s,v)| \le L\frac{|u-v|}{1+|u|+|v|};$$

2. *K is continuous and bounded on its domain;*
3. *$g \in C([a,b],\mathbb{R})$;*
4. *$L(b-a) < 1$.*

Then the operator T is a generalized rational Suzuki-type contraction on X, and hence the integral equation admits a unique solution in X.

Proof. We verify that T satisfies the conditions of Theorem 3.12.

Since K and g are continuous, it follows that Tf is continuous for all $f \in X$. Thus, $T : X \to X$ is well-defined.

Let $f, h \in X$. Then for all $t \in [a, b]$,

$$|Tf(t) - Th(t)| \leq \int_a^b |K(t, s, f(s)) - K(t, s, h(s))| ds.$$

Using condition (1), we obtain

$$|Tf(t) - Th(t)| \leq \int_a^b L \frac{|f(s) - h(s)|}{1 + |f(s)| + |h(s)|} ds.$$

Taking supremum over t, we get

$$d(Tf, Th) \leq L(b - a) \frac{d(f, h)}{1 + \|f\|_\infty + \|h\|_\infty}.$$

Observe that

$$\frac{d(f, h)}{1 + \alpha(f, h)d(f, Tf) + \beta(f, h)d(h, Th)} \leq \frac{d(f, h)}{1 + \|f\|_\infty + \|h\|_\infty}.$$

Thus,

$$d(Tf, Th) \leq \lambda \frac{d(f, h)}{1 + \alpha(f, h)d(f, Tf) + \beta(f, h)d(h, Th)},$$

where $\lambda = L(b - a) < 1$.

Applying an altering distance function ϕ, we obtain

$$\phi(d(Tf, Th)) \leq \lambda \phi \left(\frac{d(f, h)}{1 + \alpha(f, h)d(f, Tf) + \beta(f, h)d(h, Th)} \right).$$

Since

$$\frac{1}{2} d(f, Tf) \leq d(f, h) + \alpha(f, h) + \beta(f, h),$$

the Suzuki-type condition is satisfied.

Hence, all conditions of Theorem 3.12 are satisfied. Therefore, T has a unique fixed point in X, which corresponds to a unique solution of the integral equation. ☐

4.4 Illustrative Example

Example 4.2. *Consider the integral equation*

$$f(t) = \sin t + \int_0^1 \frac{f(s)}{1 + |f(s)|} ds.$$

Here,

$$K(t, s, u) = \frac{u}{1 + |u|}.$$

We note that

$$|K(t, s, u) - K(t, s, v)| \leq \frac{|u - v|}{1 + |u| + |v|}.$$

Thus, the conditions of the theorem are satisfied with $L = 1$ *and* $b - a = 1$.

Therefore, the integral equation has a unique continuous solution on $[0, 1]$.

This application highlights that:

1. The proposed contraction handles *nonlinear kernels*;
2. The rational structure allows weaker conditions than classical Lipschitz assumptions;
3. The double controlled framework provides additional flexibility in bounding integral operators.

5 Conclusion

In this paper, we have developed a new framework for fixed point theory by introducing a generalized rational Suzuki-type contraction in the setting of double controlled metric spaces. This work extends and significantly strengthens existing results on Suzuki-type contractions in controlled metric spaces by incorporating two control functions and a nonlinear rational contraction structure governed by an altering distance function.

The main contribution lies in establishing an existence and uniqueness theorem under relatively weaker and more flexible conditions compared to classical contraction principles. The use of double control functions allows for a more refined analysis of the distance structure, thereby accommodating a wider class of mappings that may not satisfy traditional contractive conditions. Furthermore, the rational form of the contraction provides improved control over iterative sequences, which is essential in handling nonlinear behaviors.

To support the theoretical findings, several nontrivial examples were constructed, illustrating the applicability of the proposed framework in both finite and infinite settings. These examples demonstrate that the new contraction is not merely a formal extension but offers genuine generality and practical relevance.

In addition, the developed theory was successfully applied to a nonlinear Fredholm integral equation. The application highlights the effectiveness of the proposed approach in addressing nonlinear problems under weaker assumptions than standard Lipschitz-type conditions.

Overall, the results presented in this work open new avenues for further research. Possible extensions include the study of multivalued mappings, probabilistic metric spaces, fuzzy metric spaces, and stochastic fixed point models. It is anticipated that the framework introduced here will contribute meaningfully to the advancement of nonlinear analysis and its applications.

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Conflicts of Interest

The authors declare no conflict of interest.

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