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Analytical Solutions of Volterra Integral Equations Involving Jacobi Polynomial Nonlinearities: A Power Series Method

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Abstract

Nonlinear Volterra integral equations model several phenomena in a variety of fields, including epidemiology, population growth theory, diffusion and heat transfer problems. In this paper, the analytical solution of a class of nonlinear Volterra integral equations of the second kind is presented using a power series method based on the generalised Cauchy product. The nonlinear terms $\Phi(U(\xi))$ in the proposed problem are given by normalised Jacobi polynomials $\mathsf{P}_{\ell}^{(\phi,\varphi)}(U(\xi))$ $(\ell = 1, 2, 3, \ldots; \phi, \varphi > -1)$ in the unknown $U(\xi)$. The proposed method first expresses the nonlinear terms as a power series in the unknown U, and then uses the generalised Cauchy product to convert the series into power series in the independent variable ξ. Subsequently, recurrence relations for the expansion coefficients of the series solution are obtained in terms of the nonlinear terms and their higher derivatives evaluated at the constant term of the series solution. Specialising the nonlinear terms to Gegenbauer or ultraspherical polynomials, Legendre polynomials, and Chebyshev polynomials, several nonlinear Volterra integral equations and their solutions are introduced. Considering the normalised Jacobi polynomials as spherical functions on rank one symmetric spaces of compact type, a number of novel nonlinear integral equations are presented. To validate the proposed method, some known nonlinear Volterra integral equations are solved and the series solutions converge to the known exact solutions. The convergence of the series solutions shows that the proposed power series method is accurate and reliable for solving such nonlinear Volterra integral problems.

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1 Introduction

Integral equations are indispensable in the modeling of many real-life phenomena in science and engineering ([25], [50]). Integral equations are classified based on the nature of the limits of integration and the presence of the unknown. An integral equation in which the unknown appears both inside and outside the integral is referred to as an *integral equation of the second kind*. An integral equation in which the unknown appears only inside the integral is called an *integral equation of the first kind*. An integral equation of the first kind (or the second kind) with constant limits of integration is known as a *Fredholm integral equation* of the first kind (or the second kind). An integral equation of the first kind (or the second kind) in which at least one limit of integration is a variable is referred to as a *Volterra integral equation* of the first kind (or the second kind) [55, Chs. 1, 2, 4, & 6]. An integral equation can be converted to an ordinary differential equation and vice-versa ([55, Sects. 5.2-5.3]). Thus integral equations can be solved by first reducing them to ordinary differential equations, and vice-versa. This paper considers Volterra integral equations of the second kind. Volterra integral equations arise in many physical phenomena such as plasma physics, epidemiology, population growth theory and genetics, telegraph model, mechanics, potential theory, water waves, the heat transfer, radiation and conduction, electrostatics, electromagnetics, acoustical and elastical waves, tomography, seismology, geometric probability, industrial replacement, blow-up and quenching phenomena, American pricing option, theory of water percolation, the Storekeeper's control problem, and transient behaviour of semiconductor devices ([3], [8], [36], [40], [45], [50, Ch. 18], [53]). Further discussions on the applications of Volterra integral equations in science and engineering are contained in, e.g., [25, Ch. 9].

Nonlinear Volterra integral equations that require advanced mathematical methods to obtain their solutions arise quite often in real-life phenomena. Thus several authors have applied advanced analytical or numerical techniques to obtain exact or approximate solutions of nonlinear Volterra integral equations ([25], [45], [50], [55]). Alkiffai, and Alaraji in [8] used a variety of integral transforms such as Sadik transform, Mahgoub transform, Kamal transform, and Laplace transform to obtain analytical solutions of Volterra integral equations. In [36], Taylor series was considered. The application of Fourier sine series and finite difference method was presented in [53]. In [12], the modified Bernstein operator for solving Volterra integral equations was considered. In [43], Chebyshev collocation method of the first kind was applied to obtain numerical solutions of Volterra integral equations. In [34], Islam and Shirivatasava obtained analytical solutions of Volterra integral equations using variational iteration method and Adomian decomposition method. Two dimensional integral equations was considered in [2] using hybrid functions approach. In [4], collocation methods based on Laguerre and Hermite polynomials were used to obtain numerical solutions of systems of integral equations. In [29], modified homotopy perturbation method for obtaining analytical solutions of nonlinear integral equations was discussed. Solutions of weakly singular multi-dimensional nonlinear Volterra integral equations using Jacob spectral method were considered in [54]. A collocation method based on eighth kind Chebyshev polynomials for approximating solutions of systems of quadratic integral equations was employed in [38]; see also [37], [39]. Mohammed and Khudair [42] obtained numerical solutions of nonlinear Volterra integral equations of the second kind via fourth-degree hat functions. Numerical solutions of highly oscillatory Volterra integral equations with general oscillators were considered using the moment free Filon type method in [56]. Kazemi [35] used triangular functions for numerical solutions of nonlinear Volterra integral equations, while Costarelli [27] obtained approximate solutions of Volterra integral equations using interpolation method based on ramp functions.

For additional relevant papers on Volterra and other classes of integral equations, see [1], [6], [7], [9], [10], [22], [23], [24], [28], [31], [33], [41], [44], [47], [48], [51], [52]. Existence and uniqueness of solutions of quadratic integral equations using the measures of the noncompactness approach coupled with the fixed-point principle of Darbo was presented in [4], [5]. See also [29], [46].

The nonlinear Volterra integral equation of the second kind is of the form

$$U(x) = F(x) + \mu \int_a^{\beta(x)} K(x,\xi)\Phi(U(\xi))d\xi, \tag{1.1}$$

where the kernel $K(x,\xi)$ and the function $F(x)$ are given continuous real-valued functions, μ $(\neq 0)$ is a parameter, $\beta(x)$ is a variable limit, and $\Phi(u(\xi))$ is the nonlinear term in $U(\xi)$. This paper examines the nonlinear Volterra integral equation of the second kind

$$U(x) = F(x) + \mu \int_0^{x} \sin(x-\xi)\mathsf{P}_\ell^{(\phi,\varphi)}(U(\xi))\, d\xi \qquad (\ell = 1,2,3,\dots) \tag{1.2}$$

using a power series method based on the generalised Cauchy product with binomial expansion coefficients. That is, in (1.1), we have set $a = 0$, $\beta(x) = x$, and $K(x,\xi) = \sin(x-\xi)$ to obtain (1.2). Here the nonlinear terms $\Phi(U(\xi))$ are normalised ℓth-degree Jacobi polynomials $\mathsf{P}_\ell^{(\phi,\varphi)}(U(\xi))$ $(\ell = 1,2,3,\dots; \phi,\varphi > -1)$ in the unknown $U(\xi)$. The proposed power series method first expresses the nonlinear terms as power series in the unknown U, and then uses the generalised Cauchy product to convert the series into power series in the independent variable ξ. Upon applying the generalised Cauchy product, one subsequently obtains recurrence relations for the expansion coefficients. The recurrence relations are expressed as a linear combination of the nonlinear terms and their higher derivatives evaluated at the constant term (initial value) of the series solution. Consideration of special cases of Jacobi polynomials such as Gegenbauer or ultraspherical polynomials, Legendre polynomials, and Chebyshev polynomials, unbundles the proposed problem (1.2) into several nonlinear Volterra integral equations. Again, in view of the normalised Jacobi polynomials as spherical functions on rank one symmetric spaces of compact type, a variety of novel nonlinear Volterra integral equations are presented. At the end of the discussion on Volterra integral equations of the second kind involving Jacobi polynomials, three known nonlinear Volterra integral equations of the second kind, with known exact (analytical) solutions, are solved to validate our main results. Results show that the series solutions converge to the exact (analytical) solutions. The convergence of the series solutions shows that the proposed power series method is accurate and reliable for solving such nonlinear Volterra integral problems.

The motivation and novelties in this paper are highlighted as follows:

(i) The nonlinearities of the proposed Volterra integral equation are given in terms of Jacobi polynomials, of which Gegenbauer, Legendre, and Chebyshev polynomials are important special cases.

(ii) Several linear and nonlinear Volterra integral equations of the second kind are introduced.

(iii) New exact (closed-form) solutions of a number of linear Volterra integral equations of the second kind are presented.

(iv) The nonlinearities in the unknown are simplified into power series in the independent variable using the generalised Cauchy product.

(v) The expansion coefficients of the series solution are remarkably expressed in terms of the nonlinear terms (Jacobi polynomials) and their higher order derivatives evaluated at the constant term of the series solution.

(vi) The proposed power series technique can handle several other nonlinear Volterra integral problems without resolving into any quasilinearisation technique.

2 Jacobi Polynomials

The ℓ-degree Jacobi polynomials $P_\ell^{(\phi,\varphi)} = P_\ell^{(\phi,\varphi)}(t)$ ($\ell = 0,1,2,\ldots$; real $\phi,\varphi > -1; -1 \leq t \leq 1$) are defined by the finite series representation ([11, Sect. 5.4.3, Eq. (5.98)])

$$P_\ell^{(\phi,\varphi)}(t) = \frac{\Gamma(\ell+\phi+1)}{\Gamma(\ell+\phi+\varphi+1)} \sum_{k=0}^{\ell} \frac{\Gamma(\ell+\phi+\varphi+k+1)}{\Gamma(\ell-k+1)\Gamma(\phi+k+1)} \frac{(t-1)^k}{2^k k!} \tag{2.1}$$

and admit the Rodrigues' representation formula

$$P_\ell^{(\phi,\varphi)}(t) = \frac{(-1)^\ell}{2^\ell \ell!} (1-t)^{-\phi} (1+t)^{-\varphi} \frac{d^\ell}{dt^\ell} \left[(1-t)^{\ell+\phi} (1+t)^{\ell+\varphi} \right]. \tag{2.2}$$

The Jacobi polynomials $y = P_\ell^{(\phi,\varphi)}(t)$ are solutions to the second-order ordinary differential equation

$$\left(1-t^2\right) \frac{d^2 y}{dt^2} - (\phi - \varphi + (\phi+\varphi+2)t) \frac{dy}{dt} + \ell(\ell+\phi+\varphi+1)y = 0. \tag{2.3}$$

The Jacobi differential equation (2.3) is a Sturm-Liouville eigenvalue value problem with the (discrete) spectrum given, respectively, by the sequence of eigenvalues and eigenfunctions

$$\lambda_\ell^{\phi,\varphi} = \ell(\ell+\phi+\varphi+1), \qquad y = P_\ell^{(\phi,\varphi)}(t), \qquad \ell = 0,1,2,\ldots. \tag{2.4}$$

As orthogonal polynomials with respect to the beta distribution $(1-t)^\phi (1+t)^\varphi \, dt$ on $[-1,1]$, the Jacobi polynomials satisfy the integral formula

$$\int_{-1}^{1} P_\ell^{(\phi,\varphi)}(t) P_m^{(\phi,\varphi)}(t) (1-t)^\phi (1+t)^\varphi \, dt = \kappa_\ell^{\phi,\varphi} \delta_{\ell m} \quad (\ell, m = 0,1,2,\ldots), \tag{2.5}$$

where the scalars $\kappa_\ell^{\phi,\varphi}$ are given by

$$\kappa_\ell^{\phi,\varphi} = \frac{2^{\phi+\varphi+1}}{2\ell+\phi+\varphi+1}\frac{\Gamma(\ell+\phi+1)\Gamma(\ell+\varphi+1)}{\Gamma(\ell+1)\Gamma(\ell+\phi+\varphi+1)}. \tag{2.6}$$

The following differentiation formula holds:

$$\frac{d^p}{dt^p}P_\ell^{(\phi,\varphi)}(t) = \frac{1}{2^p}\frac{\Gamma(\ell+p+\phi+\varphi+1)}{\Gamma(\ell+\phi+\varphi+1)}P_{\ell-p}^{(\phi+p,\varphi+p)}(t), \qquad p=1,2,\dots. \tag{2.7}$$

One has the reflection symmetry and pointwise identities

$$P_\ell^{(\phi,\varphi)}(-t) = (-1)^\ell P_\ell^{(\varphi,\phi)}(t), \quad P_\ell^{(\phi,\varphi)}(1) = \frac{\Gamma(\ell+\phi+1)}{\ell!\Gamma(\phi+1)}. \tag{2.8}$$

Gegenbauer or ultraspherical polynomials C_ℓ^ν are a special case of Jacobi polynomials and are defined in terms of Jacobi polynomials by $(\ell = 0,1,2,\dots; \nu > -1/2)$

$$C_\ell^\nu(t) = \frac{\Gamma(\ell+2\nu)\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(2\nu)\Gamma\left(\ell+\nu+\frac{1}{2}\right)}P_\ell^{\left(\nu-\frac{1}{2},\nu-\frac{1}{2}\right)}(t). \tag{2.9}$$

A normalised form of Jacobi polynomials may be defined by

$$\mathsf{P}_\ell^{(\phi,\varphi)}(t) := \frac{P_\ell^{(\phi,\varphi)}(t)}{P_\ell^{(\phi,\varphi)}(1)} = \frac{\ell!\Gamma(\phi+1)}{\Gamma(\ell+\phi+1)}P_\ell^{(\phi,\varphi)}(t), \qquad \mathsf{P}_\ell^{(\phi,\varphi)}(1) = 1; \tag{2.10}$$

and are related to normalised Gegenbauer polynomials $\mathsf{C}_m^\nu(t)$ by

$$\mathsf{C}_\ell^\nu(t) = \frac{C_\ell^\nu(t)}{C_\ell^\nu(1)} = \mathsf{P}_\ell^{(\nu-1/2,\nu-1/2)}(t), \qquad \mathsf{C}_\ell^\nu(1) = 1. \tag{2.11}$$

The Legendre polynomials are a special case of (2.11):

$$\mathsf{C}_\ell^{\frac{1}{2}}(t) = C_\ell^{\frac{1}{2}}(t) = \mathsf{P}_\ell^{(0,0)}(t) = P_\ell^{(0,0)}(t) = P_\ell(t). \tag{2.12}$$

Other notable and useful special normalised Jacobi polynomials $\mathsf{P}_\ell^{(\phi,\varphi)}$ are the following: Chebyshev polynomials of the first kind ($\phi = \varphi = -1/2$); Chebyshev polynomials of the second kind ($\phi = \varphi = 1/2$); Chebyshev polynomials of the third kind ($\phi = -1/2, \varphi = 1/2$); and the Chebyshev polynomials of the fourth kind ($\phi = 1/2, \varphi = -1/2$). These Chebyshev polynomials admit the following trigonometric function formulations ([14]):

$$\mathsf{P}_\ell^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(t) = \cos(\ell \arccos t),\ \mathsf{P}_\ell^{\left(\frac{1}{2},\frac{1}{2}\right)}(t) = \frac{\sin((\ell+1)\arccos t)}{(\ell+1)\sin(\arccos t)} \tag{2.13}$$

$$\mathsf{P}_\ell^{\left(-\frac{1}{2},\frac{1}{2}\right)}(t) = \frac{\cos\left(\left(\ell+\frac{1}{2}\right)\arccos t\right)}{\cos\left(\frac{\arccos t}{2}\right)},\ \mathsf{P}_\ell^{\left(\frac{1}{2},-\frac{1}{2}\right)}(t) = \frac{\sin\left(\left(\ell+\frac{1}{2}\right)\arccos t\right)}{(2\ell+1)\sin\left(\frac{\arccos t}{2}\right)}. \tag{2.14}$$

For further discussion on these orthogonal polynomials, see [13].

3 Preliminary Results

This section discusses some results on the series representations of normalised Jacobi polynomials $\mathsf{P}_\ell^{\phi,\varphi}(t)$. The proposed preliminary results will be useful in the sequel.

3.1 Series Representation of Jacobi Polynomials

We start with the following result, which follows from the finite series representation (2.1), and it is given as a lemma.

Lemma 3.1. *The normalised Jacobi polynomials $\mathsf{P}_\ell^{(\phi,\varphi)}(t)$ have a finite series representation*

$$\mathsf{P}_\ell^{(\phi,\varphi)}(t) = \sum_{j=0}^{\ell} J_{j,\ell}^{\phi,\varphi} t^j, \tag{3.1}$$

where the expansion (Jacobi) coefficients $J_{j,\ell}^{\phi,\varphi}$ $(0 \le j \le \ell, \ell = 0, 1, 2, \dots)$ are given by

$$J_{j,\ell}^{\phi,\varphi} = \sum_{k=j}^{\ell} \frac{(-1)^{k-j}\ell!\Gamma(\phi+1)\Gamma(\phi+\varphi+\ell+k+1)}{2^k j!(k-j)!(\ell-k)!\Gamma(\phi+k+1)\Gamma(\phi+\varphi+\ell+1)}. \tag{3.2}$$

In particular, the coefficients $J_{j,\ell}^{\phi,\varphi}$ satisfy the identity

$$\sum_{j=0}^{\ell} J_{j,\ell}^{\phi,\varphi} = 1. \tag{3.3}$$

As a consequence of Lemma 3.1 and the generalised Cauchy product, we have the following result.

Proposition 3.1 ([16]). *Let the function $U(x)$ admit the convergent power series about $x = 0$:*

$$U(x) = \sum_{k=0}^{\infty} u_k x^k, \qquad 0 \le x \le 1. \tag{3.4}$$

The normalised Jacobi polynomials $\mathsf{P}_\ell^{(\phi,\varphi)}$ have the power series formulation

$$\mathsf{P}_\ell^{(\phi,\varphi)}(U(x)) = \sum_{k=0}^{\infty} \mathfrak{J}_{k,\ell}^{\phi,\varphi} x^k, \tag{3.5}$$

where the coefficients $\mathfrak{J}_{k,\ell}^{\phi,\varphi}$ are given by

$$\mathfrak{J}_{k,\ell}^{\phi,\varphi} = \sum_{p=0}^{\ell} J_{p,\ell}^{\phi,\varphi} \mathfrak{U}_{k,p}, \tag{3.6}$$

with $\mathfrak{U}_{k,p}$ $(p = 0, 1, 2, \dots)$ *described by*

$$\mathfrak{U}_{k,0} = \delta_{k0} \quad (\delta_{00} = 1), \quad \mathfrak{U}_{k,1} = u_k, \qquad \mathfrak{U}_{k,2} = \sum_{q_1=0}^{k} u_{q_1} u_{k-q_1} \tag{3.7}$$

$$\mathfrak{U}_{k,p} = \sum_{q_{p-1}=0}^{k} \sum_{q_{p-2}=0}^{q_{p-1}} \cdots \sum_{q_1=0}^{q_2} u_{k-q_{p-1}} u_{q_{p-1}-q_{p-2}} \cdots u_{q_2-q_1} u_{q_1} \quad (p = 3, 4, \dots). \tag{3.8}$$

Here δ_{mn} *is the usual Kronecker delta and* $\mathfrak{U}_{0,p} = (u_0)^p$ $(p = 0, 1, 2, \dots)$.

3.2 The First Generalised Jacobi Coefficients

This subsection presents the first expansion coefficients $\mathfrak{U}_{k,p}$ and $\mathfrak{J}_{k,p}^{\phi,\varphi}$ appearing in Proposition 3.1. These coefficients are interestingly given in terms of the normalised Jacobi polynomials $\mathsf{P}_{\ell}^{(\phi,\varphi)}(u_0)$ and their higher order derivatives with respect to u_0.

3.2.1 The First Expansion Coefficients $\mathfrak{U}_{k,p}$ $(k = 0, 1, 2, \dots)$

Here we present explicit values for the first auxiliary coefficients $\mathfrak{U}_{k,p}$ appearing in Proposition 3.1. The finite sum formulae for these coefficients are given in (3.7). Using appropriate integer sequences, one sees that the coefficients $\mathfrak{U}_{k,p}$ can be expressed in terms of the number u_0^p and its higher order derivatives and are presented as follows:

$$\begin{aligned}
\mathfrak{U}_{0,p} =& u_0^p \\
\mathfrak{U}_{1,p} =& p u_0^{p-1} u_1 \\
\mathfrak{U}_{2,p} =& \binom{p}{2} u_0^{p-2} u_1^2 + p u_0^{p-1} u_2 \\
\mathfrak{U}_{3,p} =& \binom{p}{3} u_0^{p-3} u_1^3 + 2\binom{p}{2} u_0^{p-2} u_1 u_2 + p u_0^{p-1} u_3 \\
\mathfrak{U}_{4,p} =& \binom{p}{4} u_0^{p-4} u_1^4 + 3\binom{p}{3} u_0^{p-3} u_1^2 u_2 + \binom{p}{2} u_0^{p-2} \left(2u_1 u_3 + u_2^2\right) + p u_0^{p-1} u_4 \\
\mathfrak{U}_{5,p} =& \binom{p}{5} u_0^{p-5} u_1^5 + 4\binom{p}{4} u_0^{p-4} u_1^3 u_2 + 3\binom{p}{3} u_0^{p-3} u_1 \left(u_2^2 + u_1 u_3\right) \\
&+ 2\binom{p}{2} u_0^{p-2} \left(u_2 u_3 + u_1 u_4\right) + p u_0^{p-1} u_5 \\
\mathfrak{U}_{6,p} =& \binom{p}{6} u_0^{p-6} u_1^6 + 5\binom{p}{5} u_1^4 u_2 u_0^{p-5} + 2\binom{p}{4} u_0^{p-4} u_1^2 \left(3u_2^2 + 2u_1 u_3\right) \\
&+ \binom{p}{3} u_0^{p-3} \left(6u_1 u_2 u_3 + 3u_1^2 u_4 + u_2^3\right) + \binom{p}{2} u_0^{p-2} \left[2u_2 u_4 + 2u_1 u_5 + u_3^2\right] + p u_0^{p-1} u_6
\end{aligned} \tag{3.9}$$

$$
\begin{aligned}
\mathfrak{U}_{7,p} =& \binom{p}{7} u_1^7 u_0^{p-7} + 6\binom{p}{6} u_1^5 u_2 u_0^{p-6} + 5\binom{p}{5} u_0^{p-5}\left(2u_1^3u_2^2 + u_1^4u_3\right) \\
&+ 4\binom{p}{4} u_0^{p-4} u_1\left(3u_1u_2u_3 + u_2^3 + u_1^2u_4\right) + 3\binom{p}{3} u_0^{p-3}\left(u_1u_3^2 + 2u_1u_2u_4 + u_2^2u_3 + u_1^2u_5\right) \\
&+ 2\binom{p}{2} u_0^{p-2}\left(u_3u_4 + u_2u_5 + u_1u_6\right) + pu_0^{p-1}u_7 \\
\mathfrak{U}_{8,p} =& \binom{p}{8} u_1^8 u_0^{p-8} + 7\binom{p}{7} u_1^6 u_2 u_0^{p-7} + 15\binom{p}{6} u_1^4 u_2^2 u_0^{p-6} + 6\binom{p}{6} u_1^5 u_3 u_0^{p-6} + 10\binom{p}{5} u_1^2 u_2^3 u_0^{p-5} \\
&+ 20\binom{p}{5} u_1^3 u_2 u_3 u_0^{p-5} + 5\binom{p}{5} u_1^4 u_4 u_0^{p-5} + \binom{p}{4} u_2^4 u_0^{p-4} + 12\binom{p}{4} u_1 u_2^2 u_3 u_0^{p-4} \\
&+ 6\binom{p}{4} u_1^2 u_3^2 u_0^{p-4} + 12\binom{p}{4} u_1^2 u_2 u_4 u_0^{p-4} + 4\binom{p}{4} u_1^3 u_5 u_0^{p-4} + 3\binom{p}{3} u_2 u_3^2 u_0^{p-3} \\
&+ 6\binom{p}{3} u_1 u_3 u_4 u_0^{p-3} + 6\binom{p}{3} u_1 u_2 u_5 u_0^{p-3} + 3\binom{p}{3} u_2^2 u_4 u_0^{p-3} + 3\binom{p}{3} u_1^2 u_6 u_0^{p-3} \\
&+ \binom{p}{2} u_4^2 u_0^{p-2} + 2\binom{p}{2} u_3 u_5 u_0^{p-2} + 2\binom{p}{2} u_2 u_6 u_0^{p-2} + 2\binom{r}{2} u_1 u_7 u_0^{p-2} + pu_0^{p-1}u_8.
\end{aligned}
\tag{3.10}
$$

3.2.2 The First Coefficients $\mathfrak{J}_{k,\ell}^{\phi,\varphi}$ $(k = 0, 1, 2, \dots)$

This subsection presents explicit values for the first generalised Jacobi coefficients $\mathfrak{J}_{k,\ell}^{\phi,\varphi}$ appearing in Proposition 3.1. These coefficients are given by the finite series formula (3.6). Substituting the first coefficients $\mathfrak{U}_{k,p}$ (as illustrated in (3.9)) into the finite series (3.6), and then using Lemma 3.1, one sees that $\mathfrak{J}_{k,\ell}^{\phi,\varphi}$ can be expressed in terms of $\mathsf{P}_\ell^{(\phi,\varphi)}(u_0)$ and their higher order derivatives. Indeed using the notation

$$
\mathsf{y}(u_0) := \mathsf{P}_\ell^{(\phi,\varphi)}(u_0),
$$

we have the following first coefficients $\mathfrak{J}_{k,\ell}^{\phi,\varphi}$.

$$
\begin{aligned}
\mathfrak{J}_{0,\ell}^{\phi,\varphi} =& \mathsf{y}(u_0) \\
\mathfrak{J}_{1,\ell}^{\phi,\varphi} =& u_1\mathsf{y}'(u_0) \\
\mathfrak{J}_{2,\ell}^{\phi,\varphi} =& \frac{u_1^2}{2}\mathsf{y}''(u_0) + u_2\mathsf{y}'(u_0) \\
\mathfrak{J}_{3,\ell}^{\phi,\varphi} =& \frac{u_1^3}{6}\mathsf{y}'''(u_0) + u_1u_2\mathsf{y}''(u_0) + u_3\mathsf{y}'(u_0) \\
\mathfrak{J}_{4,\ell}^{\phi,\varphi} =& \frac{u_1^4}{24}\mathsf{y}^{(4)}(u_0) + \frac{u_1^2u_2}{2}\mathsf{y}'''(u_0) + \frac{1}{2}\left(2u_1u_3 + u_2^2\right)\mathsf{y}''(u_0) + u_4\mathsf{y}'(u_0) \\
\mathfrak{J}_{5,\ell}^{\phi,\varphi} =& \frac{u_1^5}{5!}\mathsf{y}^{(5)}(u_0) + \frac{u_1^3u_2}{6}\mathsf{y}^{(4)}(u_0) + \frac{u_1}{2}\left(u_2^2 + u_1u_3\right)\mathsf{y}'''(u_0) \\
&+ \left(u_2u_3 + u_1u_4\right)\mathsf{y}''(u_0) + u_5\mathsf{y}'(u_0)
\end{aligned}
\tag{3.11}
$$

$$\begin{aligned}
\mathfrak{J}_{6,\ell}^{\phi,\varphi} =& \frac{u_1^6}{6!}\mathsf{y}^{(6)}(u_0) + \frac{u_1^4 u_2}{4!}\mathsf{y}^{(5)}(u_0) + \frac{u_1^2}{12}\left(3u_2^2 + 2u_1u_3\right)\mathsf{y}^{(4)}(u_0) \\
&+ \frac{1}{6}\left(6u_1u_2u_3 + 3u_1^2u_4 + u_2^3\right)\mathsf{y}'''(u_0) \\
&+ \frac{1}{2}\left(2u_2u_4 + 2u_1u_5 + u_3^2\right)\mathsf{y}''(u_0) + u_6\mathsf{y}'(u_0) \\
\mathfrak{J}_{7,\ell}^{\phi,\varphi} =& \frac{u_1^7}{7!}\mathsf{y}^{(7)}(u_0) + \frac{u_1^5 u_2}{5!}\mathsf{y}^{(6)}(u_0) + \frac{u_1^3}{24}\left(2u_2^2 + u_1u_3\right)\mathsf{y}^{(5)}(u_0) \\
&+ \frac{u_1}{6}\left(3u_1u_2u_3 + u_2^3 + u_1^2u_4\right)\mathsf{y}^{(4)}(u_0) + \frac{1}{2}\left(u_1u_3^2 + 2u_1u_2u_4 + u_2^2u_3 + u_1^2u_5\right)\mathsf{y}'''(u_0) \\
&+ \left(u_3u_4 + u_2u_5 + u_1u_6\right)\mathsf{y}''(u_0) + u_7\mathsf{y}'(u_0) \\
\mathfrak{J}_{8,\ell}^{\phi,\varphi} =& \frac{u_1^8}{8!}\mathsf{y}^{(8)}(u_0) + \frac{u_2u_1^6}{6!}\mathsf{y}^{(7)}(u_0) + \frac{u_1^4}{240}\left(5u_2^2 + 2u_1u_3\right)\mathsf{y}^{(6)}(u_0) \\
&+ \frac{u_1^2}{24}\left(2u_2^3 + 4u_1u_3u_2 + u_1^2u_4\right)\mathsf{y}^{(5)}(u_0) \\
&+ \frac{1}{24}\left(u_2^4 + 12u_1u_3u_2^2 + 12u_1^2u_4u_2 + 6u_1^2u_3^2 + 4u_1^3u_5\right)\mathsf{y}^{(4)}(u_0) \\
&+ \frac{1}{2}\left(u_6u_1^2 + 2u_3u_4u_1 + 2u_2u_5u_1 + u_2u_3^2 + u_2^2u_4\right)\mathsf{y}^{(3)}(u_0) \\
&+ \frac{1}{2}\left(u_4^2 + 2u_3u_5 + 2u_2u_6 + 2u_1u_7\right)\mathsf{y}''(u_0) + u_8\mathsf{y}'(u_0). \qquad (3.12)
\end{aligned}$$

4 Power Series Solution

In this section, we present the power series algorithm for solving the proposed nonlinear Volterra integral equation (1.2). The proposed power series method is based on the generalised Cauchy product that simplifies the computational difficulty associated with the nonlinear term, as contained in Proposition 6.1.

4.1 Power Series Algorithm

This subsection presents the step-by-step procedure in the power series method of solution of the proposed integral problem (1.2). These steps are clearly highlighted as follows:

Step I Given the nonlinear Volterra integral problem (1.2) with the unknown $U(x)$, the real-valued function $F \in C[0,1]$, the kernel $\sin(x-\xi)$, and the nonlinear function $\mathsf{P}_\ell^{(\phi,\varphi)}(U(\xi))$ $(0 < \xi < x)$.

Step II Assume that the solution $U(x)$ admits a convergent power series in x with expansion coefficients $u_k \in \mathbb{R}$ $(0 < x \leq 1)$ as presented in (3.4). Also, let the given source function $F(x)$ be represented by a convergent power series in x about $x = 0$:

$$F(x) = \sum_{k=0}^{\infty} f_k x^k, \qquad 0 \leq x \leq 1, \qquad (4.1)$$

where $f_k \in \mathbb{R}$ are known expansion coefficients.

Step III Use the Maclaurin series expansion

$$\sin(x-\xi)=\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}(x-\xi)^{2m+1} \tag{4.2}$$

and Proposition 3.1.

Step IV Substitute the power series in Steps Step II and Step III into the proposed integral equation (1.2), and then apply the Cauchy product where necessary.

Step V Take the coefficients of x^k $(k=0,1,2,\dots)$ on both sides of the equation and express a recurrence relation for the expansion coefficient u_k $(k=0,1,2,\dots)$.

Step VI Compute the first coefficients u_k in terms of the initial coefficient u_0.

Step VII Substitute the obtained expansion coefficients in Step Step VI into the assumed power series solution (3.4) to obtain the required analytical solution to the given problem.

4.2 Main Results

This subsection considers the power series solution of the nonlinear Volterra integral equation (1.2). Consideration of special cases of the normalised Jacobi polynomials $\mathsf{P}_\ell^{(\phi,\varphi)}$ unbundles the proposed problem (1.2) into several novel nonlinear Volterra integral equations. In view of the preliminary results presented in Subsection 4.1, we now have the following result.

Theorem 4.1. *For $\mu\in\mathbb{R}$ $(\mu\neq 0)$, the nonlinear Volterra integral equation of the second kind*

$$U(x)=F(x)+\mu\int_0^x \sin(x-\xi)\mathsf{P}_\ell^{(\phi,\varphi)}(U(\xi))\,d\xi \qquad (\ell=1,2,3,\dots) \tag{4.3}$$

admits the series solution

$$U(x)=\sum_{k=0}^{\infty}\mathsf{u}_{k,\ell;\mu}^{\phi,\varphi}x^k, \tag{4.4}$$

where the expansion coefficients $\mathsf{u}_{k,\ell;\mu}^{\phi,\varphi}:=u_k$ $(k=0,1,2,\dots;\ell=1,2,\dots)$ are given by

$$\mathsf{u}_{0,\ell;\mu}^{\phi,\varphi}=f_0,\qquad \mathsf{u}_{1,\ell;\mu}^{\phi,\varphi}=f_1,\qquad \mathsf{u}_{k,\ell;\mu}^{\phi,\varphi}=f_k+\mathsf{J}_{k-2,\ell;\mu}^{\phi,\varphi},\qquad (k=2,3,\dots), \tag{4.5}$$

with $\mathsf{J}_{k-2,\ell;\mu}^{\phi,\varphi}$ depending on $u_0,u_2,\cdots,u_{k-2}$. Here, f_k are the expansion coefficients of the series representation of $F(x)$ and $\mathsf{J}_{q,\ell;\mu}^{\phi,\varphi}$ are given by

$$\mathsf{J}_{q,\ell;\mu}^{\phi,\varphi}=\mu\sum_{p=0}^{[q/2]}\frac{(-1)^p}{(2p+1)!}\mathfrak{J}_{q-2p,\ell}^{\phi,\varphi}B(2p+2,q-2p+1)\qquad (q=0,1,2,\dots), \tag{4.6}$$

where $\mathfrak{J}_{k,\ell}^{\phi,\varphi}$ are as described in Proposition 3.1 and $B(x,y)$ is the beta function of x and y.

Proof. Let $U(x)$ admit the power series as given in (3.4). Indeed, using (3.4), (3.5), (4.1), and (4.2) in (4.3), we have

$$\begin{aligned}
\sum_{k=0}^{\infty} u_k x^k &= \sum_{k=0}^{\infty} f_k x^k + \mu \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mathfrak{I}_{n,\ell}^{\phi,\varphi} \frac{(-1)^m}{(2m+1)!} \int_0^x (x-\xi)^{2m+1} \xi^n \, d\xi \\
&= \sum_{k=0}^{\infty} f_k x^k + \mu \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mathfrak{I}_{n,\ell}^{\phi,\varphi} \frac{(-1)^m}{(2m+1)!} \frac{\Gamma(2m+2)\Gamma(n+1)x^{2m+n+2}}{\Gamma(2m+n+3)} \\
&= \sum_{k=0}^{\infty} f_k x^k + \mu \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mathfrak{I}_{n,\ell}^{\phi,\varphi} \frac{(-1)^m}{(2m+1)!} B(2m+2, n+1) x^{2m+n+2} \\
&= \sum_{k=0}^{\infty} f_k x^k + \mu \sum_{k=0}^{\infty}\sum_{p=0}^{k} \mathfrak{I}_{k-p,\ell}^{\phi,\varphi} \frac{(-1)^p}{(2p+1)!} B(2p+2, k-p+1) x^{k+p+2}.
\end{aligned}$$

Using the identity ([11, Sect. 1.2.3. Eq. (1.18)])

$$\sum_{k=0}^{\infty}\sum_{p=0}^{k} A_{k-p,\ell} = \sum_{k=0}^{\infty}\sum_{p=0}^{[k/2]} A_{k-2p,\ell},$$

we have

$$\begin{aligned}
\sum_{k=0}^{\infty} u_k x^k &= \sum_{k=0}^{\infty} f_k x^k + \mu \sum_{k=0}^{\infty}\sum_{p=0}^{[k/2]} \frac{(-1)^p}{(2p+1)!} \mathfrak{I}_{k-2p,\ell}^{\phi,\varphi} B(2p+2, k-2p+1) x^{k+2} \\
&= \sum_{k=0}^{\infty} f_k x^k + \mu \sum_{k=2}^{\infty}\sum_{p=0}^{[(k-2)/2]} \mathfrak{I}_{k-2p-2,\ell}^{\phi,\varphi} \frac{(-1)^p}{(2p+1)!} B(2p+2, k-2p-1) x^k \\
&= \sum_{k=0}^{\infty} f_k x^k + \sum_{k=2}^{\infty} \mathsf{J}_{k-2,\ell;\mu}^{\phi,\varphi} x^k.
\end{aligned} \tag{4.7}$$

Equating the coefficients of x^k $(k = 0, 1, 2, \dots)$, we have

$$u_0 = f_0, \qquad u_1 = f_1, \qquad u_k = f_k + \mathsf{J}_{k-2,\ell;\mu}^{\phi,\varphi} \qquad (k = 2, 3, \dots; \ell = 1, 2, 3, \dots), \tag{4.8}$$

where

$$\mathsf{J}_{q,\ell;\mu}^{\phi,\varphi} = \mu \sum_{p=0}^{[q/2]} \frac{(-1)^p}{(2p+1)!} \mathfrak{I}_{q-2p,\ell}^{\phi,\varphi} B(2p+2, q-2p+1). \tag{4.9}$$

This completes the proof. □

The first coefficients $\mathsf{J}_{q,\ell;\mu}^{\phi,\varphi}$ in Theorem 4.1 are illustrated as follows:

$$
\begin{aligned}
\frac{\mathsf{J}_{0,\ell;\mu}^{\phi,\varphi}}{\mu} =& \frac{\mathfrak{J}_{0,\ell}^{\phi,\varphi}}{2} = \frac{\mathsf{y}(f_0)}{2}\\
\frac{\mathsf{J}_{1,\ell;\mu}^{\phi,\varphi}}{\mu} =& \frac{\mathfrak{J}_{1,\ell}^{\phi,\varphi}}{6} = \frac{1}{6}u_1\mathsf{y}'(f_0)\\
\frac{\mathsf{J}_{2,\ell;\mu}^{\phi,\varphi}}{\mu} =& -\frac{\mathfrak{J}_{0,\ell}^{\phi,\varphi}}{24} + \frac{\mathfrak{J}_{2,\ell}^{\phi,\varphi}}{12} = \frac{u_1^2}{24}\mathsf{y}''(f_0) + \frac{u_2}{12}\mathsf{y}'(f_0) - \frac{\mathsf{y}(f_0)}{24}\\
\frac{\mathsf{J}_{3,\ell;\mu}^{\phi,\varphi}}{\mu} =& -\frac{\mathfrak{J}_{1,\ell}^{\phi,\varphi}}{120} + \frac{\mathfrak{J}_{3,\ell}^{\phi,\varphi}}{20} = \frac{u_1^3}{120}\mathsf{y}^{(3)}(f_0) + \frac{u_2u_1}{20}\mathsf{y}''(f_0) - \frac{u_1-6u_3}{120}\mathsf{y}'(f_0)\\
\frac{\mathsf{J}_{4,\ell;\mu}^{\phi,\varphi}}{\mu} =& \frac{\mathfrak{J}_{0,\ell}^{\phi,\varphi}}{720} - \frac{\mathfrak{J}_{2,\ell}^{\phi,\varphi}}{360} + \frac{\mathfrak{J}_{4,\ell}^{\phi,\varphi}}{30}\\
=& \frac{u_1^4}{720}\mathsf{y}^{(4)}(f_0) + \frac{u_2u_1^2}{60}\mathsf{y}^{(3)}(f_0) - \frac{u_1^2-24u_3u_1-12u_2^2}{720}\mathsf{y}''(f_0)\\
&- \frac{u_2-12u_4}{360}\mathsf{y}'(f_0) + \frac{\mathsf{y}(f_0)}{720}\\
\frac{\mathsf{J}_{5,\ell;\mu}^{\phi,\varphi}}{\mu} =& \frac{\mathfrak{J}_{1,\ell}^{\phi,\varphi}}{5040} - \frac{\mathfrak{J}_{3,\ell}^{\phi,\varphi}}{840} + \frac{\mathfrak{J}_{5,\ell}^{\phi,\varphi}}{42}\\
=& \frac{u_1^5\mathsf{y}^{(5)}(f_0)}{5040} + \frac{u_2u_1^3}{252}\mathsf{y}^{(4)}(f_0) - \frac{\left(u_1^2-60u_3u_1-60u_2^2\right)u_1}{5040}\mathsf{y}^{(3)}(f_0)\\
&- \frac{u_1u_2-20u_3u_2-20u_1u_4}{840}\mathsf{y}''(f_0) + \frac{(u_1-6u_3+120u_5)}{5040}\mathsf{y}'(f_0)\\
\frac{\mathsf{J}_{6,\ell;\mu}^{\phi,\varphi}}{\mu} =& -\frac{\mathfrak{J}_{0,\ell}^{\phi,\varphi}}{40320} + \frac{\mathfrak{J}_{2,\ell}^{\phi,\varphi}}{20160} - \frac{\mathfrak{J}_{4,\ell}^{\phi,\varphi}}{1680} + \frac{\mathfrak{J}_{6,\ell}^{\phi,\varphi}}{56}\\
=& \frac{u_1^6\mathsf{y}^{(6)}(f_0)}{40320} + \frac{u_2u_1^4}{1344}\mathsf{y}^{(5)}(f_0) - \frac{\left(u_1^2-120u_3u_1-180u_2^2\right)u_1^2}{40320}\mathsf{y}^{(4)}(f_0)\\
&+ \frac{\left(10u_2^3-u_1^2u_2+60u_1u_3u_2+30u_1^2u_4\right)}{3360}\mathsf{y}^{(3)}(f_0)\\
&+ \frac{\left(u_1^2-24u_3u_1+720u_5u_1-12u_2^2+360u_3^2+720u_2u_4\right)}{40320}\mathsf{y}''(f_0)\\
&+ \frac{(u_2-12u_4+360u_6)}{20160}\mathsf{y}'(f_0) - \frac{\mathsf{y}(f_0)}{40320}\\
\frac{\mathsf{J}_{7,\ell;\mu}^{\phi,\varphi}}{\mu} =& -\frac{\mathfrak{J}_{1,\ell}^{\phi,\varphi}}{362880} + \frac{\mathfrak{J}_{3,\ell}^{\phi,\varphi}}{60480} - \frac{\mathfrak{J}_{5,\ell}^{\phi,\varphi}}{3024} + \frac{\mathfrak{J}_{7,\ell}^{\phi,\varphi}}{72}\\
=& \frac{u_1^7\mathsf{y}^{(7)}(f_0)}{362880} + \frac{u_2u_1^5}{8640}\mathsf{y}^{(6)}(f_0) - \frac{\left(u_1^2-210u_3u_1-420u_2^2\right)u_1^3}{362880}\mathsf{y}^{(5)}(f_0)\\
&+ \frac{\left(42u_2^3-u_1^2u_2+126u_1u_3u_2+42u_1^2u_4\right)u_1}{18144}\mathsf{y}^{(4)}(f_0)\\
&+ \frac{\left(u_1^3-60u_3u_1^2+2520u_5u_1^2-60u_2^2u_1+2520u_3^2u_1\right)}{362880}\mathsf{y}^{(3)}(f_0)\\
&+ \frac{\left(5040u_2u_4u_1+2520u_2^2u_3\right)}{362880}\mathsf{y}^{(3)}(f_0)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\left(u_1u_2-20u_3u_2+840u_5u_2-20u_1u_4+840u_3u_4+840u_1u_6\right)}{60480}\mathsf{y}''\left(f_0\right)\\
&-\frac{\left(u_1-6u_3+120u_5-5040u_7\right)}{362880}\mathsf{y}'\left(f_0\right)\\
\frac{\mathsf{J}^{\phi,\varphi}_{8,\ell;\mu}}{\mu}&=\frac{\mathfrak{J}^{\phi,\varphi}_{0,\ell}}{3628800}-\frac{\mathfrak{J}^{\phi,\varphi}_{2,\ell}}{1814400}+\frac{\mathfrak{J}^{\phi,\varphi}_{4,\ell}}{151200}-\frac{\mathfrak{J}^{\phi,\varphi}_{6,\ell}}{5040}+\frac{\mathfrak{J}^{\phi,\varphi}_{8,\ell}}{90}\\
&=\frac{u_1^8\mathsf{y}^{(8)}\left(f_0\right)}{3628800}+\frac{u_2u_1^6}{64800}\mathsf{y}^{(7)}\left(f_0\right)-\frac{\left(u_1^2-336u_3u_1-840u_2^2\right)u_1^4}{3628800}\mathsf{y}^{(6)}\left(f_0\right)\\
&+\frac{\left(112u_2^3-u_1^2u_2+224u_1u_3u_2+56u_1^2u_4\right)u_1^2}{120960}\mathsf{y}^{(5)}\left(f_0\right)\\
&+\frac{\left(u_1^4-120u_3u_1^3+6720u_5u_1^3-180u_2^2u_1^2+10080u_3^2u_1^2+20160u_2u_4u_1^2\right)}{3628800}\mathsf{y}^{(4)}\left(f_0\right)\\
&+\frac{\left(20160u_2^2u_3u_1+1680u_2^4\right)}{3628800}\mathsf{y}^{(4)}\left(f_0\right)\\
&-\frac{\left(10u_2^3-1680u_4u_2^2-u_1^2u_2-1680u_3^2u_2+60u_1u_3u_2-3360u_1u_5u_2\right)}{302400}\mathsf{y}^{(3)}\left(f_0\right)\\
&-\frac{\left(30u_1^2u_4-3360u_1u_3u_4-1680u_1^2u_6\right)}{302400}\mathsf{y}^{(3)}\left(f_0\right)\\
&-\frac{\left(u_1^2-24u_3u_1+720u_5u_1-40320u_7u_1-12u_2^2\right)}{3628800}\mathsf{y}''\left(f_0\right)\\
&-\frac{\left(360u_3^2-20160u_4^2+720u_2u_4\right)}{3628800}\mathsf{y}''\left(f_0\right)\\
&+\frac{\left(40320u_3u_5+40320u_2u_6\right)\mathsf{y}''\left(f_0\right)}{3628800}-\frac{\left(u_2-12u_4+360u_6-20160u_8\right)\mathsf{y}'\left(f_0\right)}{1814400}\\
&+\frac{\mathsf{y}\left(f_0\right)}{3628800}.
\end{aligned}
$$

4.3 Special Cases of Main Results

This subsection gives special cases of our main result Theorem 4.1. It also presents a preliminary result on the higher order derivatives of the normalised Jacobi polynomials $\mathsf{P}_\ell^{(\phi,\varphi)}(t)$ evaluated at $t=1$.

Lemma 4.1 ([15], [17], [18], [19]). *For $\ell = 1, 2, \ldots,$ the normalised Jacobi polynomials $\mathsf{P}_\ell^{(\phi,\varphi)}$ satisfy the differentiation formula*

$$
\mathsf{y}^{(q)}(1) := \frac{d^q}{dt^q}\mathsf{P}_\ell^{(\phi,\varphi)}(t)\bigg|_{t=1} = \sum_{j=1}^{q}\mathsf{c}_j^q(\phi,\varphi)[\ell(\ell+\phi+\varphi+1)]^j \quad (q=1,2,\ldots), \tag{4.10}
$$

where

$$
\mathsf{c}_j^q(\phi,\varphi) = \frac{\mathsf{s}_j^q(\phi,\varphi)}{2^q(\phi+1)(\phi+2)\cdots(\phi+q)}, \qquad \mathsf{s}_q^q(\phi,\varphi)=1. \tag{4.11}
$$

Here $\mathsf{s}_j^q(\phi,\varphi)$ are a class of Jacobi coefficients which coincide with the Jacobi-Stirling numbers of the first kind (see, e.g., [15], [30, Table 2]). The first values $\mathsf{y}^{(q)}(1)$ are illustrated as follows ($\lambda_\ell^{\phi,\varphi}=\ell(\ell+\phi+\varphi+1)$):

$$\mathsf{y}'(1)=\frac{\lambda_\ell^{\phi,\varphi}}{2(\phi+1)} \tag{4.12}$$

$$\mathsf{y}''(1)=\frac{\left(\lambda_\ell^{\phi,\varphi}\right)^2}{4(\phi+1)(\phi+2)}-\frac{(\phi+\varphi+2)\lambda_\ell^{\phi,\varphi}}{4(\phi+1)(\phi+2)} \tag{4.13}$$

$$\begin{aligned}\mathsf{y}'''(1)=&\frac{\left(\lambda_\ell^{\phi,\varphi}\right)^3}{8(\phi+1)(\phi+2)(\phi+3)}-\frac{(3\phi+3\varphi+8)\left(\lambda_\ell^{\phi,\varphi}\right)^2}{8(\phi+1)(\phi+2)(\phi+3)}\\&+\frac{2(\phi+\varphi+3)(\phi+\varphi+2)\lambda_\ell^{\phi,\varphi}}{8(\phi+1)(\phi+2)(\phi+3)}\end{aligned} \tag{4.14}$$

$$\begin{aligned}\mathsf{y}^{(4)}(1)=&\frac{\left(\lambda_\ell^{\phi,\varphi}\right)^4-2(3\phi+3\varphi+10)\left(\lambda_\ell^{\phi,\varphi}\right)^3}{16(\phi+1)(\phi+2)(\phi+3)(\phi+4)}\\&+\frac{\left(11\phi^2+11\varphi^2+22\phi\varphi+70\phi+70\varphi+108\right)\left(\lambda_\ell^{\phi,\varphi}\right)^2}{16(\phi+1)(\phi+2)(\phi+3)(\phi+4)}\\&-\frac{6(\phi+\varphi+2)(\phi+\varphi+3)(\phi+\varphi+4)\lambda_\ell^{\phi,\varphi}}{16(\phi+1)(\phi+2)(\phi+3)(\phi+4)}.\end{aligned} \tag{4.15}$$

We now give a special case of Theorem 4.1.

Theorem 4.2. *The nonlinear Volterra integral equation of the second kind*

$$U(x)=1+\int_0^x \sin(x-\xi)\mathsf{P}_\ell^{(\phi,\varphi)}(U(\xi))\,d\xi \qquad (\ell=1,2,3,\dots) \tag{4.16}$$

admits the series solution

$$U(x)=\sum_{k=0}^{\infty}\mathsf{u}_{k,\ell;1}^{\phi,\varphi}x^k, \tag{4.17}$$

where the expansion coefficients $\mathsf{u}_{k,\ell;1}^{\phi,\varphi}:=u_k$ $(k=0,1,2,\dots;\ell=1,2,\dots)$ are given by

$$\mathsf{u}_{0,\ell;1}^{\phi,\varphi}=1,\qquad \mathsf{u}_{1,\ell;1}^{\phi,\varphi}=0,\qquad \mathsf{u}_{k,\ell;1}^{\phi,\varphi}=\mathsf{J}_{k-2,\ell;1}^{\phi,\varphi},\qquad (k=2,3,\dots). \tag{4.18}$$

Here, $f_0=1,\ f_k=0\ (k=1,2,\dots)$; and $\mathsf{J}_{q,\ell;1}^{\phi,\varphi}$ are given by

$$\mathsf{J}_{q,\ell;1}^{\phi,\varphi}=\sum_{p=0}^{[q/2]}(-1)^p\frac{\mathfrak{J}_{q-2p,\ell}^{\phi,\varphi}}{(2p+1)!}B(2p+2,q-2p+1)\qquad (q=0,1,2,\dots), \tag{4.19}$$

where $\mathfrak{J}_{k,\ell}^{\phi,\varphi}$ are as described in Proposition 3.1 and $B(x,y)$ is the beta function of x and y.

The expansion coefficients $\mathsf{J}_{q,\ell;1}^{\phi,\varphi}$ as illustrated in (4.19) can be simplified further to give the following result.

Corollary 4.1. *The nonlinear Volterra integral equation of the second kind*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\mathsf{P}_\ell^{(\phi,\varphi)}(U(\xi))\,d\xi \qquad (\ell = 1,2,3,\dots) \tag{4.20}$$

admits the series solution

$$\begin{aligned}U(x) =& 1 + \frac{x^2}{2} + \left(\mathcal{C}_{4,\ell}^{\phi,\varphi} - \frac{1}{24}\right)x^4 + \left(\mathcal{C}_{6,\ell}^{\phi,\varphi} + \frac{1}{720}\right)x^6 + \left(\mathcal{C}_{8,\ell}^{\phi,\varphi} - \frac{1}{40320}\right)x^8 \\ &+ \left(\mathcal{C}_{10,\ell}^{\phi,\varphi} + \frac{1}{3628800}\right)x^{10} + \cdots,\end{aligned} \tag{4.21}$$

where

$$\begin{aligned}\mathcal{C}_{4,\ell}^{\phi,\varphi} =& \frac{\lambda_\ell^{\phi,\varphi}}{48(\phi+1)} \\ \mathcal{C}_{6,\ell}^{\phi,\varphi} =& -\frac{(7\phi+3\varphi+14)\lambda_\ell^{\phi,\varphi}}{2880(\phi+1)(\phi+2)} + \frac{(4\phi+5)\left(\lambda_\ell^{\phi,\varphi}\right)^2}{2880(\phi+1)^2(\phi+2)} \\ \mathcal{C}_{8,\ell}^{\phi,\varphi} =& \frac{\left(13\phi^2+16\phi\varphi+65\phi+5\varphi^2+43\varphi+78\right)}{53760(\phi+1)(\phi+2)(\phi+3)}\lambda_\ell^{\phi,\varphi} \\ &- \frac{\left(35\phi^2+21\phi\varphi+143\phi+33\varphi+124\right)}{107520(\phi+1)^2(\phi+2)(\phi+3)}\left(\lambda_\ell^{\phi,\varphi}\right)^2 + \frac{\left(34\phi^2+107\phi+75\right)\left(\lambda_\ell^{\phi,\varphi}\right)^3}{322560(\phi+1)^3(\phi+2)(\phi+3)}\end{aligned} \tag{4.22}$$

$$\begin{aligned}\mathcal{C}_{10,\ell}^{\phi,\varphi} =& -\frac{\left(943\phi^3+2007\phi^2\varphi+8487\phi^2+1395\phi\varphi^2+12654\phi\varphi+24518\phi+315\varphi^3\right)}{29030400(\phi+1)(\phi+2)(\phi+3)(\phi+4)}\lambda_\ell^{\phi,\varphi} \\ &- \frac{\left(4635\varphi^2+19134\varphi+22632\right)}{29030400(\phi+1)(\phi+2)(\phi+3)(\phi+4)}\lambda_\ell^{\phi,\varphi} \\ &+ \frac{\left(1237\phi^4+1684\phi^3\varphi+12269\phi^3+563\phi^2\varphi^2+11936\phi^2\varphi+43316\phi^2\right)}{19353600(\phi+1)^2(\phi+2)^2(\phi+3)(\phi+4)}\left(\lambda_\ell^{\phi,\varphi}\right)^2 \\ &+ \frac{\left(2251\phi\varphi^2+26354\phi\varphi+64384\phi+2306\varphi^2+18436\varphi+33864\right)}{19353600(\phi+1)^2(\phi+2)^2(\phi+3)(\phi+4)}\left(\lambda_\ell^{\phi,\varphi}\right)^2 \\ &- \frac{\left(1168\phi^4+777\phi^3\varphi+10322\phi^3+4659\phi^2\varphi+32093\phi^2+8949\phi\varphi\right)}{29030400(\phi+1)^3(\phi+2)^2(\phi+3)(\phi+4)}\left(\lambda_\ell^{\phi,\varphi}\right)^3 \\ &- \frac{(41812\phi+5310\varphi+19140)}{29030400(\phi+1)^3(\phi+2)^2(\phi+3)(\phi+4)}\left(\lambda_\ell^{\phi,\varphi}\right)^3 \\ &+ \frac{\left(496\phi^4+3902\phi^3+10817\phi^2+12415\phi+5010\right)}{58060800(\phi+1)^4(\phi+2)^2(\phi+3)(\phi+4)}\left(\lambda_\ell^{\phi,\varphi}\right)^4.\end{aligned} \tag{4.23}$$

Proof. One notes that $u_1 = 0$, and as a result, $u_{2r+1} = 0$ $(r = 1, 2,)$. Thus the first expansion coefficients $\mathsf{u}_{k,\ell;1}^{\phi,\varphi} = \mathsf{J}_{k-2,\ell;1}^{\phi,\varphi}$ $(k = 2, 3, \dots)$ of the series solution of the nonlinear Volterra integral equation (4.20) are presented as follows:

$$\begin{aligned}
\mathsf{u}_{2,\ell;1}^{\phi,\varphi} =& \frac{\mathsf{y}(1)}{2} = \frac{1}{2} \\
\mathsf{u}_{4,\ell;1}^{\phi,\varphi} =& \frac{\mathsf{y}'(1)}{24} - \frac{1}{24} \\
\mathsf{u}_{6,\ell;1}^{\phi,\varphi} =& \frac{\mathsf{y}''(1)}{240} + \frac{1}{720}\mathsf{y}'(1)^2 - \frac{\mathsf{y}'(1)}{360} + \frac{1}{720} \\
\mathsf{u}_{8,\ell;1}^{\phi,\varphi} =& \frac{\mathsf{y}^{(3)}(1)}{2688} - \frac{\mathsf{y}''(1)}{2240} + \frac{\mathsf{y}'(1)^3}{40320} - \frac{\mathsf{y}'(1)^2}{13440} + \frac{\mathsf{y}'(1)}{13440} + \frac{\mathsf{y}'(1)\mathsf{y}''(1)}{2240} - \frac{1}{40320} \\
\mathsf{u}_{10,\ell;1}^{\phi,\varphi} =& \frac{\mathsf{y}^{(4)}(1)}{34560} - \frac{\mathsf{y}^{(3)}(1)}{16128} + \frac{\mathsf{y}''(1)^2}{43200} + \frac{\mathsf{y}''(1)}{44800} + \frac{\mathsf{y}'(1)^4}{3628800} - \frac{\mathsf{y}'(1)^3}{907200} + \frac{\mathsf{y}'(1)^2}{604800} - \frac{\mathsf{y}'(1)}{907200} \\
&+ \frac{\mathsf{y}^{(3)}(1)\mathsf{y}'(1)}{16128} + \frac{\mathsf{y}'(1)^2\mathsf{y}''(1)}{44800} - \frac{\mathsf{y}'(1)\mathsf{y}''(1)}{22400} + \frac{1}{3628800}.
\end{aligned} \tag{4.24}$$

Upon using the explicit values of the derivative formulae in (4.12) and simplifying, we obtain the results as required. □

A special case of Corollary 4.1 is the following result on the series solution of nonlinear Volterra integral equations with ultraspherical polynomial nonlinearities.

Corollary 4.2. *The nonlinear Volterra integral equation of the second kind*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\mathsf{P}_\ell^{(\phi,\phi)}(U(\xi))\,d\xi \qquad (\ell = 1, 2, 3, \dots) \tag{4.25}$$

admits the series solution

$$\begin{aligned}
U(x) =& 1 + \frac{x^2}{2} + \left(\mathcal{C}_{4,\ell}^{\phi,\phi} - \frac{1}{24}\right)x^4 + \left(\mathcal{C}_{6,\ell}^{\phi,\phi} + \frac{1}{720}\right)x^6 + \left(\mathcal{C}_{8,\ell}^{\phi,\phi} - \frac{1}{40320}\right)x^8 \\
&+ \left(\mathcal{C}_{10,\ell}^{\phi,\phi} + \frac{1}{3628800}\right)x^{10} + \cdots,
\end{aligned} \tag{4.26}$$

where

$$\mathcal{C}_{4,\ell}^{\phi,\phi} = \frac{\lambda_\ell^{\phi,\phi}}{48(\phi+1)} \tag{4.27}$$

$$\mathcal{C}_{6,\ell}^{\phi,\phi} = -\frac{(10\phi+14)\lambda_\ell^{\phi,\phi}}{2880(\phi+1)(\phi+2)} + \frac{(4\phi+5)\left(\lambda_\ell^{\phi,\phi}\right)^2}{2880(\phi+1)^2(\phi+2)} \tag{4.28}$$

$$\begin{aligned}
\mathcal{C}_{8,\ell}^{\phi,\phi} =& \frac{\left(34\phi^2+108\phi+78\right)\lambda_\ell^{\phi,\phi}}{53760(\phi+1)(\phi+2)(\phi+3)} - \frac{\left(56\phi^2+176\phi+124\right)\left(\lambda_\ell^{\phi,\phi}\right)^2}{107520(\phi+1)^2(\phi+2)(\phi+3)} \\
&+ \frac{\left(34\phi^2+107\phi+75\right)\left(\lambda_\ell^{\phi,\phi}\right)^3}{322560(\phi+1)^3(\phi+2)(\phi+3)} \\
\mathcal{C}_{10,\ell}^{\phi,\phi} =& -\frac{\left(4660\phi^3+25776\phi^2+43652\phi+22632\right)}{29030400(\phi+1)(\phi+2)(\phi+3)(\phi+4)}\lambda_\ell^{\phi,\phi} \\
&+ \frac{\left(3484\phi^4+26456\phi^3+71976\phi^2+82820\phi+33864\right)}{19353600(\phi+1)^2(\phi+2)^2(\phi+3)(\phi+4)}\left(\lambda_\ell^{\phi,\phi}\right)^2 \\
&- \frac{\left(1945\phi^4+14981\phi^3+41042\phi^2+47122\phi+19140\right)}{29030400(\phi+1)^3(\phi+2)^2(\phi+3)(\phi+4)}\left(\lambda_\ell^{\phi,\phi}\right)^3 \\
&+ \frac{\left(496\phi^4+3902\phi^3+10817\phi^2+12415\phi+5010\right)}{58060800(\phi+1)^4(\phi+2)^2(\phi+3)(\phi+4)}\left(\lambda_\ell^{\phi,\phi}\right)^4 .
\end{aligned} \tag{4.29}$$

We have the following three special cases of Corollary 4.2. The nonlinear terms of these Volterra integral equations are Chebyshev polynomials of the first kind, Chebyshev polynomials of the second kind, and Legendre polynomials, respectively.

Corollary 4.3. *The nonlinear Volterra integral equation of the second kind*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\cos(\ell \arccos U(\xi))\, d\xi \qquad (\ell = 1,2,3,\dots) \tag{4.30}$$

admits the series solution

$$\begin{aligned}
U(x) =& 1 + \frac{x^2}{2} + \frac{(\ell-1)(\ell+1)}{24}x^4 + \frac{(\ell-1)(\ell+1)\left(2\ell^2-1\right)}{720}x^6 \\
&+ \frac{(\ell-1)(\ell+1)\left(8\ell^4-12\ell^2+1\right)}{40320}x^8 \\
&+ \frac{(\ell-1)(\ell+1)\left(160\ell^6-463\ell^4+378\ell^2-3\right)}{10886400}x^{10} + \cdots .
\end{aligned} \tag{4.31}$$

Corollary 4.4. *The nonlinear Volterra integral equation of the second kind*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\frac{\sin((\ell+1)\arccos U(\xi))}{(\ell+1)\sin(\arccos U(\xi))}\, d\xi \qquad (\ell = 1,2,3,\dots) \tag{4.32}$$

admits the series solution

$$\begin{aligned}U(x) =&1+\frac{x^2}{2}+\frac{(\ell-1)(\ell+3)}{72}x^4+\frac{(\ell-1)(\ell+3)\left(14\ell^2+28\ell-15\right)}{32400}x^6\\&+\frac{(\ell-1)(\ell+3)\left(548\ell^4+2192\ell^3-232\ell^2-4848\ell+315\right)}{38102400}x^8\\&+\frac{(\ell-1)(\ell+3)\left(25672\ell^6+154032\ell^5+91983\ell^4-658948\ell^3\right)}{51438240000}x^{10}\\&-\frac{(\ell-1)(\ell+3)\left(351054\ell^2-1026540\ell+4725\right)}{51438240000}x^{10}+\cdots.\end{aligned}\tag{4.33}$$

Corollary 4.5. *The nonlinear Volterra integral equation of the second kind*

$$U(x)=1+\int_0^x\sin(x-\xi)P_\ell(U(\xi))\,d\xi\qquad(\ell=1,2,3,\dots)\tag{4.34}$$

admits the series solution

$$\begin{aligned}U(x) =&1+\frac{x^2}{2}+\frac{(\ell-1)(\ell+2)}{48}x^4+\frac{(\ell-1)(\ell+2)\left(5\ell^2+5\ell-4\right)}{5760}x^6\\&+\frac{(\ell-1)(\ell+2)\left(25\ell^4+50\ell^3-49\ell^2-74\ell+8\right)}{645120}x^8\\&+\frac{(\ell-1)(\ell+2)\left(835\ell^6+2505\ell^5-2205\ell^4-8585\ell^3+2802\ell^2+7512\ell-64\right)}{464486400}x^{10}\\&+\cdots.\end{aligned}\tag{4.35}$$

4.4 Volterra Integral Equations Involving Jacobi Linearities

This subsection considers Volterra integral equations of the second kind with linearities $\mathsf{P}_1^{(\phi,\varphi)}(U)$ for different values of the Jacobi parameters ϕ,φ. Interestingly, the series solutions of the linear Volterra integral equations considered converge to the exact or closed-form solutions.

Corollary 4.6. *The linear Volterra integral equation of the second kind*

$$U(x)=1+\int_0^x\sin(x-\xi)\left(\frac{\phi+\varphi+2}{2(\phi+1)}U(\xi)+\frac{\phi-\varphi}{2(\phi+1)}\right)d\xi\tag{4.36}$$

admits the analytical solution

$$\begin{aligned}U(x) =&1+\frac{x^2}{2}-\frac{\phi-\varphi}{48(\phi+1)}x^4+\frac{(\phi-\varphi)^2}{2880(\phi+1)^2}x^6-\frac{(\phi-\varphi)^3}{322560(\phi+1)^3}x^8\\&+\frac{(\phi-\varphi)^4}{58060800(\phi+1)^4}x^{10}+\cdots\\=&1+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\left(\frac{\phi-\varphi}{\phi+1}\right)^{k-1}}{2^{k-1}(2k)!}x^{2k}\\=&1-\frac{2(\phi+1)\left(\cos\left(x\sqrt{\frac{\phi-\varphi}{2(\phi+1)}}\right)-1\right)}{\phi-\varphi}\qquad(\phi\neq\varphi),\end{aligned}\tag{4.37}$$

which is the exact solution. For the special case $\phi = \varphi$, we see that the linear Volterra equation of the second kind

$$U(x) = 1 + \int_0^x \sin(x-\xi)U(\xi)\,d\xi \tag{4.38}$$

admits the exact solution

$$U(x) = 1 + \frac{x^2}{2}. \tag{4.39}$$

Remark 4.1. One recovers (4.38)-(4.39) from Corollaries 4.3- 4.5 by setting $\ell = 1$.

Example 4.1 ($\phi = -1/2, \varphi = 1/2$)**.** *Here one sees that the linear Volterra integral equation*

$$U(x) = 1 + \int_0^x \sin(x-\xi)(2U(\xi)-1)\,d\xi \tag{4.40}$$

admits the exact solution

$$U(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \frac{x^8}{40320} + \frac{x^{10}}{3628800} + \cdots = \cosh x. \tag{4.41}$$

Example 4.2 ($\phi = 1/2, \varphi = -1/2$)**.** *Indeed the linear Volterra integral equation*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\left(\frac{2}{3}U(\xi) + \frac{1}{3}\right)d\xi \tag{4.42}$$

has the exact solution

$$\begin{aligned} U(x) =& 1 + \frac{x^2}{2} - \frac{x^4}{72} + \frac{x^6}{6480} - \frac{x^8}{1088640} + \frac{x^{10}}{293932800} + \cdots \\ =& 1 + \frac{x^2}{2} - \frac{x^4}{4!\cdot 3} + \frac{x^6}{6!\cdot 9} - \frac{x^8}{8!\cdot 27} + \frac{x^{10}}{10!\cdot 81} + \cdots \\ =& 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n}}{3^{n-1}(2n)!} = 4 - 3\cos\left(\frac{x}{\sqrt{3}}\right). \end{aligned} \tag{4.43}$$

Example 4.3. *One has another important special linear Volterra integral equation by setting $\phi = 7, \varphi = 3$:*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\left(\frac{3}{4}U(\xi) + \frac{1}{4}\right)d\xi \tag{4.44}$$

admits the exact solution

$$\begin{aligned} U(x) =& 1 + \frac{x^2}{2} - \frac{x^4}{96} + \frac{x^6}{11520} - \frac{x^8}{2580480} + \frac{x^{10}}{928972800} + \cdots \\ =& 1 + \frac{x^2}{2} - \frac{x^4}{4!\cdot 4} + \frac{x^6}{6!\cdot 16} - \frac{x^8}{8!\cdot 64} + \frac{x^{10}}{10!\cdot 256} + \cdots \\ =& 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n}}{4^{n-1}(2n)!} = 5 - 4\cos\left(\frac{x}{2}\right). \end{aligned} \tag{4.45}$$

Remark 4.2. For different values of the Jacobi parameters ϕ and φ ($\phi \neq \varphi$), several linear Volterra integral equations of the second kind and their exact solutions can easily be obtained from Corollary 4.6.

4.5 Volterra Integral Equations with Jacobi Polynomial Nonlinearities

In this subsection, we present series solutions of special cases of nonlinear Volterra integral equation (4.16) by considering the first degrees $\ell = 2, 3$. Special Volterra integral equations that involve Gegenbauer, Legendre, and Chebyshev polynomial nonlinearities are also considered.

4.5.1 Volterra Integral Equations with Jacobi Quadratic Nonlinearities

Here we present series solutions of Volterra integral equations with 2nd-degree Jacobi polynomials. Volterra integral equations involving Gegenbauer, Legendre, and Chebyshev quadratic nonlinearities are important special cases.

Corollary 4.7. *The nonlinear Volterra integral equation of the second kind*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\left(J_{2,2}^{\phi,\varphi}U^2(\xi) + J_{1,2}^{\phi,\varphi}U(\xi) + J_{0,2}^{\phi,\varphi}\right)d\xi, \tag{4.46}$$

with

$$\begin{aligned} J_{2,2}^{\phi,\varphi} &= \frac{(\phi+\varphi+3)(\phi+\varphi+4)}{4(\phi+1)(\phi+2)},\ J_{1,2}^{\phi,\varphi} = \frac{(\phi-\varphi)(\phi+\varphi+3)}{2(\phi+1)(\phi+2)} \\ J_{0,2}^{\phi,\varphi} &= -\frac{4+(\phi+\varphi)-(\phi-\varphi)^2}{4(\phi+1)(\phi+2)}, \end{aligned} \tag{4.47}$$

admits the series solution

$$\begin{aligned} U(x) =& 1 + \frac{x^2}{2} + \frac{\varphi+2}{24(\phi+1)}x^4 + \frac{5\varphi^2\phi + 7\varphi^2 + 6\varphi\phi^2 + 35\varphi\phi + 37\varphi + 3\phi^3 + 24\phi^2 + 65\phi + 52}{1440(\phi+1)^2(\phi+2)}x^6 \\ &+ \frac{(\varphi+2)\left(11\varphi^2 + 71\varphi + 9\phi^3 + 18\varphi\phi^2 + 72\phi^2 + 10\varphi^2\phi + 85\varphi\phi + 175\phi + 116\right)}{40320(\phi+1)^3(\phi+2)}x^8 \\ &+ \frac{42\phi^6 + 168\phi^5\varphi + 672\phi^5 + 333\phi^4\varphi^2 + 2424\phi^4\varphi + 4608\phi^4 + 330\phi^3\varphi^3}{7257600(\phi+1)^4(\phi+2)^2}x^{10} \\ &+ \frac{3726\phi^3\varphi^2 + 13716\phi^3\varphi + 17016\phi^3 + 125\phi^2\varphi^4 + 2317\phi^2\varphi^3 + 14265\phi^2\varphi^2}{7257600(\phi+1)^4(\phi+2)^2}x^{10} \\ &+ \frac{36700\phi^2\varphi + 34598\phi^2 + 335\phi\varphi^4 + 4405\phi\varphi^3 + 21210\phi\varphi^2 + 44800\phi\varphi}{7257600(\phi+1)^4(\phi+2)^2}x^{10} \\ &+ \frac{35480\phi + 212\varphi^4 + 2434\varphi^3 + 10386\varphi^2 + 19624\varphi + 13952}{7257600(\phi+1)^4(\phi+2)^2}x^{10} + \cdots. \end{aligned} \tag{4.48}$$

One can also set $\phi = \varphi$ in Corollary 4.7 to give the following result.

Corollary 4.8. *The Volterra integral equation with Gegenbauer quadratic nonlinearities*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\left(\frac{2\phi+3}{2(\phi+1)}U^2(\xi) - \frac{1}{2(\phi+1)}\right)d\xi \tag{4.49}$$

admits the series solution

$$U(x) =1 + \frac{x^2}{2} + \frac{\phi+2}{24(\phi+1)}x^4 + \frac{7\phi^2+19\phi+13}{720(\phi+1)^2}x^6 + \frac{(\phi+2)\left(37\phi^2+94\phi+58\right)}{40320(\phi+1)^3}x^8 + \frac{499\phi^4+2741\phi^3+5643\phi^2+5144\phi+1744}{3628800(\phi+1)^4}x^{10} + \cdots . \tag{4.50}$$

Other special cases of Corollary 4.7 are the following, given as examples.

Example 4.4. *The Volterra integral equation with the Legendre quadratic nonlinearity*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\left(\frac{3}{2}U^2(\xi) - \frac{1}{2}\right) d\xi \tag{4.51}$$

admits the series solution

$$U(x) =1 + \frac{x^2}{2} + \frac{1}{12}x^4 + \frac{13}{720}x^6 + \frac{29}{10080}x^8 + \frac{109}{226800}x^{10} + \cdots . \tag{4.52}$$

Example 4.5. *The Volterra integral equation with Chebyshev quadratic nonlinearity of the first kind*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\left(2U^2(\xi) - 1\right) d\xi \tag{4.53}$$

has the series solution

$$U(x) =1 + \frac{x^2}{2} + \frac{1}{8}x^4 + \frac{7}{240}x^6 + \frac{27}{4480}x^8 + \frac{1447}{1209600}x^{10} + \cdots . \tag{4.54}$$

Example 4.6. *In the case of Chebyshev quadratic nonlinearity of the second kind, one sees that the Volterra integral equation*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\left(\frac{4}{3}U^2(\xi) - \frac{1}{3}\right) d\xi \tag{4.55}$$

admits the series solution

$$U(x) =1 + \frac{x^2}{2} + \frac{5}{72}x^4 + \frac{97}{6480}x^6 + \frac{457}{217728}x^8 + \frac{97609}{293932800}x^{10} + \cdots . \tag{4.56}$$

Example 4.7. *The Volterra integral equation with Chebyshev quadratic nonlinearity of the third kind*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\left(4U^2(\xi) - 2U(\xi) - 1\right) d\xi \tag{4.57}$$

has the series solution

$$U(x) =1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \frac{49}{720}x^6 + \frac{169}{8064}x^8 + \frac{22201}{3628800}x^{10} + \cdots . \tag{4.58}$$

Example 4.8. *The Volterra integral equation with Chebyshev quadratic nonlinearity of the fourth kind*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\left(\frac{4}{5}U^2(\xi) + \frac{2}{5}U(\xi) - \frac{1}{5}\right) d\xi \tag{4.59}$$

admits the series solution

$$U(x) =1 + \frac{x^2}{2} + \frac{1}{24}x^4 + \frac{29}{3600}x^6 + \frac{149}{201600}x^8 + \frac{8641}{90720000}x^{10} + \cdots . \tag{4.60}$$

Example 4.9. *Another interesting nonlinear integral equation can be obtained by setting $\phi = 7$ and $\varphi = 3$:*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\left(\frac{91}{144}U^2(\xi) + \frac{13}{36}U(\xi) + \frac{1}{144}\right) d\xi \tag{4.61}$$

with the series solution

$$U(x) = 1 + \frac{x^2}{2} + \frac{5}{192}x^4 + \frac{803}{138240}x^6 + \frac{1481}{4128768}x^8 + \frac{19279027}{401316249600}x^{10} + \cdots . \tag{4.62}$$

4.5.2 Volterra Integral Equations with Jacobi Cubic Nonlinearities

We give series solutions of Volterra integral equations with 3rd-degree Jacobi polynomials. Volterra integral equations involving Gegenbauer, Legendre, and Chebyshev cubic nonlinearities are important special cases.

Corollary 4.9. *Consider the nonlinear Volterra integral equation*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\left(J_{3,3}^{\phi,\varphi}U^3(\xi) + J_{2,3}^{\phi,\varphi}U^2(\xi) + J_{1,3}^{\phi,\varphi}U(\xi) + J_{0,3}^{\phi,\varphi}\right) d\xi, \tag{4.63}$$

with

$$\begin{aligned} J_{3,3}^{\phi,\varphi} &= \frac{(\phi+\varphi+4)(\phi+\varphi+5)(\phi+\varphi+6)}{8(\phi+1)(\phi+2)(\phi+3)} \\ J_{2,3}^{\phi,\varphi} &= \frac{3(\phi-\varphi)(\phi+\varphi+4)(\phi+\varphi+5)}{8(\phi+1)(\phi+2)(\phi+3)} \\ J_{1,3}^{\phi,\varphi} &= \frac{3\left((\phi-\varphi)^2-\phi-\varphi-6\right)(\phi+\varphi+4)}{8(\phi+1)(\phi+2)(\phi+3)} \\ J_{0,3}^{\phi,\varphi} &= \frac{(\phi-\varphi)^3-(\phi-\varphi)(3\phi+3\varphi+16)}{8(\phi+1)(\phi+2)(\phi+3)}. \end{aligned} \tag{4.64}$$

The series solution is given by

$$\begin{aligned} y(x) =& 1 + \frac{1}{2}x^2 + \frac{3\varphi+\phi+10}{48(\phi+1)}x^4 + \frac{36\varphi^2+282\varphi+19\phi^3+42\varphi\phi^2+202\phi^2}{2880(\phi+1)^2(\phi+2)}x^6 \\ &+ \frac{27\varphi^2\phi+270\varphi\phi+662\phi+560}{2880(\phi+1)^2(\phi+2)}x^6 \\ &+ \frac{1224\varphi^3+14958\varphi^2+60660\varphi+199\phi^5+819\varphi\phi^4+4049\phi^4+1053\varphi^2\phi^3}{322560(\phi+1)^3(\phi+2)(\phi+3)}x^8 \\ &+ \frac{11673\varphi\phi^3+30318\phi^3+441\varphi^3\phi^2+9315\varphi^2\phi^2+56196\varphi\phi^2+103426\phi^2}{322560(\phi+1)^3(\phi+2)(\phi+3)}x^8 \\ &+ \frac{1611\varphi^3\phi+22734\varphi^2\phi+104544\varphi\phi+157100\phi+81600}{322560(\phi+1)^3(\phi+2)(\phi+3)}x^8 + \cdots . \end{aligned} \tag{4.65}$$

We conclude this section by presenting several examples of Volterra integral equations with Jacobi cubic nonlinearities.

Example 4.10. *The nonlinear Volterra integral equation of the second kind with* 3*rd-degree Gegenbauer polynomials*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\left(\frac{2\phi+5}{2(\phi+1)}U^3(\xi) - \frac{3}{2(\phi+1)}U(\xi) - \frac{(\phi-1)\phi}{8(\phi+2)(\phi+3)}\right) d\xi. \tag{4.66}$$

has the series solution

$$\begin{aligned} y(x) =& 1 + \frac{1}{2}x^2 + \frac{2\phi+5}{24(\phi+1)}x^4 + \frac{(2\phi+5)(11\phi+14)}{720(\phi+1)^2}x^6 + \frac{(2\phi+5)\left(157\phi^2+488\phi+340\right)}{40320(\phi+1)^3}x^8 \\ &+ \frac{(2\phi+5)^2\left(1921\phi^2+4583\phi+2671\right)}{3628800(\phi+1)^4}x^{10} + \cdots . \end{aligned} \tag{4.67}$$

Example 4.11. *The Volterra integral equation with Chebyshev cubic nonlinearity of the first kind*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\left(4U^3(\xi) - 3U(\xi)\right) d\xi. \tag{4.68}$$

admits the series solution

$$y(x) = 1 + \frac{1}{2}x^2 + \frac{1}{3}x^4 + \frac{17}{90}x^6 + \frac{541}{5040}x^8 + \frac{3439x^{10}}{56700} + \cdots . \tag{4.69}$$

Example 4.12. *The Volterra integral equation with Chebyshev cubic nonlinearity of the second kind*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\left(2U^3(\xi) - U(\xi)\right) d\xi. \tag{4.70}$$

admits the series solution

$$y(x) = 1 + \frac{1}{2}x^2 + \frac{1}{3}x^4 + \frac{17}{90}x^6 + \frac{541}{5040}x^8 + \frac{2419x^{10}}{226800} + \cdots . \tag{4.71}$$

Example 4.13. *The Volterra integral equation with Chebyshev cubic nonlinearity of the third kind*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\left(8U^3(\xi) - 4U^2(\xi) - 4U(\xi) + 1\right) d\xi. \tag{4.72}$$

admits the series solution

$$y(x) = 1 + \frac{1}{2}x^2 + \frac{11}{24}x^4 + \frac{241}{720}x^6 + \frac{9971}{40320}x^8 + \frac{659881}{3628800}x^{10} + \cdots . \tag{4.73}$$

Example 4.14. *The Volterra integral equation with Chebyshev cubic nonlinearity of the fourth kind*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\left(\frac{8}{7}U^3(\xi) + \frac{4}{7}U^2(\xi) - \frac{4}{7}U(\xi) - \frac{1}{7}\right) d\xi. \tag{4.74}$$

has the series solution

$$y(x) = 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{11}{240}x^6 + \frac{437}{31360}x^8 + \frac{37141}{8467200}x^{10} + \cdots . \tag{4.75}$$

Example 4.15. *Another nonlinear Volterra integral equation can be obtained by setting* $\phi = 7$, $\varphi = 3$:

$$U(x) = 1 + \int_0^x \sin(x-\xi)\left(\frac{7}{12}U^3(\xi) + \frac{7}{16}U^2(\xi) - \frac{1}{48}\right) d\xi. \tag{4.76}$$

with the series solution

$$y(x) = 1 + \frac{1}{2}x^2 + \frac{13}{192}x^4 + \frac{1009}{46080}x^6 + \frac{94597}{20643840}x^8 + \frac{15688681}{14863564800}x^{10} + \cdots . \tag{4.77}$$

5 Several Other Nonlinear Volterra Integral Equations

In this section, we establish several nonlinear Volterra integral equations of the second kind involving spherical functions on rank one symmetric spaces of compact type ([32]), simply abbreviated R1SSCT. Interestingly, spherical functions on rank one symmetric spaces of compact type coincide with normalised Jacobi polynomials $\mathsf{P}_\ell^{(\phi,\varphi)}$ ([17], [26]). One recalls that N-dimensional rank one symmetric spaces of compact type include the sphere $\mathbb{S}^n$ (of real dimension $N = n$), the complex projective space $\mathsf{P}^n(\mathbb{C})$ (of real dimension $N = 2n$), the quaternionic projective space $\mathsf{P}^n(\mathbb{H})$ (of real dimension $N = 4n$), and the Cayley projective plane $\mathsf{P}^2(\mathrm{Cay})$ (of real dimension $N = 16$). For further and detailed geometric discussion on these symmetric spaces, see, e.g., [49, Ch. 3].

The following remark relates and connects spherical functions on R1SSCT to our discussion:

(a) The spherical functions on the sphere $\mathbb{S}^n$ are normalised Gegenbauer polynomials $\mathsf{C}_\ell^{(n-1)/2}$. In particular, the spherical functions on the circle $\mathbb{S}^1$ are Chebyshev polynomials of the first kind $\mathsf{P}_\ell^{(-1/2,-1/2)}$; the spherical functions on the sphere $\mathbb{S}^2$ are Legendre polynomials P_ℓ; while the spherical functions on the sphere $\mathbb{S}^3$ are Chebyshev polynomials of the second kind $\mathsf{P}_\ell^{(1/2,1/2)}$.

(b) The spherical functions on the complex projective space $\mathsf{P}^n(\mathbb{C})$ are Jacobi polynomials $\mathsf{P}_\ell^{(n-1,0)}$.

(c) The spherical functions on the quaternionic projective space $\mathsf{P}^n(\mathbb{H})$ are Jacobi polynomials $\mathsf{P}_\ell^{(2n-1,1)}$.

(d) The spherical functions on the Cayley projective plane $\mathsf{P}^2(\mathrm{Cay})$ are Jacobi polynomials $\mathsf{P}_\ell^{(7,3)}$.

Note that the Volterra integral equations with Jacobi nonlinearities $\mathsf{P}_\ell^{(7,3)}$ $(\ell = 2, 3)$ have been considered in Section 4.5.

5.1 Nonlinear Volterra Integral Equations with $\mathsf{C}_\ell^{\frac{n-1}{2}}$ $(\ell = 2, 3, \dots)$

In this subsection, we introduce nonlinear Volterra integral equations involving Gegenbauer polynomials $\mathsf{C}_\ell^{(n-1)/2}$. Upon setting $n = 1, 2, 3, \dots$; $\ell = 2, 3, \dots$; several nonlinear Volterra integral equations with their solutions are presented. To this end, setting $\phi = (n-2)/2$ in Corollary 4.2, we have the following result.

Corollary 5.1. *The nonlinear Volterra integral equation of the second kind*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\mathsf{C}_\ell^{\frac{n-1}{2}}(U(\xi))\,d\xi \qquad (\ell = 2, 3, \dots) \tag{5.1}$$

admits the series solution $(\lambda = \ell(\ell + n - 1))$

$$\begin{aligned}
U(x) =& 1 + \frac{x^2}{2} - \frac{n-\lambda}{24n}x^4 + \frac{(n-\lambda)\left(n^2 - 2\lambda(2n+1) + 2n\right)}{720n^2(n+2)}x^6 \\
&- \frac{(n-\lambda)\left(n^4 + 6n^3 + 2\lambda^2\left(17n^2 + 39n + 4\right) - 2\lambda\left(25n^2 + 57n + 8\right)n + 8n^2\right)}{40320n^3(n+2)(n+4)}x^8 \\
&+ \frac{(n-\lambda)\left(n^7 + 14n^6 + 68n^5 + 136n^4 + 96n^3\right)}{3628800n^4(n+2)^2(n+4)(n+6)}x^{10} \\
&- \frac{4(n-\lambda)\lambda^3\left(124n^4 + 959n^3 + 2087n^2 + 1006n + 24\right)}{3628800n^4(n+2)^2(n+4)(n+6)}x^{10} \\
&+ \frac{3n(n-\lambda)\lambda^2\left(483n^4 + 3522n^3 + 7576n^2 + 4528n + 96\right)}{3628800n^4(n+2)^2(n+4)(n+6)}x^{10} \\
&- \frac{6n^2(n-\lambda)\lambda(n+2)\left(194n^3 + 981n^2 + 1006n + 24\right)}{3628800n^4(n+2)^2(n+4)(n+6)}x^{10} + \cdots . \qquad (5.2)
\end{aligned}$$

We give some special cases of Corollary 5.1.

Example 5.1. *Let the nonlinear term be given by the normalised Gegenbauer polynomials* $\mathsf{C}_2^{(n-1)/2}$. *Then the nonlinear Volterra integral equation of the second kind*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\left(\frac{n+1}{n}U^2(\xi) - \frac{1}{n}\right)d\xi \qquad (5.3)$$

admits the series solution

$$\begin{aligned}
U(x) =& 1 + \frac{x^2}{2} + \frac{n+2}{24n}x^4 + \frac{7n^2 + 10n + 4}{720n^2}x^6 + \frac{(n+2)\left(37n^2 + 40n + 4\right)}{40320n^3}x^8 \\
&+ \frac{499n^4 + 1490n^3 + 1656n^2 + 680n + 16}{3628800n^4}x^{10} + \cdots . \qquad (5.4)
\end{aligned}$$

Example 5.2. *In the case of Gegenbauer cubic nonlinearities* $\mathsf{C}_3^{(n-1)/2}$, *we see that the nonlinear Volterra integral equation of the second kind*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\left(\frac{n+3}{n}U^3(\xi) - \frac{3}{n}U(\xi)\right)d\xi \qquad (5.5)$$

admits the series solution

$$\begin{aligned}
U(x) =& 1 + \frac{x^2}{2} + \frac{n+3}{12n}x^4 + \frac{(n+3)(11n+6)}{360n^2}x^6 + \frac{(n+3)\left(157n^2 + 348n + 36\right)}{20160n^3}x^8 \\
&+ \frac{(n+3)^2\left(1921n^2 + 1482n + 36\right)}{907200n^4}x^{10} + \cdots . \qquad (5.6)
\end{aligned}$$

5.2 Nonlinear Volterra Integral Equations with $\mathsf{P}_\ell^{(n-1,0)}$ $(\ell = 2, 3, \ldots)$

This subsection introduces nonlinear Volterra integral equations involving the spherical functions $\mathsf{P}_\ell^{(n-1,0)}$ $(n = 1, 2, 3, \ldots; \ell = 2, 3, \ldots)$. Setting $\phi = n - 1$, $\varphi = 0$ in Theorem 4.2, we have the following result.

Corollary 5.2. *The nonlinear Volterra integral equation of the second kind*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\mathsf{P}_\ell^{(n-1,0)}(U(\xi))\,d\xi \qquad (\ell = 2,3,\dots) \tag{5.7}$$

admits the series solution $(\lambda = \ell(\ell+n))$

$$\begin{aligned} U(x) =& 1 + \frac{x^2}{2} - \frac{2n-\lambda}{48n}x^4 + \frac{4n^3+4n^2+\lambda^2(4n+1)-7\lambda n(n+1)}{2880n^2(n+1)}x^6 \\ &- \frac{8n^5+24n^4+16n^3-\lambda^3\left(34n^2+39n+2\right)}{322560n^3(n+1)(n+2)}x^8 \\ &- \frac{3\lambda^2 n\left(35n^2+73n+16\right)-78\lambda n^2(n+1)(n+2)}{322560n^3(n+1)(n+2)}x^8 \\ &+ \frac{16n^8+112n^7+272n^6+272n^5+96n^4-1886\lambda n^3(n+1)^2(n+2)(n+3)}{58060800n^4(n+1)^2(n+2)(n+3)}x^{10} \\ &+ \frac{3\lambda^2 n^2(n+1)\left(1237n^3+6084n^2+7847n+1764\right)}{58060800n^4(n+1)^2(n+2)(n+3)}x^{10} \\ &- \frac{2\lambda^3 n(n+1)\left(1168n^3+4482n^2+3653n+267\right)}{58060800n^4(n+1)^2(n+2)(n+3)}x^{10} \\ &+ \frac{\lambda^4\left(496n^4+1918n^3+2087n^2+503n+6\right)}{58060800n^4(n+1)^2(n+2)(n+3)}x^{10} + \cdots . \end{aligned} \tag{5.8}$$

We present some special cases of Corollary 5.2.

Example 5.3. *The nonlinear Volterra integral equation of the second kind with Jacobi quadratic nonlinearities*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\left(\frac{(n+2)(n+3)}{4n(n+1)}U^2(\xi) + \frac{(n-1)(n+2)}{2n(n+1)}U(\xi) + \frac{n^2-3n-2}{4n(n+1)}\right)d\xi \tag{5.9}$$

has the series solution

$$\begin{aligned} U(x) =& 1 + \frac{x^2}{2} + \frac{1}{12n}x^4 + \frac{3n^3+15n^2+26n+8}{1440n^2(n+1)}x^6 + \frac{9n^3+45n^2+58n+4}{20160n^3(n+1)}x^8 \\ &+ \frac{21n^6+210n^5+939n^4+2232n^3+2554n^2+1004n+16}{3628800n^4(n+1)^2}x^{10} + \cdots . \end{aligned} \tag{5.10}$$

Example 5.4. *The Volterra integral equation of the second kind with Jacobi cubic nonlinearities*

$$U(x) = 1 + \int_0^x \sin(x-\xi)\left(J_{3,3}^{n-1,0}U^3(\xi) + J_{2,3}^{n-1,0}U^2(\xi) + J_{1,3}^{n-1,0}U(\xi) + J_{0,3}^{n-1,0}\right)d\xi, \tag{5.11}$$

with

$$\begin{aligned} J_{3,3}^{n-1,0} =& \frac{(n+3)(n+4)(n+5)}{8n(n+1)(n+2)}, \quad J_{2,3}^{n-1,0} = \frac{3(n-1)(n+3)(n+4)}{8n(n+1)(n+2)} \\ J_{1,3}^{n-1,0} =& \frac{3(n-4)(n+3)}{8n(n+2)}, \quad J_{0,3}^{n-1,0} = \frac{(n-1)\left(n^2-5n-12\right)}{8n(n+1)(n+2)}, \end{aligned}$$

has the series solution

$$\begin{aligned}U(x) =&1+\frac{x^2}{2}+\frac{n+9}{48n}x^4+\frac{19n^3+145n^2+315n+81}{2880n^2(n+1)}x^6\\&+\frac{199n^5+3054n^4+16112n^3+34776n^2+26001n+1458}{322560n^3(n+1)(n+2)}x^8\\&+\frac{4861n^7+91732n^6+687515n^5+2540018n^4+4680171n^3}{58060800n^4(n+1)^2(n+2)}x^{10}\\&+\frac{+3819960n^2+983421n+13122}{58060800n^4(n+1)^2(n+2)}x^{10}+\cdots.\end{aligned} \tag{5.12}$$

5.3 Nonlinear Volterra Integral Equations with $\mathsf{P}_\ell^{(2n-1,1)}$ $(\ell=2,3,\dots)$

This subsection introduces nonlinear Volterra integral equations involving the spherical functions $\mathsf{P}_\ell^{(2n-1,1)}$ $(n=1,2,3,\dots;\ell=2,3,\dots)$. Setting $\phi=2n-1$, $\varphi=1$ in Theorem 4.2, we have the following result.

Corollary 5.3. *The nonlinear Volterra integral equation of the second kind*

$$U(x)=1+\int_0^x \sin(x-\xi)\mathsf{P}_\ell^{(2n-1,1)}(U(\xi))\,d\xi \qquad (\ell=2,3,\dots) \tag{5.13}$$

admits the series solution $(\lambda=\ell(\ell+2n+1))$

$$\begin{aligned}U(x) =&1+\frac{x^2}{2}-\frac{4n-\lambda}{96n}x^4+\frac{32n^3+16n^2+\lambda^2(8n+1)-4\lambda n(7n+5)}{11520n^2(2n+1)}x^6\\&-\frac{128n^5+192n^4+64n^3-\lambda^3\left(68n^2+39n+1\right)}{2580480n^3(n+1)(2n+1)}x^8\\&-\frac{12\lambda^2 n\left(35n^2+47n+7\right)-24\lambda n^2(n+1)(26n+29)}{2580480n^3(n+1)(2n+1)}x^8\\&+\frac{2048n^8+7168n^7+8704n^6+4352n^5+768n^4}{928972800n^4(n+1)(2n+1)^2(2n+3)}x^{10}\\&-\frac{32\lambda n^3(n+1)(2n+1)(2n+3)(943n+1475)}{928972800n^4(n+1)(2n+1)^2(2n+3)}x^{10}\\&+\frac{24\lambda^2n^2\left(4948n^4+18010n^3+21378n^2+9135n+1179\right)}{928972800n^4(n+1)(2n+1)^2(2n+3)}x^{10}\\&-\frac{4\lambda^3 n\left(9344n^4+25708n^3+20926n^2+5882n+255\right)}{928972800n^4(n+1)(2n+1)^2(2n+3)}x^{10}\\&+\frac{\lambda^4\left(3968n^4+7672n^3+4174n^2+503n+3\right)}{928972800n^4(n+1)(2n+1)^2(2n+3)}x^{10}+\cdots.\end{aligned} \tag{5.14}$$

We set $\ell=2,3$ in Corollary 5.3 to obtain the following.

Example 5.5. *The nonlinear Volterra integral equation of the second kind*

$$U(x)=1+\int_0^x \sin(x-\xi)\left(\frac{(2n+3)(2n+4)}{8n(2n+1)}U^2(\xi)+\frac{(n-1)(2n+3)}{2n(2n+1)}U(\xi)+\frac{2n-5}{4(2n+1)}\right)d\xi \tag{5.15}$$

admits the series solution

$$\begin{aligned}U(x) =&1+\frac{x^2}{2}+\frac{1}{16n}x^4+\frac{4n^3+14n^2+18n+3}{960n^2(2n+1)}x^6+\frac{3\left(8n^3+28n^2+26n+1\right)}{35840n^3(2n+1)}x^8\\&+\frac{448n^6+3136n^5+10120n^4+17184n^3+13374n^2+3024n+27}{19353600n^4(2n+1)^2}x^{10}+\cdots.\end{aligned}\tag{5.16}$$

Example 5.6. *The nonlinear Volterra integral equation of the second kind*

$$U(x)=1+\int_0^x \sin(x-\xi)\left(J_{3,3}^{2n-1,1}U^3(\xi)+J_{2,3}^{2n-1,1}U^2(\xi)+J_{1,3}^{2n-1,1}U(\xi)+J_{0,3}^{2n-1,1}\right)d\xi,\tag{5.17}$$

with

$$J_{3,3}^{2n-1,1}=\frac{(n+2)(n+3)(2n+5)}{8n(n+1)(2n+1)},\quad J_{2,3}^{2n-1,1}=\frac{3(n-1)(n+2)(2n+5)}{8n(n+1)(2n+1)}\tag{5.18}$$

$$J_{1,3}^{2n-1,1}=\frac{3(n+2)\left(2n^2-5n-1\right)}{8n(n+1)(2n+1)},\quad J_{0,3}^{2n-1,1}=\frac{(n-1)\left(2n^2-7n-6\right)}{8n(n+1)(2n+1)},\tag{5.19}$$

admits the series solution

$$\begin{aligned}U(x) =&1+\frac{x^2}{2}+\frac{n+6}{48n}x^4+\frac{38n^3+187n^2+264n+36}{2880n^2(2n+1)}x^6\\&+\frac{398n^5+3873n^4+12781n^3+16866n^2+7236n+216}{322560n^3(n+1)(2n+1)}x^8\\&+\frac{19444n^7+231656n^6+1088945n^5+2492005n^4+2775732n^3}{58060800n^4(n+1)(2n+1)^2}x^{10}\\&+\frac{1300968n^2+182304n+1296}{58060800n^4(n+1)(2n+1)^2}x^{10}+\cdots.\end{aligned}\tag{5.20}$$

6 Some Known Nonlinear Volterra Integral Equations

In this section, we validate our results on the power series solutions of nonlinear Volterra integral equations involving Jacobi polynomial nonlinearities by considering some of the known nonlinear Volterra integral equations having same kernel. Three nonlinear Volterra integral equations - two of which involve one-term polynomial nonlinearities and the third contains rational function nonlinearity in the unknown variable, are taken from the literature to demonstrate the effectiveness, accuracy, and reliability of the present method. The solutions of the proposed Volterra integral equations using the present method are compared with the exact or existing solutions. The problems are presented as examples.

To this end, we first present results on the generalised Cauchy product, which is a special polynomial case of Proposition 3.1, before presenting the proposed examples.

Proposition 6.1 (Generalised Cauchy product [16], [20], [21])**.** *Assume the function $V(x)$ admits a convergent power series about $x=0$:*

$$V(x)=\sum_{k=0}^{\infty}v_kx^k,\qquad 0<x\leq 1,\tag{6.1}$$

where the expansion coefficients $v_k \in \mathbb{R}$ *are to be determined. One has*

$$V^j(x) = \left(\sum_{k=0}^{\infty} v_k x^k\right)^j = \sum_{k=0}^{\infty} \mathfrak{V}_{k,j} x^k \qquad (j = 1, 2, \dots), \tag{6.2}$$

where the coefficients $\mathfrak{V}_{k,j}$ $(k = 0, 1, 2, \dots; j = 1, 2, \dots)$ *are given by*

$$\mathfrak{V}_{k,1} = v_k, \quad \mathfrak{V}_{k,2} = \sum_{p=0}^{k} v_p v_{k-p}. \tag{6.3}$$

$$\mathfrak{V}_{k,j} = \sum_{p_{j-1}=0}^{k} \sum_{p_{j-2}=0}^{p_{j-1}} \cdots \sum_{p_1=0}^{p_2} v_{k-p_{j-1}} v_{p_{j-1}-p_{j-2}} \cdots v_{p_2-p_1} v_{p_1}, \quad j = 3, 4, \dots. \tag{6.4}$$

In particular,

$$\mathfrak{V}_{k,0} = \begin{cases} 1, & k = 0, \\ 0, & k = 1, 2, \cdots, \end{cases} \qquad \mathfrak{V}_{0,j} = (v_0)^j \quad (j = 0, 1, 2, \dots). \tag{6.5}$$

Example 6.1. *Consider the nonlinear Volterra integral equation ([27], [42])*

$$V(x) = 1 + \sin^2 x - 3 \int_0^x \sin(x - \xi) V^2(\xi)\, d\xi. \tag{6.6}$$

The exact solution is $V(x) = \cos x$. *Indeed we have the recurrence relation*

$$v_0 = g_0, \qquad v_1 = g_1, \qquad v_k = g_k + \mathsf{V}_{k-2,2} \qquad (k = 2, 3, \dots). \tag{6.7}$$

Here, g_k *are the expansion coefficients of the series representation of*

$$G(x) := 1 + \sin^2 x$$

and $\mathsf{V}_{q,2}$ *are given by*

$$\mathsf{V}_{q,2} = -3 \sum_{p=0}^{[q/2]} (-1)^p \frac{\mathfrak{V}_{q-2p,2}}{(2p+1)!} B(2p+2, q-2p+1) \qquad (q = 0, 1, 2, \dots). \tag{6.8}$$

The function $G(x)$ *has the power series expansion*

$$G(x) = 1 + x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315} + \frac{2x^{10}}{14175} + \cdots = 1 + \sum_{k=0}^{\infty} \sum_{p=0}^{k} \frac{(-1)^k x^{2k+2}}{(2p+1)!(2k-2p+1)!}. \tag{6.9}$$

Thus we obtain the expansion coefficients

$$\begin{aligned} &v_0 = 1, \quad v_1 = 0, \qquad v_2 = g_2 + \mathsf{V}_{0,2} = 1 - \frac{3\mathfrak{V}_{0,2}}{2} = -\frac{1}{2}, \qquad v_3 = g_3 + \mathsf{V}_{1,2} = -\frac{\mathfrak{V}_{1,2}}{2} = 0, \\ &v_4 = g_4 + \mathsf{V}_{2,2} = -\frac{1}{3} + \frac{\mathfrak{V}_{0,2}}{8} - \frac{\mathfrak{V}_{2,2}}{4} = \frac{1}{24}, \quad v_5 = g_5 + \mathsf{V}_{3,2} = \frac{\mathfrak{V}_{1,2}}{40} - \frac{3\mathfrak{V}_{3,2}}{20} = 0, \\ &v_6 = g_6 + \mathsf{V}_{4,2} = \frac{2}{45} - \frac{\mathfrak{V}_{0,2}}{240} + \frac{\mathfrak{V}_{2,2}}{120} - \frac{\mathfrak{V}_{4,2}}{10} = -\frac{1}{720} \\ &v_7 = g_7 + \mathsf{V}_{5,2} = -\frac{\mathfrak{V}_{1,2}}{1680} + \frac{\mathfrak{V}_{3,2}}{280} - \frac{\mathfrak{V}_{5,2}}{14} = 0. \end{aligned} \tag{6.10}$$

In general, it follows that

$$v_{2k} = g_{2k} + \mathsf{V}_{2k-2,2} = \frac{(-1)^k}{(2k)!} \qquad (k = 1, 2, 3, \dots). \tag{6.11}$$

Hence, we obtain the series solution

$$V(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = \cos x, \tag{6.12}$$

which is the exact solution.

Example 6.2. *Consider the nonlinear Volterra integral equation of the second kind ([35], [42])*

$$V(x) = G(x) + \int_0^x \sin(x - \xi)(\xi + V^2(\xi))\, d\xi, \tag{6.13}$$

where

$$G(x) = \frac{x^3}{8} - \frac{29}{4} \cos x + \sin x + \frac{29}{4} - \frac{x^6}{64} + \frac{7x^4}{32} - \frac{29}{8} x^2. \tag{6.14}$$

The exact solution is $V(x) = x + x^3/8$*. Now, in this case, the function* $G(x)$ *admits the power series expansion*

$$G(x) = x - \frac{x^3}{24} - \frac{x^4}{12} + \frac{x^5}{120} - \frac{x^6}{180} - \frac{x^7}{5040} - \frac{29x^8}{161280} + \frac{x^9}{362880} + \frac{29x^{10}}{14515200} + \dots = \sum_{k=0}^{\infty} g_k x^k. \tag{6.15}$$

On the other hand, the nonlinear term gives

$$\Phi(V(\xi)) = \xi + V^2(\xi) = \xi + \sum_{k=0}^{\infty} \mathfrak{V}_{k,2} \xi^k = \mathfrak{V}_{0,2} + (1 + \mathfrak{V}_{1,2})\, \xi + \sum_{k=2}^{\infty} \mathfrak{V}_{k,2} \xi^k. \tag{6.16}$$

Using (6.16)*, we have the integral formula*

$$\begin{aligned}\int_0^x \sin(x - \xi)(\xi + V^2(\xi))\, d\xi =& \mathfrak{V}_{0,2} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2}}{(2m+2)!} + (1 + \mathfrak{V}_{1,2}) \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+3}}{(2m+3)!} \\ &+ \sum_{m=0}^{\infty} \sum_{p=0}^{m} \frac{(-1)^p \mathfrak{V}_{m-p+2,2}}{(2p+1)!} B(2p+2, m-p+3) x^{m+p+4},\end{aligned} \tag{6.17}$$

where $B(r, s)$ *is the beta function of* r *and* s*. Simplifying the last series on the right of* (6.17)*, we have*

$$\begin{aligned}&\sum_{m=0}^{\infty} \sum_{p=0}^{m} \frac{(-1)^p \mathfrak{V}_{m-p+2,2}}{(2p+1)!} B(2p+2, m-p+3) x^{m+p+4} \\ &= \sum_{m=4}^{\infty} \mathsf{V}_{m-4,2} x^m = \sum_{m=2}^{\infty} \mathsf{V}_{2m-4,2} x^{2m} + \sum_{m=2}^{\infty} \mathsf{V}_{2m-3,2} x^{2m+1},\end{aligned} \tag{6.18}$$

where

$$\mathsf{V}_{q,2} = \sum_{p=0}^{[q/2]} \frac{(-1)^p}{(2p+1)!} \mathfrak{V}_{q-2p+2,2} B(2p+2, q-2p+3) \qquad (q = 0, 1, 2, \dots). \tag{6.19}$$

We have

$$\int_0^x \sin(x-\xi)(\xi + V^2(\xi))\,d\xi = \mathfrak{V}_{0,2}\frac{x^2}{2} + (1+\mathfrak{V}_{1,2})\,\frac{x^3}{6} + \sum_{m=2}^{\infty} \mathsf{B}_{2m}x^{2m} + \sum_{m=2}^{\infty} \mathsf{C}_{2m+1}x^{2m+1}, \tag{6.20}$$

where

$$\mathsf{B}_{2m} = \mathfrak{V}_{0,2}\frac{(-1)^{m-1}}{(2m)!} + \mathsf{V}_{2m-4,2}, \quad \mathsf{C}_{2m+1} = (1+\mathfrak{V}_{1,2})\,\frac{(-1)^{m-1}}{(2m+1)!} + \mathsf{V}_{2m-3,2}. \tag{6.21}$$

Hence, we have the first expansion coefficients

$$v_0 = g_0 = 0, \quad v_1 = g_1 = 1, \quad v_2 = g_2 + \mathfrak{V}_{0,2} = g_2 + v_0^2 = 0, \quad v_3 = g_3 + \frac{1+\mathfrak{V}_{1,2}}{6} = \frac{1}{8}. \tag{6.22}$$

The values of the higher expansion coefficients are given as

$$v_{2m} = g_{2m} + \mathsf{V}_{2m-4,2}, \quad v_{2m+1} = g_{2m+1} + \frac{(-1)^{m-1}}{(2m+1)!} + \mathsf{V}_{2m-3,2} \qquad (m = 2, 3, \dots); \tag{6.23}$$

with the following first expansion coefficients:

$$v_4 = g_4 + \mathsf{V}_{0,2} = -\frac{1}{12} + \frac{\mathfrak{V}_{2,2}}{12} = 0, \quad v_5 = g_5 + \mathsf{V}_{1,2} - \frac{1}{120} = 0 \tag{6.24}$$

$$v_6 = g_6 + \mathsf{V}_{2,2} = -\frac{1}{180} + \frac{\mathfrak{V}_{4,2}}{30} - \frac{\mathfrak{V}_{2,2}}{360} = 0, \quad v_7 = g_7 + \frac{1}{5040} + \mathsf{V}_{3,2} = \frac{\mathfrak{V}_{5,2}}{42} - \frac{\mathfrak{V}_{3,2}}{840} = 0. \tag{6.25}$$

In general, one sees that $v_k = 0$ $(k = 4, 5, 6, \dots)$. Hence, we obtain the series solution

$$V(x) = \sum_{k=0}^{\infty} v_k x^k = x + \frac{1}{8}x^3, \tag{6.26}$$

which is the exact solution.

Example 6.3. *Consider the nonlinear Volterra integral equation of the second kind ([55, Ch. 6, Problem 1, pp. 236-237])*

$$V(x) = 1 + \int_0^x \sin(x-\xi)\frac{1}{1+V^3(\xi)}\,d\xi. \tag{6.27}$$

The series solution, using the method of successive approximations ([55, Ch. 6, Problem 1, pp. 236-237]), is

$$V(x) = 1 + x^2/4 - 7x^4/192 + \cdots .$$

In this case, the nonlinear term is given by the geometric series in V^3. Indeed one has

$$\Phi(V(\xi)) = \frac{1}{1+V^3(\xi)} = \sum_{n=0}^{\infty}(-1)^n V^{3n}(\xi) = \sum_{n=0}^{\infty}(-1)^n \sum_{k=0}^{\infty} \mathfrak{V}_{k,3n}\xi^k = \sum_{k=0}^{\infty} \mathcal{V}_k \xi^k, \tag{6.28}$$

where

$$\mathcal{V}_k = \sum_{n=0}^{\infty}(-1)^n \mathfrak{V}_{k,3n}, \qquad \mathcal{V}_0 = \sum_{n=0}^{\infty}(-1)^n \mathfrak{V}_{0,3n} = \sum_{n=0}^{\infty}(-1)^n v_0^{3n} = \Phi(v_0). \tag{6.29}$$

Thus, from Theorem 4.2, we get the formula

$$\int_0^x \frac{\sin(x-\xi)}{1+V^3(\xi)}\,d\xi = \sum_{k=2}^{\infty} \mathscr{V}_{k-2} x^k, \tag{6.30}$$

where

$$\mathscr{V}_q = \sum_{p=0}^{[q/2]} \frac{(-1)^p}{(2p+1)!} \mathcal{V}_{q-2p} B(2p+2, q-2p+1) \qquad (q = 0, 1, 2, \ldots). \tag{6.31}$$

Hence, we obtain the recurrence relation

$$v_0 = 1, \quad v_1 = 0, \quad v_k = \mathscr{V}_{k-2} \qquad (k = 2, 3, 4, \ldots). \tag{6.32}$$

Using the differentiation formula

$$\Phi^{(m)}(1) = \frac{d^m}{dV^m}\left(\frac{1}{1+V^3}\right)\Bigg|_{V=1} \qquad (m = 0, 1, 2, \ldots), \tag{6.33}$$

we compute the first expansion coefficients v_k $(k = 2, 3, \ldots)$ as follows:

$$\begin{aligned}
v_2 =& \mathscr{V}_0 = \frac{\mathcal{V}_0}{2} = \frac{\Phi(1)}{2} = \frac{1}{4} \\
v_4 =& \frac{v_1^2}{24}\Phi''(1) + \frac{v_2}{12}\Phi'(1) - \frac{\Phi(1)}{24} = \frac{1}{48}\Phi'(1) - \frac{1}{48} = -\frac{7}{192} \\
v_6 =& \frac{v_1^4}{720}\Phi^{(4)}(1) + \frac{v_2 v_1^2}{60}\Phi^{(3)}(1) - \frac{v_1^2 - 24 v_3 v_1 - 12 v_2^2}{720}\Phi''(1) - \frac{v_2 - 12 v_4}{360}\Phi'(1) + \frac{\Phi(1)}{720} \\
=& \frac{\Phi''(1)}{960} - \frac{11\Phi'(1)}{5760} + \frac{\Phi(1)}{720} = \frac{67}{23040} \\
v_8 =& \frac{v_1^6 \Phi^{(6)}(1)}{40320} + \frac{v_2 v_1^4 \Phi^{(5)}(1)}{1344} - \frac{\left(v_1^2 - 120 v_3 v_1 - 180 v_2^2\right) v_1^2 \Phi^{(4)}(1)}{40320} \\
&- \frac{\left(-10 v_2^3 + v_1^2 v_2 - 60 v_1 v_3 v_2 - 30 v_1^2 v_4\right)\Phi^{(3)}(1)}{3360} \\
&+ \frac{\left(v_1^2 - 24 v_3 v_1 + 720 v_5 v_1 - 12 v_2^2 + 360 v_3^2 + 720 v_2 v_4\right)\Phi''(1)}{40320} \\
&+ \frac{\left(v_2 - 12 v_4 + 360 v_6\right)\Phi'(1)}{20160} - \frac{\Phi(1)}{40320} \\
=& \frac{\Phi^{(3)}(1)}{21504} - \frac{13\Phi''(1)}{71680} + \frac{37\Phi'(1)}{430080} - \frac{\Phi(1)}{40320} = -\frac{649}{5160960},
\end{aligned}$$

where we have used the derivative values

$$\Phi(1) = \frac{1}{2}, \quad \Phi'(1) = -\frac{3}{4}, \quad \Phi''(1) = \frac{3}{4}, \quad \Phi'''(1) = \frac{15}{8}. \tag{6.34}$$

Hence, we obtain the series solution

$$V(x) = 1 + \frac{x^2}{4} - \frac{7x^4}{192} + \frac{67}{23040}x^6 - \frac{649}{5160960}x^8 + \cdots. \tag{6.35}$$

7 Concluding Remarks

This paper has successfully applied a power series method based on the generalised Cauchy product to obtain series solutions of a class of nonlinear Volterra integral equations of the second kind. The integral equations considered have Jacobi polynomial nonlinearities. By specialising the Jacobi parameters to different values, we obtained several linear and nonlinear Volterra integral equations with Gegenbauer, Chebyshev, and Legendre polynomial nonlinearities. We also relate our discussion to spectral theory of the Laplacian on symmetric spaces by considering normalised Jacobi polynomials as spherical functions on rank one symmetric spaces of compact type. Thus several nonlinear Volterra integral equations of the second kind involving spherical functions on the sphere, the complex projective space, the quaternionic projective space, and the Cayley projective plane, as well as their series solutions were presented. It is also worth mentioning that several Volterra integral equations of the second kind with Jacobi linearities and their exact solutions were introduced.

To validate our results on the series solutions of Volterra integral equations with Jacobi polynomial nonlinearities, we applied the present power series method to known nonlinear Volterra integral equations in the literature. These known integral equations have same kernel as in our main problem. Three known nonlinear Volterra integral equations of the second kind were considered. In the first two examples of the known problems, our series solutions converge to the exact solutions, while the series solution in the third example agrees with the known series solution.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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