Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 16, Number 5, September 2026, Pages 991-1007
https://doi.org/10.34198/ejms.16526.63.9911007

A Chebyshev Derived Numerical Block Method for Solving Fourth-Order ODEs

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Abstract

This study presents a numerical block method by using Chebyshev polynomials of the first kind as the basis function for the direct solution of initial-value problems for fourth-order ordinary differential equations, without the need to reduce the problems to a first-order system of differential equations. The method was derived by applying interpolation and collocation procedures to a Chebyshev approximating polynomial. The unknown parameters were obtained using the Gaussian elimination method and then substituted into the approximate solution to obtain the continuous scheme, and the resulting scheme was evaluated at the selected point to yield the discrete scheme. The method derived had an order of seven and was convergent, consistent, and zero-stable, and, as shown by the region of absolute stability, was p-stable. Four numerical examples were solved to test both the accuracy and usability of the derived method. The computational results show that the derived method was highly efficient because it produced low errors. The numerical results of the derived method, when compared with those of the reported works in the literature, are found to be better, as they produce lower errors.

1 Introduction

This study presents an approximate solution for general fourth-order ordinary differential equations expressed as follows:

$$w^{iv}(x) = f(x, w, w', w'', w'''), \quad w(x_0) = \theta_0, \quad w'(x_0) = \theta_1, \quad w''(x_0) = \theta_2, \quad w'''(x_0) = \theta_3 \tag{1}$$

where f is a continuous real-valued function.

In both engineering, life sciences, and physical sciences, mathematical modeling often results in both linear and nonlinear IVPs of high-order. A prominent example of a fourth-order IVP is the static deflection of both uniform and cantilever beams, where specific boundary conditions such as an embedded left end and a free right end apply (see [1]). According to [2–5] and numerous others, the earlier approach has been the reduction of equation (1) to systems of first order ordinary differential equations and solving them using the existing known methods. Even though this approach has been very effective, it does have some drawbacks.

Received: April 24, 2026; Revised & Accepted: June 13, 2026; Published online: August 24, 2026
2020 Mathematics Subject Classification: 33B15, 33B99, 47A30.
Keywords and phrases: Chebyshev polynomials, hybrid points, linear multistep methods (LMMs).
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For instance, the subroutines that provide the beginning values for the implementation of the techniques are usually sophisticated computer programs, which increase computational work and computing time. Furthermore, [6] observed that reduction methods often fail to incorporate intrinsic properties of specific Ordinary Differential Equations (ODEs), such as the oscillatory behavior of a solution. To overcome the drawbacks associated with reduction methodologies, direct numerical approaches have been introduced as a robust alternative. However, the literature reveals that relatively few direct techniques are specifically tailored for fourth-order ODEs. Notable existing contributions include the Block Hybrid Collocation Method (BHCM) using three off-grid points developed by [7], and a four-step implicit block approach utilizing three generalized off-step points proposed by [8]. Additionally, [9] introduced an algorithmic collocation method for approximating fourth-order IVPs, while [1] presented a Runge-Kutta type approach. According to [9–11, 23], single-step methods are particularly efficient in terms of accuracy when they incorporate hybrid points. [24] presented a novel six-step block method that was derived using a linear block algorithm (LBA) for the general solution of fourth-order ODEs to overcome the need to reduce them to a system of first order ODEs. [25] presented a numerical method that was used to solve fourth-order ODEs by interpolating at all points except one, and this proved to be effective as the error in the numerical computation was very small. Motivated by the success of [23], this study proposes a fully hybrid block method to directly solve equation (1). This new method approximates the solution over the interval $[x_n, x_{n+\frac{1}{8}}]$ and incorporates six intermediate points.

2 Mathematical Derivation

Let the exact solution $w(x)$ of the fourth-order initial value problem of ordinary differential equation (1) be approximated by a Chebyshev polynomial of the type

$$w(x) = \sum_{j=0}^{(p+r)-1} a_j T_j(x). \tag{2}$$

The first, second, third and fourth derivatives of equation (2) are given as:

$$w'(x) = \sum_{j=1}^{(p+r)-1} a_j T_j'(x), \tag{3}$$

$$w''(x) = \sum_{j=2}^{(p+r)-1} a_j T_j''(x), \tag{4}$$

$$w'''(x) = \sum_{j=3}^{(p+r)-1} a_j T_j'''(x), \tag{5}$$

$$w^{(iv)}(x) = \sum_{j=4}^{(p+r)-1} a_j T_j^{(iv)}(x). \tag{6}$$

Equations (1) and (6) yield a differential system:

$$f(x, w, w', w'', w''') = \sum_{j=4}^{(p+r)-1} a_j T_j^{(iv)}(x), \tag{7}$$

where f is both continuous and differentiable, and the parameters a_j's in (2), (3), (4), (5), and (7) are linear terms to be determined. To get the system of algebraic equations in equations (8), (9), (10), (11) and (12), $x = x_n$ was applied to equations (2), (3), (4), and (5), while $x = x_{n+j}$, $j = 0\left(\frac{1}{48}\right)\frac{1}{8}$ was applied to equation (7).

$$w_n = \sum_{j=0}^{10} a_j T_j(x_n), \tag{8}$$

$$w'_n = \sum_{j=1}^{10} a_j T_j(x_n), \tag{9}$$

$$w''_n = \sum_{j=2}^{10} a_j T_j(x_n), \tag{10}$$

$$w'''_n = \sum_{j=3}^{10} a_j T_j(x_n), \tag{11}$$

$$f_{n+j} = \sum_{j=4}^{10} a_j T_j^{(iv)}(x_{n+j}), \quad j = 0\left(\frac{1}{48}\right)\frac{1}{8}. \tag{12}$$

By having $x_{n+\frac{j}{48}} = x_n + \frac{jh}{48}$, (8), (9), (10), (11), and (12) were solved using CAS in Wolfram Mathematica to obtain the parameters a_j's for $j = 0, 1, 2, \ldots, 10$, which were then substituted back into (2) and, after some simplifications, yielded the following continuous scheme:

$$w(t) = \alpha_0 w_n + \alpha_1 w'_n + \alpha_2 w''_n + \alpha_3 w'''_n + h^4 \sum_{j=0}^{6} \beta_j f_{n+\frac{j}{48}} \tag{13}$$

where $x = x_{n+t} = x_n + th$, and α_j's and β_j's are

$$\alpha_0 = 1$$

$$\alpha_1 = ht$$

$$\alpha_2 = \frac{1}{2}h^2t^2$$

$$\alpha_3 = \frac{5}{48}h^3t^3$$

$$\beta_0 = \frac{h^4t^4}{24} - \frac{49h^4t^5}{50} + \frac{3248h^4t^6}{225} - \frac{672h^4t^7}{5} + 768h^4t^8 - \frac{12288h^4t^9}{5} + \frac{589824h^4t^{10}}{175}$$

$$\beta_1 = \frac{12h^4t^5}{5} - \frac{1392h^4t^6}{25} + \frac{22272h^4t^7}{35} - \frac{142848h^4t^8}{35} + \frac{98304h^4t^9}{7} - \frac{3538944h^4t^{10}}{175}$$

$$\beta_2 = -3h^4t^5 + \frac{468h^4t^6}{5} - \frac{44256h^4t^7}{35} + \frac{315648h^4t^8}{35} - \frac{233472h^4t^9}{7} + \frac{1769472h^4t^{10}}{35}$$

$$\beta_3 = \frac{8h^4t^5}{3} - \frac{4064h^4t^6}{45} + \frac{47616h^4t^7}{35} - \frac{371712h^4t^8}{35} + \frac{294912h^4t^9}{7} - \frac{2359296h^4t^{10}}{35}$$

$$\beta_4 = -\frac{3}{2}h^4t^5 + \frac{264h^4t^6}{5} - \frac{29472h^4t^7}{35} + \frac{246528h^4t^8}{35} - \frac{208896h^4t^9}{7} + \frac{1769472h^4t^{10}}{35}$$

$$\beta_5 = \frac{12h^4t^5}{25} - \frac{432h^4t^6}{25} + \frac{9984h^4t^7}{35} - \frac{87552h^4t^8}{35} + \frac{393216h^4t^9}{35} - \frac{3538944h^4t^{10}}{175}$$

$$\beta_6 = -\frac{1}{24}h^4t^5 + \frac{548h^4t^6}{225} - \frac{288h^4t^7}{7} + \frac{13056h^4t^8}{35} - \frac{12288h^4t^9}{7} + \frac{589824h^4t^{10}}{175}$$

Evaluating (13) at $t = \frac{1}{48}\left(\frac{1}{48}\right)\frac{1}{8}$ yields the discrete Chebyshev-derived numerical block formula

$$\begin{aligned} w_{n+\frac{1}{48}} = w_n + \frac{hw_n'}{48} + \frac{h^2w_n''}{4608} + \frac{h^3w_n'''}{663552} \\ + \frac{h^4}{19263179980800}\big(95929f_n + 112028f_{n+\frac{1}{48}} - 115165f_{n+\frac{1}{24}} \\ + 97320f_{n+\frac{1}{16}} - 53465f_{n+\frac{1}{12}} + 16876f_{n+\frac{5}{48}} - 2323f_{n+\frac{1}{8}}\big) \end{aligned} \tag{14}$$

$$\begin{aligned} w_{n+\frac{1}{24}} = w_n + \frac{hw_n'}{24} + \frac{h^2w_n''}{1152} + \frac{h^3w_n'''}{82944} \\ + \frac{h^4}{75246796800}\big(4127f_n + 8782f_{n+\frac{1}{48}} - 6965f_{n+\frac{1}{24}} + 5820f_{n+\frac{1}{16}} \\ - 3175f_{n+\frac{1}{12}} + 998f_{n+\frac{5}{48}} - 137f_{n+\frac{1}{8}}\big) \end{aligned} \tag{15}$$

$$\begin{aligned} w_{n+\frac{1}{16}} &= w_n + \frac{hw_n'}{16} + \frac{h^2w_n''}{512} + \frac{h^3w_n'''}{24576} \\ &+ \frac{h^4}{26424115200}\big(5471f_n + 15228f_{n+\frac{1}{48}} - 8775f_{n+\frac{1}{24}} + 8120f_{n+\frac{1}{16}} \\ &- 4455f_{n+\frac{1}{12}} + 1404f_{n+\frac{5}{48}} - 193f_{n+\frac{1}{8}}\big) \end{aligned} \tag{16}$$

$$\begin{aligned} w_{n+\frac{1}{12}} &= w_n + \frac{hw_n'}{12} + \frac{h^2w_n''}{288} + \frac{h^3w_n'''}{10368} \\ &+ \frac{h^4}{2351462400}\big(1220f_n + 3904f_{n+\frac{1}{48}} - 1580f_{n+\frac{1}{24}} + 1920f_{n+\frac{1}{16}} \\ &- 1015f_{n+\frac{1}{12}} + 320f_{n+\frac{5}{48}} - 44f_{n+\frac{1}{8}}\big) \end{aligned} \tag{17}$$

$$\begin{aligned} w_{n+\frac{5}{48}} &= w_n + \frac{5hw_n'}{48} + \frac{25h^2w_n''}{4608} + \frac{125h^3w_n'''}{663552} \\ &+ \frac{125h^4}{770527199232}\big(6457f_n + 22460f_{n+\frac{1}{48}} - 6325f_{n+\frac{1}{24}} + 11400f_{n+\frac{1}{16}} \\ &- 5225f_{n+\frac{1}{12}} + 1708f_{n+\frac{5}{48}} - 235f_{n+\frac{1}{8}}\big) \end{aligned} \tag{18}$$

$$\begin{aligned} w_{n+\frac{1}{8}} &= w_n + \frac{hw_n'}{8} + \frac{h^2w_n''}{128} + \frac{h^3w_n'''}{3072} \\ &+ \frac{h^4}{103219200}\big(191f_n + 702f_{n+\frac{1}{48}} - 135f_{n+\frac{1}{24}} + 380f_{n+\frac{1}{16}} \\ &- 135f_{n+\frac{1}{12}} + 54f_{n+\frac{5}{48}} - 7f_{n+\frac{1}{8}}\big) \end{aligned} \tag{19}$$

Equation (19) is the main method.

Additional Methods

In order to have a complete block, the first, second, and third derivatives of equation (13) were also evaluated at $t = \frac{1}{48}\left(\frac{1}{48}\right)\frac{1}{8}$ and are given below:

$$\begin{aligned} w'_{n+\frac{1}{48}} &= w'_n + \frac{hw_n''}{48} + \frac{h^2w_n'''}{4608} \\ &+ \frac{h^3}{401316249600}\big(343801f_n + 506604f_{n+\frac{1}{48}} - 494715f_{n+\frac{1}{24}} \\ &+ 414160f_{n+\frac{1}{16}} - 226605f_{n+\frac{1}{12}} + 71364f_{n+\frac{5}{48}} - 9809f_{n+\frac{1}{8}}\big) \end{aligned} \tag{20}$$

$$\begin{aligned} w'_{n+\frac{1}{24}} &= w'_n + \frac{hw_n''}{24} + \frac{h^2w_n'''}{1152} \\ &+ \frac{h^3}{3135283200}\big(13774f_n + 35976f_{n+\frac{1}{48}} - 24465f_{n+\frac{1}{24}} + 20800f_{n+\frac{1}{16}} \\ &- 11370f_{n+\frac{1}{12}} + 3576f_{n+\frac{5}{48}} - 491f_{n+\frac{1}{8}}\big) \end{aligned} \tag{21}$$

$$\begin{aligned} w'_{n+\frac{1}{16}} &= w'_n + \frac{hw''_n}{16} + \frac{h^2 w'''_n}{512} \\ &+ \frac{h^3}{550502400}\big(5877f_n + 19188f_{n+\frac{1}{48}} - 8055f_{n+\frac{1}{24}} + 8960f_{n+\frac{1}{16}} \\ &- 4905f_{n+\frac{1}{12}} + 1548f_{n+\frac{5}{48}} - 213f_{n+\frac{1}{8}}\big) \end{aligned} \tag{22}$$

$$\begin{aligned} w'_{n+\frac{1}{12}} &= w'_n + \frac{hw''_n}{12} + \frac{h^2 w'''_n}{288} \\ &+ \frac{h^3}{195955200}\big(3863f_n + 13992f_{n+\frac{1}{48}} - 3390f_{n+\frac{1}{24}} + 6800f_{n+\frac{1}{16}} \\ &- 3255f_{n+\frac{1}{12}} + 1032f_{n+\frac{5}{48}} - 142f_{n+\frac{1}{8}}\big) \end{aligned} \tag{23}$$

$$\begin{aligned} w'_{n+\frac{5}{48}} &= w'_n + \frac{5hw''_n}{48} + \frac{25h^2 w'''_n}{4608} \\ &+ \frac{125h^3}{16052649984}\big(4045f_n + 15564f_{n+\frac{1}{48}} - 2055f_{n+\frac{1}{24}} + 8560f_{n+\frac{1}{16}} \\ &- 2865f_{n+\frac{1}{12}} + 1092f_{n+\frac{5}{48}} - 149f_{n+\frac{1}{8}}\big) \end{aligned} \tag{24}$$

$$\begin{aligned} w'_{n+\frac{1}{8}} &= w'_n + \frac{hw''_n}{8} + \frac{h^2 w'''_n}{128} \\ &+ \frac{h^3}{4300800}\big(198f_n + 792f_{n+\frac{1}{48}} - 45f_{n+\frac{1}{24}} + 480f_{n+\frac{1}{16}} \\ &- 90f_{n+\frac{1}{12}} + 72f_{n+\frac{5}{48}} - 7f_{n+\frac{1}{8}}\big) \end{aligned} \tag{25}$$

$$\begin{aligned} w''_{n+\frac{1}{48}} &= w''_n + \frac{hw'''_n}{48} \\ &+ \frac{h^2}{278691840}\big(28549f_n + 57750f_{n+\frac{1}{48}} - 51453f_{n+\frac{1}{24}} + 42484f_{n+\frac{1}{16}} \\ &- 23109f_{n+\frac{1}{12}} + 7254f_{n+\frac{5}{48}} - 995f_{n+\frac{1}{8}}\big) \end{aligned} \tag{26}$$

$$\begin{aligned} w''_{n+\frac{1}{24}} &= w''_n + \frac{hw'''_n}{24} \\ &+ \frac{h^2}{4354560}\big(1027f_n + 3492f_{n+\frac{1}{48}} - 1680f_{n+\frac{1}{24}} + 1576f_{n+\frac{1}{16}} \\ &- 873f_{n+\frac{1}{12}} + 276f_{n+\frac{5}{48}} - 38f_{n+\frac{1}{8}}\big) \end{aligned} \tag{27}$$

$$\begin{aligned} w'''_{n+\frac{1}{16}} &= w'''_n + \frac{h}{107520}\big(685f_n + 3240f_{n+\frac{1}{48}} + 1161f_{n+\frac{1}{24}} + 2176f_{n+\frac{1}{16}} \\ &- 729f_{n+\frac{1}{12}} + 216f_{n+\frac{5}{48}} - 29f_{n+\frac{1}{8}}\big) \end{aligned} \tag{28}$$

$$\begin{aligned} w'''_{n+\frac{1}{12}} &= w'''_n + \frac{h}{22680}\big(143f_n + 696f_{n+\frac{1}{48}} + 192f_{n+\frac{1}{24}} + 752f_{n+\frac{1}{16}} \\ &+ 87f_{n+\frac{1}{12}} + 24f_{n+\frac{5}{48}} - 4f_{n+\frac{1}{8}}\big) \end{aligned} \tag{29}$$

$$
\begin{aligned}
w'''_{n+\frac{5}{48}} = w'''_n + \frac{5h}{580608}\big(&743f_n + 3480f_{n+\frac{1}{48}} + 1275f_{n+\frac{1}{24}} + 3200f_{n+\frac{1}{16}} \\
&+ 2325f_{n+\frac{1}{12}} + 1128f_{n+\frac{5}{48}} - 55f_{n+\frac{1}{8}}\big)
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
w'''_{n+\frac{1}{8}} = w'''_n + \frac{h}{6720}\big(&41f_n + 216f_{n+\frac{1}{48}} + 27f_{n+\frac{1}{24}} + 272f_{n+\frac{1}{16}} \\
&+ 27f_{n+\frac{1}{12}} + 216f_{n+\frac{5}{48}} + 41f_{n+\frac{1}{8}}\big)
\end{aligned} \tag{31}
$$

3 Analysis of the Properties of the Method

In this section, the basic properties of the derived method are analyzed and presented.

3.1 Order of the Method

Since $w(x)$ is both continuous and differentiable, and following [2], the linear difference operator equivalent to the formulas in (14)-(31) is defined by:

$$
\begin{aligned}
&\zeta_{\frac{r}{48}}\{w(x):h\} \\
&= w\left(x_n + \frac{r}{48}h\right) - \left\{\sum_{b=0}^{8} \alpha_b\left(\frac{r}{48}\right) w^{(b)}(x)h^b - h^4 \sum_{b=0}^{8} \beta_b\left(\frac{r}{48}\right) w^{(4)}\left(x_n + \frac{r}{48}h\right), r = 1,2,3,4,5,6\right\}.
\end{aligned} \tag{32}
$$

The Taylor series expansion about the point x, after applying (16), provides a formula for the local truncation error, since $w(x)$ is the valid solution to (1), written as:

$$
\begin{aligned}
\zeta_{\frac{r}{48}}\{w(x):h\} =& c_0 w(x) + c_1 h w'(x) + c_2 h^2 w''(x) + \cdots + c_{p+3}h^{p+3}w^{(p+3)}(x) \\
&+ c_{p+4}h^{p+4}w^{(p+4)}(x).
\end{aligned} \tag{33}
$$

The term c_{p+4} is called the error constant, and it implies that the local truncation error is defined as:

$$
\zeta_{\frac{r}{48}}\{w(x):h\} = c_{p+4}h^{p+4}w^{(p+4)}(x_n) + Oh^{p+5}. \tag{34}
$$

Since $c_0 = c_1 = \cdots = c_{p+3} = 0$ and $c_{p+4} \neq 0$ (refer to [14]), the formulas in (14)-(31) have uniform order $p = 7$.

3.2 Consistency of the Method

Definition 1 (according to [2])**.** The linear multistep method is said to be consistent if it has order $p \geq 1$. Obviously, the presented method is consistent.

3.3 Zero Stability of the Block Method

According to [3] and [15], a block method is zero-stable if the zeros of the characteristic polynomial satisfy the following conditions:

$|\eta_i| \leq 1$, and the root for which $|\eta_i| = 1$ has multiplicity less than the order of the differential equation. Considering

$$A_1 W_{\mu+4} = A_0 W_\mu + h(B_0 W_{\mu+1}) + h^2(B_1 W_{\mu+2}) + h^3(B_2 W_{\mu+3}) + h^4(B_2 F_\mu) \tag{35}$$

where

$$\begin{aligned}
W_\mu &= \left(y_{n-\frac{1}{48}}, y_{n-\frac{1}{24}}, y_{n-\frac{1}{16}}, y_{n-\frac{1}{12}}, y_{n-\frac{5}{48}}, y_n\right)^{\mathrm{T}}\\
W_{\mu+1} &= \left(y_{n-\frac{1}{48}}, y_{n-\frac{1}{24}}, y_{n-\frac{1}{16}}, y_{n-\frac{1}{12}}, y_{n-\frac{5}{48}}, y'_n\right)^{\mathrm{T}}\\
W_{\mu+2} &= \left(y_{n-\frac{1}{48}}, y_{n-\frac{1}{24}}, y_{n-\frac{1}{16}}, y_{n-\frac{1}{12}}, y_{n-\frac{5}{48}}, y''_n\right)^{\mathrm{T}}\\
W_\mu &= \left(y_{n-\frac{1}{48}}, y_{n-\frac{1}{24}}, y_{n-\frac{1}{16}}, y_{n-\frac{1}{12}}, y_{n-\frac{5}{48}}, y'''_n\right)^{\mathrm{T}}\\
W_{\mu+4} &= \left(y_{n+\frac{1}{48}}, y_{n+\frac{1}{24}}, y_{n+\frac{1}{16}}, y_{n+\frac{1}{12}}, y_{n+\frac{5}{48}}, y_{n+\frac{1}{8}}\right)^{\mathrm{T}}\\
F_\mu &= \left(f_{n+\frac{1}{48}}, f_{n+\frac{1}{24}}, f_{n+\frac{1}{16}}, f_{n+\frac{1}{12}}, f_{n+\frac{5}{48}}, f_{n+\frac{1}{8}}\right)^{\mathrm{T}}
\end{aligned}$$

with

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad B_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{48}\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{24}\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{16}\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{12}\\ 0 & 0 & 0 & 0 & 0 & \frac{5}{48}\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{8} \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{4608}\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{1152}\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{512}\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{288}\\ 0 & 0 & 0 & 0 & 0 & \frac{25}{4608}\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{128} \end{pmatrix} \qquad B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{663552}\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{82944}\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{24576}\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{10368}\\ 0 & 0 & 0 & 0 & 0 & \frac{125}{663552}\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3072} \end{pmatrix}$$

$$\begin{pmatrix} \frac{95929}{19263179980800} & \frac{4001}{687970713600} & -\frac{23033}{3852635996160} & \frac{811}{160526499840} & -\frac{10693}{3852635996160} & \frac{4219}{4815794995200} & -\frac{2323}{19263179980800} \\ \frac{4127}{75246796800} & \frac{4391}{37623398400} & -\frac{199}{2149908480} & \frac{97}{1254113280} & -\frac{127}{3009871872} & \frac{499}{37623398400} & -\frac{137}{75246796800} \\ \frac{5471}{26424115200} & \frac{423}{734003200} & -\frac{39}{117440512} & \frac{29}{94371840} & -\frac{99}{587202560} & \frac{39}{734003200} & -\frac{193}{26424115200} \\ \frac{61}{117573120} & \frac{61}{36741600} & -\frac{79}{117573120} & \frac{1}{1224720} & -\frac{29}{67184640} & \frac{1}{7348320} & -\frac{11}{587865600} \\ \frac{807125}{770527199232} & \frac{701875}{192631799808} & -\frac{790625}{770527199232} & \frac{59375}{32105299968} & -\frac{653125}{770527199232} & \frac{7625}{27518828544} & -\frac{29375}{770527199232} \\ \frac{191}{103219200} & \frac{39}{5734400} & -\frac{3}{2293760} & \frac{19}{5160960} & -\frac{3}{2293760} & \frac{3}{5734400} & -\frac{1}{14745600} \end{pmatrix}$$

Zero-stability refers to the stability of the difference system in (35) as $h \to 0$. As such, if $h \to 0$ in (35), we obtain

$$A_1 W_{\mu+4} = A_0 W_\mu \tag{36}$$

The characteristic equation of (36) is given by $\rho(r) = r^5(r-1)$. Solving for the roots of $\rho(r) = 0$, we obtain $r = 1, 0, 0$.

The linear multistep methods (14-19), which by extension are represented in (36), are said to be zero-stable if the roots of the polynomial $\rho(r) = 0$ satisfy the strict root condition, i.e., if all its roots lie inside the unit circle, $|r| < 1$, with those on the boundary being simple.

3.4 Convergence

According to [16], for a numerical method to converge, it must be both consistent and zero-stable. To ensure that the proposed method is consistent, it must have order $p \geq 1$ (see [17]). We can infer that the proposed method is convergent because Section 3.3 demonstrated that it is zero-stable.

3.5 Region of Absolute Stability of the Method

This section investigates the range of values for which the proposed method is stable.

Definition 2 [2]**.** The block method CDNBM is p-stable if the periodicity interval of the method is $(0, +\infty)$.

Proposition 1. The proposed method is p-stable.

This proposition is ascertained by first considering the characteristic polynomial of the primary formulas of (14)-(19), which is given as

$$p(r)=\begin{bmatrix} \sqrt[48]{r} & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & \sqrt[24]{r} & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & \sqrt[16]{r} & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & \sqrt[12]{r} & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & r^{5/48} & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \sqrt[8]{r} & -1 \end{bmatrix}$$

$$\sigma(r)=\begin{bmatrix} \frac{95929}{19263179980800} & \frac{4001\sqrt[48]{r}}{687970713600} & -\frac{23033\sqrt[24]{r}}{3852635996160} & \frac{811\sqrt[16]{r}}{160526499840} & -\frac{10693\sqrt[12]{r}}{3852635996160} & \frac{4219r^{5/48}}{4815794995200} & -\frac{2323\sqrt[8]{r}}{19263179980800} \\ \frac{4127}{75246796800} & \frac{4391\sqrt[48]{r}}{37623398400} & -\frac{199\sqrt[24]{r}}{2149908480} & \frac{97\sqrt[16]{r}}{1254113280} & -\frac{127\sqrt[12]{r}}{3009871872} & \frac{499r^{5/48}}{37623398400} & -\frac{137\sqrt[8]{r}}{75246796800} \\ \frac{5471}{26424115200} & \frac{423\sqrt[48]{r}}{734003200} & -\frac{39\sqrt[24]{r}}{117440512} & \frac{29\sqrt[16]{r}}{94371840} & -\frac{99\sqrt[12]{r}}{587202560} & \frac{39r^{5/48}}{734003200} & -\frac{193\sqrt[8]{r}}{26424115200} \\ \frac{61}{117573120} & \frac{61\sqrt[48]{r}}{36741600} & -\frac{79\sqrt[24]{r}}{117573120} & \frac{\sqrt[16]{r}}{1224720} & -\frac{29\sqrt[12]{r}}{67184640} & \frac{r^{5/48}}{7348320} & -\frac{11\sqrt[8]{r}}{587865600} \\ \frac{807125}{770527199232} & \frac{701875\sqrt[48]{r}}{192631799808} & -\frac{790625\sqrt[24]{r}}{770527199232} & \frac{59375\sqrt[16]{r}}{32105299968} & -\frac{653125\sqrt[12]{r}}{770527199232} & \frac{7625r^{5/48}}{27518828544} & -\frac{29375\sqrt[8]{r}}{770527199232} \\ \frac{191}{103219200} & \frac{39\sqrt[48]{r}}{5734400} & -\frac{3\sqrt[24]{r}}{2293760} & \frac{19\sqrt[16]{r}}{5160960} & -\frac{3\sqrt[12]{r}}{2293760} & \frac{3r^{5/48}}{5734400} & -\frac{\sqrt[8]{r}}{14745600} \end{bmatrix}$$

The stability polynomial corresponding to (14)-(19) is given by

$$\pi(r,z)=p(r)+zq(r) \tag{37}$$

where $z=\lambda^4h^4$ and $\lambda=\frac{\partial f}{\partial w}$. Note that the stability polynomial (37) is obtained by applying the formulas in (14)-(19) to the scalar test function

$$w^{(4)}=-\lambda^4 w.$$

Replacing r with $e^{i\theta}$ and solving the stability polynomial (37) for $\Pi(r,z)=0$ yields, after some simplifications, the image below:

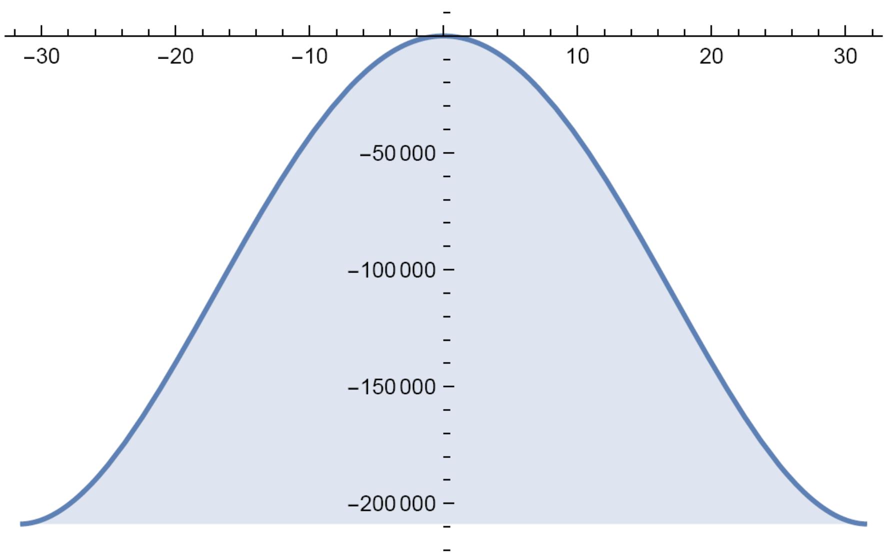

Figure 1: Region of absolute stability.

4 Numerical Experiments

Four sample problems were used by the authors to test how well the proposed method performed.

5 Problem 1

Considering the general fourth-order IVP of an ordinary differential equation

$$w^{(4)}(x) = w'''(x) + w''(x) + w'(x) + 2w(x), w(0) = 0, w'(0) = 0, w''(0) = 0, w'''(0) = 30 \quad (38)$$

Source [7]
with the exact solution $w(x) = 2e^{2x} - 5e^{-x} + 3\cos x - 9\sin x$ (see [7] and [6]). The solutions to Problem (1) were obtained within $[0, 2]$ over 10 iterations and were compared with the exact solution, as presented in Figure 2. It is clear from Table (1) and Figure (2) that the method shows good performance compared with the methods in [7] and [6].

Table 1: Solution of Problem 1 obtained using the proposed method.

x	w-computed	w-exact	Error in New Method	Error in [23]	Error in [7]	Error in [6]
0.2	0.0421714	0.0421714	3.747×10^{-16}	8.70415×10^{-14}	3.5129×10^{-13}	2.319×10^{-13}
0.4	0.3579	0.3579	3.10862×10^{-15}	8.04246×10^{-13}	4.1833×10^{-12}	2.2603×10^{-12}
1.8	62.9237	62.9237	1.07114×10^{-12}	2.93019×10^{-10}	5.4334×10^{-10}	9.1180×10^{-09}
2.0	99.0875	99.0875	1.87761×10^{-12}	5.1155×10^{-10}	8.0796×10^{-10}	1.7409×10^{-08}

From Table (1), it is clear that the new method performed well, as it gave the lowest errors when compared with the other methods. Hence, the method is reliable.

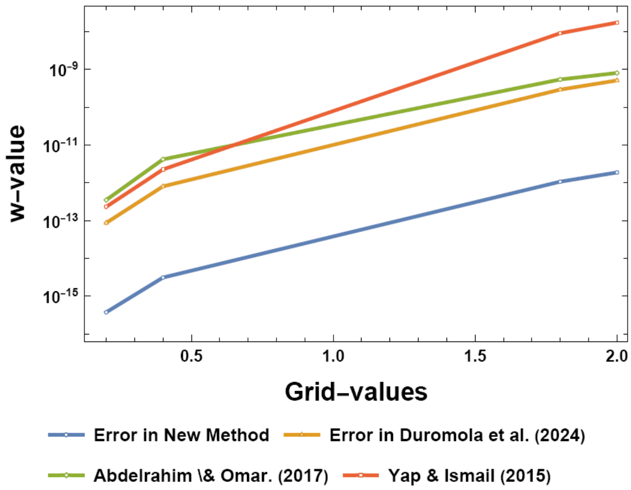

Figure 2: Graph of the comparison of Problem 1.

6 Problem 2

$$w^{(iv)} = -w''(x)$$

Theoretical Solution

$$w(t) = -5e^{-t} + 2e^{2t} - 9\sin(t) + 3\cos(t)$$

Source [23]

Table 2: Comparison results of Problem 2.

x	Error in New Method	Error in [23]	Error in [1]
0.0	0.000	0.0000	0.0000
0.2	4.33681×10^{-19}	2.60208×10^{-18}	3.40060×10^{-15}
0.4	1.73472×10^{-18}	4.33680×10^{-18}	7.40519×10^{-14}
1.6	2.60209×10^{-17}	5.37764×10^{-17}	5.09100×10^{-11}
1.8	2.9490×10^{-17}	6.76542×10^{-17}	9.85949×10^{-11}
2.0	3.46945×10^{-17}	7.97972×10^{-17}	1.83206×10^{-10}

From Table (2), it is clear that the new method performed well, as it gave the lowest errors when compared with the other methods. Hence, the method is reliable.

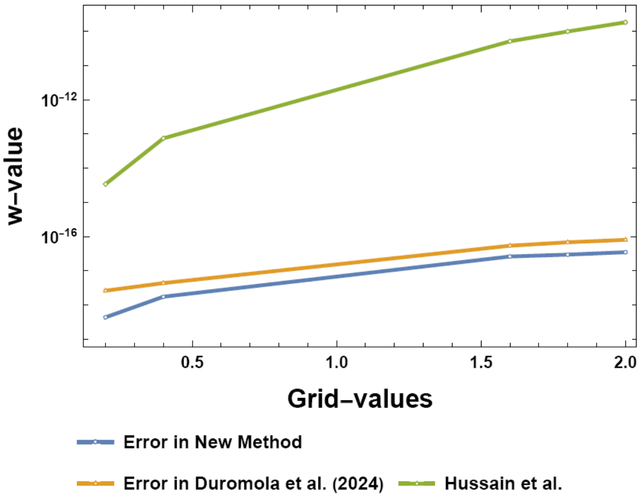

Figure 3: Graph of the comparison of Problem 2.

Problem 3

The third sample equation considered is a nonlinear fourth-order ordinary differential equation

$$w^{(4)} = \frac{3\sin(w)(3+2\sin^2(w))}{\cos^7(w)}, \quad w(0)=0,\ w'(0)=1,\ w''(0)=0,\ w'''(0)=1$$

Source [23]
with the exact solution $w(x) = \arcsin(x)$ (see [20, 21]). The problem was integrated over the interval $[0, \frac{\pi}{4}]$. Table (3) and Figure (4) show the w-approximations generated by the derived method at the grid points $\left(\frac{3\pi}{200}, \frac{\pi}{50}, \frac{\pi}{40}, \frac{3\pi}{100}, \frac{\pi}{20}\right)$ as compared to those generated by NDSolve in Wolfram Mathematica. This confirms that the present method is a good alternative for solving nonlinear problems.

Table 3: Computed results of Problem 3.

x	w-approx	W-Exact	NDSolve	w-Error	NDSolveError
$\frac{3\pi}{200}$	0.0471413	0.0471413	0.0471413	2.0817×10^{-17}	1.7452×10^{-8}
$\frac{\pi}{50}$	0.0628733	0.0628733	0.0628732	1.3878×10^{-17}	7.3618×10^{-8}
$\frac{\pi}{40}$	0.0786208	0.0786208	0.0786206	2.7756×10^{-17}	2.2496×10^{-7}
$\frac{3\pi}{100}$	0.0943879	0.0943879	0.0943873	4.1633×10^{-17}	5.6069×10^{-7}
$\frac{\pi}{20}$	0.157733	0.157733	0.157726	2.7756×10^{-17}	7.2795×10^{-6}

From Table (3), it is clear that the new method performed well, as it gave lower errors when compared with NDSolve from Wolfram Mathematica. Hence, the method is reliable.

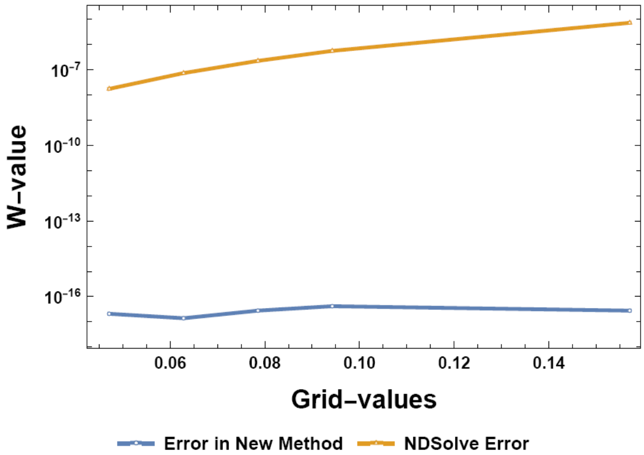

Figure 4: Graph of the comparison of Problem 3.

Problem 4

Consider a nonlinear, non-homogeneous fourth-order ordinary differential equation:

$$w^{(4)} = w^2 + \cos^2(x) + \sin(x) - 1, \quad w(0) = 0,\ w'(0) = 1,\ w''(0) = 0,\ w'''(0) = -1$$

Source [23]
whose exact solution is $w(x) = \sin(x)$ (see [21]). The problem was integrated over the interval $[0, 1]$. The results are presented in Table (4) and Figure (5).

Table 4: Computed results of Problem 4.

x	Error in [23]	Error in New Method
0.	0.	0.
0.02	2.4887×10^{-16}	3.46945×10^{-18}
0.04	3.1200×10^{-14}	6.93889×10^{-18}
0.16	4.4589×10^{-10}	2.77556×10^{-17}
0.18	9.9195×10^{-10}	8.32667×10^{-17}
0.2	2.0216×10^{-9}	5.55112×10^{-17}

From Table (4), it is clear that the new method performed well, as it gave the lowest errors when compared with the other methods. Hence, the method is reliable.

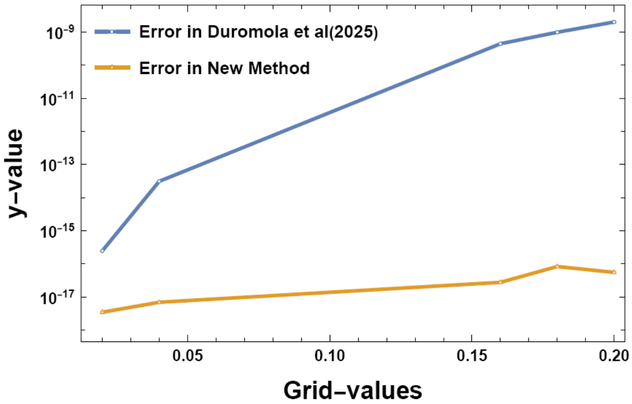

Figure 5: Graph of the comparison of Problem 4.

7 Conclusion

This work derived and presented a numerical hybrid block method using Chebyshev polynomials as the basis function with a theoretical order of seven to solve initial value problems of fourth-order ordinary differential equations. The method satisfied the basic properties of linear multistep methods.

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