



Sixteen General Theorems of Cyclic Integral Inequalities

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Abstract

This paper extends classical discrete cyclic inequalities to a continuous setting involving cyclic sequences of measurable functions. By integrating cyclic symmetry with functional analysis, we establish an integral inequality framework that provides global bounds on combinations of adjacent functional terms. Utilizing fundamental results including the Minkowski, Young, Cauchy-Schwarz, Hardy, and Jensen inequalities, we present sixteen general families of cyclic integral inequalities. Detailed proofs, special cases, and concluding remarks are also provided.

1 Introduction

Inequalities are fundamental tools in mathematical analysis. An important subclass consists of cyclic inequalities, which remain invariant under cyclic permutations of their variables. A classic benchmark is the Nesbitt inequality, which states that, for any $a, b, c > 0$,

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

While numerous generalizations and refinements of this result have been explored [1, 2, 5, 6, 8, 9, 11], most existing literature focuses primarily on discrete real domains. For recent developments in this direction, see [3, 4, 10].

This paper extends these classical discrete frameworks to a continuous setting by considering a cyclic sequence of measurable functions $f_1, f_2, \dots, f_n$, with $n \in \mathbb{N} \setminus \{0, 1\}$, defined on a real interval $I \subseteq \mathbb{R}$, with indices evaluated modulo n . Consequently, the sequence forms a closed loop such that $f_{n+1} = f_1$, $f_{n+2} = f_2$, and so forth. Building upon these functions, we transition to an integral inequality framework. A simple illustrative example is described below. Let $n \in \mathbb{N} \setminus \{0, 1\}$, $a, b \in \mathbb{R}$ with $b > a$, and $f_1, f_2, \dots, f_n : I = [a, b] \rightarrow \mathbb{R}$ be a cyclic sequence of measurable functions. We suppose that, for any $i = 1, 2, \dots, n$, $|f_i|$ is integrable. Then we have

$$\sum_{i=1}^n \int_a^b |f_i(x) + f_{i+1}(a+b-x)| dx \leq 2 \sum_{i=1}^n \int_a^b |f_i(x)| dx.$$

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The proof follows directly from the triangle inequality, the change of variables $y = a + b - x$, and the cyclic symmetry, which implies that $\sum_{i=1}^n \int_a^b |f_{i+1}(x)| dx = \sum_{i=1}^n \int_a^b |f_i(x)| dx$. This result can be interpreted as follows: the total L^1 norm of the sums between each function and the symmetric reflection of its successor is bounded by twice the total L^1 norm of the functions in the cycle.

More generally, unlike pointwise comparisons, integral formulations capture the global behavior of sequences of functions over their domain. They thereby smooth out localized fluctuations to provide robust bounds on cyclic combinations, such as sums, products, and quotients, of adjacent terms. This framework establishes a natural link between cyclic symmetry and functional analysis, enabling the integration of classical tools (notably the Minkowski, Young, Cauchy-Schwarz, Hardy, and Jensen inequalities) into cyclic structures. Building on recent parametric extensions of cyclic inequalities, we state and prove sixteen general families of cyclic integral inequalities that offer a wide class of new bounds.

The remainder of the paper is organized as follows: Section 2 develops an important aspect of the considered framework. Section 3 presents the general theorems, accompanied by their proofs and some cases of particular interest. A conclusion is provided in Section 4.

2 Framework

To maximize generality, the main theorems are formulated using two cyclic sequences of measurable functions, $f_1, f_2, \dots, f_n$, as previously introduced, and $g_1, g_2, \dots, g_n$, that share common integrability properties.

For example, in the first theorem, we deal with an integral assumption of the following form: for any $i = 1, 2, \dots, n$,

$$\int_a^b g_i(x)^p dx = \int_a^b f_i(x)^p dx,$$

where $p \geq 1$ (see Theorem 3.1).

First, it is obviously satisfied for $g_i = f_i$, which serves as a natural baseline.

Second, symmetry and standard changes of variables frequently guarantee this integral identity. For instance, taking $g_i(x) = f_i(a + b - x)$, the change of variables $y = a + b - x$ gives

$$\int_a^b g_i(x)^p dx = \int_a^b f_i(a + b - x)^p dx = - \int_b^a f_i(y)^p dy = \int_a^b f_i(x)^p dx.$$

Finally, functions do not strictly require a global pointwise relationship or structural symmetry to satisfy this integral identity. For instance, taking $a = 0$, $b = 1$, $p = 1$, $f_i(x) = 2x$, and $g_i(x) = 3x^2$, we have

$$\int_a^b f_i(x)^p dx = \int_0^1 2x dx = 1 = \int_0^1 3x^2 dx = \int_a^b g_i(x)^p dx.$$

Here, f_i and g_i exhibit completely different local behaviors: $f_i(x) > g_i(x)$ on $[0, 2/3)$, while $g_i(x) > f_i(x)$ on $(2/3, 1]$, demonstrating that the identity depends solely on overall mass rather than pointwise dominance.

Introducing $g_1, g_2, \dots, g_n$ thus enhances the generality of the results.

3 General Theorems

For a concise summary of the core inequalities, the reader is referred to Table 1 in Section 4. The remainder of this section is devoted to the formal statements, detailed proofs, and notable special cases of the sixteen theorems. The proofs rely on fundamental tools from classical analysis, including the Young, Cauchy-Schwarz, Minkowski, Hardy, and Jensen inequalities. For their general statements, we refer to [7].

3.1 First theorem

The first theorem is stated below.

Theorem 3.1. *Let $n \in \mathbb{N} \setminus \{0, 1\}$, $a, b \in \mathbb{R}$ with $b > a$, $p \geq 1$, and $f_1, f_2, \dots, f_n : [a, b] \rightarrow (0, \infty)$ and $g_1, g_2, \dots, g_n : [a, b] \rightarrow (0, \infty)$ be two cyclic sequences of measurable functions. We suppose that, for any $i = 1, 2, \dots, n$, f_i^p and g_i^p are integrable, and*

$$\int_a^b g_i(x)^p dx = \int_a^b f_i(x)^p dx.$$

Then we have

$$\sum_{i=1}^n \left(\int_a^b (f_i(x) + g_{i+1}(x))^p dx \right)^{1/p} \leq 2 \sum_{i=1}^n \left(\int_a^b f_i(x)^p dx \right)^{1/p}.$$

Proof. By the Minkowski inequality and the equality $\int_a^b g_{i+1}(x)^p dx = \int_a^b f_{i+1}(x)^p dx$, we have

$$\begin{aligned} \left(\int_a^b (f_i(x) + g_{i+1}(x))^p dx \right)^{1/p} &\leq \left(\int_a^b f_i(x)^p dx \right)^{1/p} + \left(\int_a^b g_{i+1}(x)^p dx \right)^{1/p} \\ &= \left(\int_a^b f_i(x)^p dx \right)^{1/p} + \left(\int_a^b f_{i+1}(x)^p dx \right)^{1/p}. \end{aligned}$$

Summing cyclically both sides, we obtain

$$\begin{aligned} &\sum_{i=1}^n \left(\int_a^b (f_i(x) + g_{i+1}(x))^p dx \right)^{1/p} \\ &\leq \sum_{i=1}^n \left(\int_a^b f_i(x)^p dx \right)^{1/p} + \sum_{i=1}^n \left(\int_a^b f_{i+1}(x)^p dx \right)^{1/p}. \end{aligned} \tag{3.1}$$

Due to the cyclic symmetry, we derive

$$\sum_{i=1}^n \left(\int_a^b f_{i+1}(x)^p dx \right)^{1/p} = \sum_{i=1}^n \left(\int_a^b f_i(x)^p dx \right)^{1/p}.$$

Therefore, we get

$$\sum_{i=1}^n \left(\int_a^b f_i(x)^p dx \right)^{1/p} + \sum_{i=1}^n \left(\int_a^b f_{i+1}(x)^p dx \right)^{1/p} = 2 \sum_{i=1}^n \left(\int_a^b f_i(x)^p dx \right)^{1/p}. \quad (3.2)$$

It follows from Equations (3.1) and (3.2) that

$$\sum_{i=1}^n \left(\int_a^b (f_i(x) + g_{i+1}(x))^p dx \right)^{1/p} \leq 2 \sum_{i=1}^n \left(\int_a^b f_i(x)^p dx \right)^{1/p}.$$

This completes the proof. $\square$

In particular, choosing $p = 2$ in this theorem, the main inequality reduces to

$$\sum_{i=1}^n \left(\int_a^b (f_i(x) + g_{i+1}(x))^2 dx \right)^{1/2} \leq 2 \sum_{i=1}^n \left(\int_a^b f_i(x)^2 dx \right)^{1/2}.$$

Notably, the case $p = 1$ holds with equality.

3.2 Second theorem

The second theorem is presented below.

Theorem 3.2. Let $m, n \in \mathbb{N} \setminus \{0, 1\}$, $a, b \in \mathbb{R}$ with $b > a$, and $f_1, f_2, \dots, f_n : [a, b] \rightarrow (0, \infty)$ and $g_1, g_2, \dots, g_n : [a, b] \rightarrow (0, \infty)$ be two cyclic sequences of measurable functions. We suppose that, for any $i = 1, 2, \dots, n$, f_i^{m+1} and g_i^{m+1} are integrable, and

$$\int_a^b g_i(x)^{m+1} dx = \int_a^b f_i(x)^{m+1} dx.$$

Then we have

$$\sum_{i=1}^n \int_a^b f_i(x) \prod_{k=1}^m g_{i+k}(x) dx \leq \sum_{i=1}^n \int_a^b f_i(x)^{m+1} dx.$$

Proof. By the arithmetic mean-geometric mean inequality, we have

$$f_i(x) \prod_{k=1}^m g_{i+k}(x) \leq \frac{1}{m+1} \left(f_i(x)^{m+1} + \sum_{k=1}^m g_{i+k}(x)^{m+1} \right).$$

Integrating both sides and using the equality $\int_a^b g_{i+k}(x)^{m+1} dx = \int_a^b f_{i+k}(x)^{m+1} dx$, we obtain

$$\begin{aligned} \int_a^b f_i(x) \prod_{k=1}^m g_{i+k}(x) dx &\leq \frac{1}{m+1} \left(\int_a^b f_i(x)^{m+1} dx + \sum_{k=1}^m \int_a^b g_{i+k}(x)^{m+1} dx \right) \\ &= \frac{1}{m+1} \left(\int_a^b f_i(x)^{m+1} dx + \sum_{k=1}^m \int_a^b f_{i+k}(x)^{m+1} dx \right). \end{aligned}$$

Summing cyclically, we get

$$\sum_{i=1}^n \int_a^b f_i(x) \prod_{k=1}^m g_{i+k}(x) dx \leq \frac{1}{m+1} \left(\sum_{i=1}^n \int_a^b f_i(x)^{m+1} dx + \sum_{k=1}^m \sum_{i=1}^n \int_a^b f_{i+k}(x)^{m+1} dx \right). \quad (3.3)$$

Due to the cyclic symmetry, for any $k = 1, 2, \dots, n$, we have

$$\sum_{i=1}^n \int_a^b f_{i+k}(x)^{m+1} dx = \sum_{i=1}^n \int_a^b f_i(x)^{m+1} dx.$$

Therefore, we derive

$$\begin{aligned} &\frac{1}{m+1} \left(\sum_{i=1}^n \int_a^b f_i(x)^{m+1} dx + \sum_{k=1}^m \sum_{i=1}^n \int_a^b f_{i+k}(x)^{m+1} dx \right) \\ &= \frac{1}{m+1} \left(\sum_{i=1}^n \int_a^b f_i(x)^{m+1} dx + m \sum_{i=1}^n \int_a^b f_i(x)^{m+1} dx \right) \\ &= \sum_{i=1}^n \int_a^b f_i(x)^{m+1} dx. \end{aligned} \quad (3.4)$$

It follows from Equations (3.3) and (3.4) that

$$\sum_{i=1}^n \int_a^b f_i(x) \prod_{k=1}^m g_{i+k}(x) dx \leq \sum_{i=1}^n \int_a^b f_i(x)^{m+1} dx.$$

This ends the proof. □

In particular, choosing $m = 2$ in this theorem, the main inequality reduces to

$$\sum_{i=1}^n \int_a^b f_i(x) g_{i+1}(x) g_{i+2}(x) dx \leq \sum_{i=1}^n \int_a^b f_i(x)^3 dx.$$

Remark 3.1. Theorem 3.2 provides a continuous framework for multi-variable cyclic inequalities. By representing discrete index shifts through the product $\prod_{k=1}^m g_{i+k}(x)$, this result extends the structural symmetry of classical Nesbitt-type inequalities to continuous spaces of measurable functions, serving as a functional foundation for multi-parameter cyclic bounds.

3.3 Third theorem

The third theorem is established below.

Theorem 3.3. Let $n \in \mathbb{N} \setminus \{0, 1\}$, $a, b \in \mathbb{R}$ with $b > a$, and $f_1, f_2, \dots, f_n : [a, b] \rightarrow (0, \infty)$ and $g_1, g_2, \dots, g_n : [a, b] \rightarrow (0, \infty)$ be two cyclic sequences of measurable functions. We suppose that, for any $i = 1, 2, \dots, n$, f_i and g_i^2 are integrable, and

$$\int_a^b g_i(x) dx = \int_a^b f_i(x) dx.$$

Then we have

$$\sum_{i=1}^n \int_a^b \frac{g_i(x)^2}{\int_a^b f_{i+1}(t) dt} dx \geq \frac{1}{b-a} \sum_{i=1}^n \int_a^b f_i(x) dx.$$

Proof. We can write

$$\sum_{i=1}^n \int_a^b \frac{g_i(x)^2}{\int_a^b f_{i+1}(t) dt} dx = \sum_{i=1}^n \frac{\int_a^b g_i(x)^2 dx}{\int_a^b f_{i+1}(x) dx}. \quad (3.5)$$

By the Cauchy-Schwarz inequality and the equality $\int_a^b g_i(x) dx = \int_a^b f_i(x) dx$, we get

$$\int_a^b g_i(x)^2 dx \geq \frac{1}{b-a} \left(\int_a^b g_i(x) dx \right)^2 = \frac{1}{b-a} \left(\int_a^b f_i(x) dx \right)^2.$$

Therefore, we have

$$\sum_{i=1}^n \frac{\int_a^b g_i(x)^2 dx}{\int_a^b f_{i+1}(x) dx} \geq \frac{1}{b-a} \sum_{i=1}^n \frac{\left(\int_a^b f_i(x) dx \right)^2}{\int_a^b f_{i+1}(x) dx}. \quad (3.6)$$

It follows from the Titu lemma that

$$\sum_{i=1}^n \frac{\left(\int_a^b f_i(x) dx \right)^2}{\int_a^b f_{i+1}(x) dx} \geq \frac{\left(\sum_{i=1}^n \int_a^b f_i(x) dx \right)^2}{\sum_{i=1}^n \int_a^b f_{i+1}(x) dx}. \quad (3.7)$$

Due to the cyclic symmetry, we derive

$$\sum_{i=1}^n \int_a^b f_{i+1}(x) dx = \sum_{i=1}^n \int_a^b f_i(x) dx.$$

Therefore, we get

$$\frac{\left(\sum_{i=1}^n \int_a^b f_i(x) dx \right)^2}{\sum_{i=1}^n \int_a^b f_{i+1}(x) dx} = \frac{\left(\sum_{i=1}^n \int_a^b f_i(x) dx \right)^2}{\sum_{i=1}^n \int_a^b f_i(x) dx} = \sum_{i=1}^n \int_a^b f_i(x) dx. \quad (3.8)$$

It follows from Equations (3.5), (3.6), (3.7), and (3.8) that

$$\sum_{i=1}^n \int_a^b \frac{g_i(x)^2}{\int_a^b f_{i+1}(t) dt} dx \geq \frac{1}{b-a} \sum_{i=1}^n \int_a^b f_i(x) dx.$$

This concludes the proof. $\square$

3.4 Fourth theorem

The fourth theorem is stated below. It can be viewed as a variant of Theorem 3.3.

Theorem 3.4. Let $n \in \mathbb{N} \setminus \{0, 1\}$, $a, b \in \mathbb{R}$ with $b > a$, and $f_1, f_2, \dots, f_n : [a, b] \rightarrow (0, \infty)$ and $g_1, g_2, \dots, g_n : [a, b] \rightarrow (0, \infty)$ be two cyclic sequences of measurable functions. We suppose that, for any $i = 1, 2, \dots, n$, f_i^2/g_{i+1} , f_i and g_i are integrable, and

$$\int_a^b g_i(x) dx = \int_a^b f_i(x) dx.$$

Then we have

$$\sum_{i=1}^n \int_a^b \frac{f_i(x)^2}{g_{i+1}(x)} dx \geq \sum_{i=1}^n \int_a^b f_i(x) dx.$$

Proof. Applying the (continuous) Titu lemma and using the equality $\int_a^b g_{i+1}(x) dx = \int_a^b f_{i+1}(x) dx$, we get

$$\int_a^b \frac{f_i(x)^2}{g_{i+1}(x)} dx \geq \frac{\left(\int_a^b f_i(x) dx\right)^2}{\int_a^b g_{i+1}(x) dx} = \frac{\left(\int_a^b f_i(x) dx\right)^2}{\int_a^b f_{i+1}(x) dx}.$$

Summing cyclically both sides, we obtain

$$\sum_{i=1}^n \int_a^b \frac{f_i(x)^2}{g_{i+1}(x)} dx \geq \sum_{i=1}^n \frac{\left(\int_a^b f_i(x) dx\right)^2}{\int_a^b f_{i+1}(x) dx}. \quad (3.9)$$

It follows from the (discrete) Titu lemma that

$$\sum_{i=1}^n \frac{\left(\int_a^b f_i(x) dx\right)^2}{\int_a^b f_{i+1}(x) dx} \geq \frac{\left(\sum_{i=1}^n \int_a^b f_i(x) dx\right)^2}{\sum_{i=1}^n \int_a^b f_{i+1}(x) dx}. \quad (3.10)$$

Due to the cyclic symmetry, we have

$$\sum_{i=1}^n \int_a^b f_{i+1}(x) dx = \sum_{i=1}^n \int_a^b f_i(x) dx.$$

Therefore, we derive

$$\frac{\left(\sum_{i=1}^n \int_a^b f_i(x) dx\right)^2}{\sum_{i=1}^n \int_a^b f_{i+1}(x) dx} = \frac{\left(\sum_{i=1}^n \int_a^b f_i(x) dx\right)^2}{\sum_{i=1}^n \int_a^b f_i(x) dx} = \sum_{i=1}^n \int_a^b f_i(x) dx. \quad (3.11)$$

It follows from Equations (3.9), (3.10), and (3.11) that

$$\sum_{i=1}^n \int_a^b \frac{f_i(x)^2}{g_{i+1}(x)} dx \geq \sum_{i=1}^n \int_a^b f_i(x) dx.$$

This completes the proof. □

Remark 3.2. The reliance on the continuous Titu lemma in the proof of Theorem 3.4 naturally invites a generalization via the continuous Radon inequality. Specifically, by considering the integrability of $f_i(x)^p/g_{i+1}(x)^{p-1}$ for $p > 1$, one can extend this cyclic structure beyond cyclic quadratic forms to more general L^p norm spaces, providing robust lower bounds for a wider family of fractional cyclic integral combinations.

3.5 Fifth theorem

The fifth theorem is presented below.

Theorem 3.5. Let $n \in \mathbb{N} \setminus \{0, 1\}$, $a, b \in \mathbb{R}$ with $b > a$, and $f_1, f_2, \dots, f_n : [a, b] \rightarrow \mathbb{R}$ and $g_1, g_2, \dots, g_n : [a, b] \rightarrow \mathbb{R}$ be two cyclic sequences of continuously differentiable functions. We suppose that, for any $i = 1, 2, \dots, n$, $f_i(a) = 0$, $|f'_i|$ and $|g'_i|$ are integrable, and

$$\int_a^b |g'_i(x)| dx = \int_a^b |f'_i(x)| dx.$$

Then we have

$$\sum_{i=1}^n \int_a^b |f_i(x)g'_{i+1}(x)| dx \leq \sum_{i=1}^n \left(\int_a^b |f'_i(x)| dx \right)^2.$$

Proof. Since $f_i(a) = 0$, we can write

$$f_i(x) = \int_a^x f'_i(t) dt + f_i(a) = \int_a^x f'_i(t) dt.$$

Therefore, we have

$$|f_i(x)| = \left| \int_a^x f'_i(t) dt \right| \leq \int_a^x |f'_i(t)| dt \leq \int_a^b |f'_i(t)| dt.$$

Multiplying by $|g'_{i+1}(x)|$, integrating both sides and using the equality $\int_a^b |g'_{i+1}(x)| dx = \int_a^b |f'_{i+1}(x)| dx$, we obtain

$$\begin{aligned} \int_a^b |f_i(x)g'_{i+1}(x)| dx &\leq \int_a^b \left(\int_a^b |f'_i(t)| dt \right) |g'_{i+1}(x)| dx \\ &= \left(\int_a^b |f'_i(t)| dt \right) \left(\int_a^b |g'_{i+1}(x)| dx \right) \\ &= \left(\int_a^b |f'_i(x)| dx \right) \left(\int_a^b |f'_{i+1}(x)| dx \right). \end{aligned}$$

Summing cyclically both sides, we derive

$$\sum_{i=1}^n \int_a^b |f_i(x)g'_{i+1}(x)| dx \leq \sum_{i=1}^n \left(\int_a^b |f'_i(x)| dx \right) \left(\int_a^b |f'_{i+1}(x)| dx \right). \quad (3.12)$$

By the arithmetic mean-geometric mean inequality, we get

$$\left(\int_a^b |f'_i(x)|dx\right) \left(\int_a^b |f'_{i+1}(x)|dx\right) \leq \frac{1}{2} \left(\left(\int_a^b |f'_i(x)|dx\right)^2 + \left(\int_a^b |f'_{i+1}(x)|dx\right)^2 \right).$$

Summing cyclically both sides, we derive

$$\begin{aligned} & \sum_{i=1}^n \left(\int_a^b |f'_i(x)|dx\right) \left(\int_a^b |f'_{i+1}(x)|dx\right) \\ & \leq \frac{1}{2} \left(\sum_{i=1}^n \left(\int_a^b |f'_i(x)|dx\right)^2 + \sum_{i=1}^n \left(\int_a^b |f'_{i+1}(x)|dx\right)^2 \right). \end{aligned} \quad (3.13)$$

Due to the cyclic symmetry, we have

$$\sum_{i=1}^n \left(\int_a^b |f'_{i+1}(x)|dx\right)^2 = \sum_{i=1}^n \left(\int_a^b |f'_i(x)|dx\right)^2.$$

Therefore, we obtain

$$\frac{1}{2} \left(\sum_{i=1}^n \left(\int_a^b |f'_i(x)|dx\right)^2 + \sum_{i=1}^n \left(\int_a^b |f'_{i+1}(x)|dx\right)^2 \right) = \sum_{i=1}^n \left(\int_a^b |f'_i(x)|dx\right)^2. \quad (3.14)$$

It follows from Equations (3.12), (3.13), and (3.14) that

$$\sum_{i=1}^n \int_a^b |f_i(x)g'_{i+1}(x)|dx \leq \sum_{i=1}^n \left(\int_a^b |f'_i(x)|dx\right)^2.$$

This ends the proof. $\square$

3.6 Sixth theorem

The sixth theorem is established below. The inclusion of a convex non-decreasing function plays a central role in this framework.

Theorem 3.6. *Let $n \in \mathbb{N} \setminus \{0, 1\}$, $a, b \in \mathbb{R}$ with $b > a$, $p, q > 0$, $f_1, f_2, \dots, f_n : [a, b] \rightarrow (0, \infty)$ and $g_1, g_2, \dots, g_n : [a, b] \rightarrow (0, \infty)$ be two cyclic sequences of measurable functions, and $\phi : (0, \infty) \rightarrow \mathbb{R}$ be a function that is both convex and non-decreasing. We suppose that, for any $i = 1, 2, \dots, n$, $\phi(f_i^{p+q})$ and $\phi(g_i^{p+q})$ are integrable, and*

$$\int_a^b \phi(g_i(x)^{p+q}) dx = \int_a^b \phi(f_i(x)^{p+q}) dx.$$

Then we have

$$\sum_{i=1}^n \int_a^b \phi(f_i(x)^p g_{i+1}(x)^q) dx \leq \sum_{i=1}^n \int_a^b \phi(f_i(x)^{p+q}) dx.$$

Proof. By the Young inequality, we get

$$f_i(x)^p g_{i+1}(x)^q \leq \frac{p}{p+q} f_i(x)^{p+q} + \frac{q}{p+q} g_{i+1}(x)^{p+q}.$$

Since ϕ is non-decreasing, we have

$$\phi(f_i(x)^p g_{i+1}(x)^q) \leq \phi\left(\frac{p}{p+q} f_i(x)^{p+q} + \frac{q}{p+q} g_{i+1}(x)^{p+q}\right).$$

By the Jensen inequality, we obtain

$$\phi\left(\frac{p}{p+q} f_i(x)^{p+q} + \frac{q}{p+q} g_{i+1}(x)^{p+q}\right) \leq \frac{p}{p+q} \phi(f_i(x)^{p+q}) + \frac{q}{p+q} \phi(g_{i+1}(x)^{p+q}).$$

Therefore, we have

$$\phi(f_i(x)^p g_{i+1}(x)^q) \leq \frac{p}{p+q} \phi(f_i(x)^{p+q}) + \frac{q}{p+q} \phi(g_{i+1}(x)^{p+q}).$$

Integrating both sides and using the equality $\int_a^b \phi(g_{i+1}(x)^{p+q}) dx = \int_a^b \phi(f_{i+1}(x)^{p+q}) dx$, we get

$$\begin{aligned} \int_a^b \phi(f_i(x)^p g_{i+1}(x)^q) dx &\leq \frac{p}{p+q} \int_a^b \phi(f_i(x)^{p+q}) dx + \frac{q}{p+q} \int_a^b \phi(g_{i+1}(x)^{p+q}) dx \\ &= \frac{p}{p+q} \int_a^b \phi(f_i(x)^{p+q}) dx + \frac{q}{p+q} \int_a^b \phi(f_{i+1}(x)^{p+q}) dx. \end{aligned}$$

Summing cyclically both sides, we derive

$$\begin{aligned} &\sum_{i=1}^n \int_a^b \phi(f_i(x)^p g_{i+1}(x)^q) dx \\ &\leq \frac{p}{p+q} \sum_{i=1}^n \int_a^b \phi(f_i(x)^{p+q}) dx + \frac{q}{p+q} \sum_{i=1}^n \int_a^b \phi(f_{i+1}(x)^{p+q}) dx. \end{aligned} \quad (3.15)$$

Due to the cyclic symmetry, we have

$$\sum_{i=1}^n \int_a^b \phi(f_{i+1}(x)^{p+q}) dx = \sum_{i=1}^n \int_a^b \phi(f_i(x)^{p+q}) dx.$$

Therefore, we obtain

$$\begin{aligned} &\frac{p}{p+q} \sum_{i=1}^n \int_a^b \phi(f_i(x)^{p+q}) dx + \frac{q}{p+q} \sum_{i=1}^n \int_a^b \phi(f_{i+1}(x)^{p+q}) dx \\ &= \frac{p}{p+q} \sum_{i=1}^n \int_a^b \phi(f_i(x)^{p+q}) dx + \frac{q}{p+q} \sum_{i=1}^n \int_a^b \phi(f_i(x)^{p+q}) dx \\ &= \left(\frac{p}{p+q} + \frac{q}{p+q}\right) \sum_{i=1}^n \int_a^b \phi(f_i(x)^{p+q}) dx = \sum_{i=1}^n \int_a^b \phi(f_i(x)^{p+q}) dx. \end{aligned} \quad (3.16)$$

It follows from Equations (3.15) and (3.16) that

$$\sum_{i=1}^n \int_a^b \phi(f_i(x)^p g_{i+1}(x)^q) dx \leq \sum_{i=1}^n \int_a^b \phi(f_i(x)^{p+q}) dx.$$

This concludes the proof. $\square$

In particular, choosing $\phi(x) = \exp(x)$ in this theorem, the main inequality reduces to

$$\sum_{i=1}^n \int_a^b \exp(f_i(x)^p g_{i+1}(x)^q) dx \leq \sum_{i=1}^n \int_a^b \exp(f_i(x)^{p+q}) dx.$$

Beyond the exponential function, other standard convex non-decreasing functions on $(0, \infty)$ include $f(x) = x^r$ with $r \geq 1$, $f(x) = \ln(1 + e^{sx})$ with $s > 0$, $f(x) = x \ln(1 + x)$, and $f(x) = x \arctan(x)$, along with any positive linear combinations and convex non-decreasing compositions of these functions.

3.7 Seventh theorem

The seventh theorem is stated below. This result is closely related to Theorem 3.6 and likewise relies on a convex non-decreasing function.

Theorem 3.7. Let $n \in \mathbb{N} \setminus \{0, 1\}$, $a, b \in \mathbb{R}$ with $b > a$, $p, q > 0$, $f_1, f_2, \dots, f_n : [a, b] \rightarrow (0, \infty)$ and $g_1, g_2, \dots, g_n : [a, b] \rightarrow (0, \infty)$ be two cyclic sequences of measurable functions, and $\phi : (0, \infty) \rightarrow \mathbb{R}$ be a function that is both convex and non-decreasing. We suppose that, for any $i = 1, 2, \dots, n$, $\phi(f_i^{p+q})$ and $\phi(g_i^{p+q})$ are integrable, and

$$\int_a^b \phi(g_i(x)^{p+q}) dx = \int_a^b \phi(f_i(x)^{p+q}) dx.$$

Then we have

$$\sum_{i=1}^n \phi\left(\frac{1}{b-a} \int_a^b f_i(x)^p g_{i+1}(x)^q dx\right) \leq \frac{1}{b-a} \sum_{i=1}^n \int_a^b \phi(f_i(x)^{p+q}) dx.$$

Proof. By the Jensen inequality, we have

$$\phi\left(\frac{1}{b-a} \int_a^b f_i(x)^p g_{i+1}(x)^q dx\right) \leq \frac{1}{b-a} \int_a^b \phi(f_i(x)^p g_{i+1}(x)^q) dx.$$

Summing cyclically both sides, we obtain

$$\sum_{i=1}^n \phi\left(\frac{1}{b-a} \int_a^b f_i(x)^p g_{i+1}(x)^q dx\right) \leq \frac{1}{b-a} \sum_{i=1}^n \int_a^b \phi(f_i(x)^p g_{i+1}(x)^q) dx.$$

It follows from Theorem 3.6 that

$$\sum_{i=1}^n \int_a^b \phi(f_i(x)^p g_{i+1}(x)^q) dx \leq \sum_{i=1}^n \int_a^b \phi(f_i(x)^{p+q}) dx.$$

Therefore, we derive

$$\sum_{i=1}^n \phi\left(\frac{1}{b-a} \int_a^b f_i(x)^p g_{i+1}(x)^q dx\right) \leq \frac{1}{b-a} \sum_{i=1}^n \int_a^b \phi(f_i(x)^{p+q}) dx.$$

This completes the proof. □

In particular, choosing $\phi(x) = \exp(x)$ in this theorem, the main inequality reduces to

$$\sum_{i=1}^n \exp \left(\frac{1}{b-a} \int_a^b f_i(x)^p g_{i+1}(x)^q dx \right) \leq \frac{1}{b-a} \sum_{i=1}^n \int_a^b \exp(f_i(x)^{p+q}) dx.$$

3.8 Eighth theorem

The eighth theorem is presented below.

Theorem 3.8. Let $n \in \mathbb{N} \setminus \{0, 1\}$, $a, b \in \mathbb{R}$ with $b > a$, and $f_1, f_2, \dots, f_n : [a, b] \rightarrow (0, \infty)$ and $g_1, g_2, \dots, g_n : [a, b] \rightarrow (0, \infty)$ be two cyclic sequences of measurable functions. We suppose that, for any $i = 1, 2, \dots, n$, f_i and g_i are integrable, and

$$\int_a^b g_i(x) dx = \int_a^b f_i(x) dx.$$

Then we have

$$\sum_{i=1}^n \int_a^b \int_a^b \frac{g_i(x)}{f_{i+1}(y)} dx dy \geq n(b-a)^2.$$

Proof. By the Fubini-Tonelli theorem and the equality $\int_a^b g_i(x) dx = \int_a^b f_i(x) dx$, we can write

$$\begin{aligned} \sum_{i=1}^n \int_a^b \int_a^b \frac{g_i(x)}{f_{i+1}(y)} dx dy &= \sum_{i=1}^n \left(\int_a^b g_i(x) dx \right) \left(\int_a^b \frac{1}{f_{i+1}(y)} dy \right) \\ &= \sum_{i=1}^n \left(\int_a^b f_i(x) dx \right) \left(\int_a^b \frac{1}{f_{i+1}(y)} dy \right). \end{aligned} \quad (3.17)$$

The arithmetic mean-geometric mean yields

$$\begin{aligned} \sum_{i=1}^n \left(\int_a^b f_i(x) dx \right) \left(\int_a^b \frac{1}{f_{i+1}(y)} dy \right) &\geq n \left(\prod_{i=1}^n \left(\int_a^b f_i(x) dx \right) \left(\int_a^b \frac{1}{f_{i+1}(y)} dy \right) \right)^{1/n} \\ &= n \left(\left(\prod_{i=1}^n \left(\int_a^b f_i(x) dx \right) \right) \left(\prod_{i=1}^n \left(\int_a^b \frac{1}{f_{i+1}(y)} dy \right) \right) \right)^{1/n}. \end{aligned} \quad (3.18)$$

Due to the cyclic symmetry, we have

$$\prod_{i=1}^n \left(\int_a^b \frac{1}{f_{i+1}(y)} dy \right) = \prod_{i=1}^n \left(\int_a^b \frac{1}{f_i(y)} dy \right).$$

Therefore, we derive

$$\begin{aligned} &\left(\prod_{i=1}^n \left(\int_a^b f_i(x) dx \right) \right) \left(\prod_{i=1}^n \left(\int_a^b \frac{1}{f_{i+1}(y)} dy \right) \right) \\ &= \left(\prod_{i=1}^n \left(\int_a^b f_i(x) dx \right) \right) \left(\prod_{i=1}^n \left(\int_a^b \frac{1}{f_i(y)} dy \right) \right) \\ &= \prod_{i=1}^n \left(\int_a^b f_i(x) dx \right) \left(\int_a^b \frac{1}{f_i(x)} dx \right). \end{aligned} \quad (3.19)$$

The Cauchy-Schwarz inequality gives

$$\begin{aligned} \left(\int_a^b f_i(x) dx \right) \left(\int_a^b \frac{1}{f_i(x)} dx \right) &\geq \left(\int_a^b \sqrt{f_i(x)} \frac{1}{\sqrt{f_i(x)}} dx \right)^2 \\ &= \left(\int_a^b dx \right)^2 = (b-a)^2. \end{aligned}$$

Taking the cyclic product both sides, we get

$$\prod_{i=1}^n \left(\int_a^b f_i(x) dx \right) \left(\int_a^b \frac{1}{f_i(x)} dx \right) \geq (b-a)^{2n}. \quad (3.20)$$

It follows from Equations (3.17), (3.18), (3.19), and (3.20) that

$$\sum_{i=1}^n \int_a^b \int_a^b \frac{g_i(x)}{f_{i+1}(y)} dx dy \geq n(b-a)^2.$$

This ends the proof. $\square$

3.9 Ninth theorem

The ninth theorem is established below. The inclusion of a submultiplicative non-decreasing function plays a central role in this framework.

Theorem 3.9. Let $n \in \mathbb{N} \setminus \{0, 1\}$, $a, b \in \mathbb{R}$ with $b > a$, $p, q > 0$, $f_1, f_2, \dots, f_n : [a, b] \rightarrow (0, \infty)$ and $g_1, g_2, \dots, g_n : [a, b] \rightarrow (0, \infty)$ be two cyclic sequences of measurable functions, and $\phi : (0, \infty) \rightarrow (0, \infty)$ be a function that is both submultiplicative and non-decreasing. We suppose that, for any $i = 1, 2, \dots, n$, f_i^{p+q} and g_i^{p+q} are integrable, and

$$\int_a^b g_i(x)^{p+q} dx = \int_a^b f_i(x)^{p+q} dx.$$

Then we have

$$\begin{aligned} &\prod_{i=1}^n \phi \left(\int_a^b f_i(x)^p g_{i+1}(x)^q dx \right) \\ &\leq \prod_{i=1}^n \phi \left(\left(\int_a^b f_i(x)^{p+q} dx \right)^{p/(p+q)} \right) \phi \left(\left(\int_a^b f_i(x)^{p+q} dx \right)^{q/(p+q)} \right). \end{aligned}$$

Proof. By the Hölder inequality and the equality $\int_a^b g_{i+1}(x)^{p+q} dx = \int_a^b f_{i+1}(x)^{p+q} dx$, we have

$$\begin{aligned} \int_a^b f_i(x)^p g_{i+1}(x)^q dx &\leq \left(\int_a^b f_i(x)^{p+q} dx \right)^{p/(p+q)} \left(\int_a^b g_{i+1}(x)^{p+q} dx \right)^{q/(p+q)} \\ &= \left(\int_a^b f_i(x)^{p+q} dx \right)^{p/(p+q)} \left(\int_a^b f_{i+1}(x)^{p+q} dx \right)^{q/(p+q)}. \end{aligned}$$

Since ϕ is non-decreasing, we get

$$\phi \left(\int_a^b f_i(x)^p g_{i+1}(x)^q dx \right) \leq \phi \left(\left(\int_a^b f_i(x)^{p+q} dx \right)^{p/(p+q)} \left(\int_a^b f_{i+1}(x)^{p+q} dx \right)^{q/(p+q)} \right).$$

Moreover, since ϕ is submultiplicative, we obtain

$$\begin{aligned} & \phi \left(\left(\int_a^b f_i(x)^{p+q} dx \right)^{p/(p+q)} \left(\int_a^b f_{i+1}(x)^{p+q} dx \right)^{q/(p+q)} \right) \\ & \leq \phi \left(\left(\int_a^b f_i(x)^{p+q} dx \right)^{p/(p+q)} \right) \phi \left(\left(\int_a^b f_{i+1}(x)^{p+q} dx \right)^{q/(p+q)} \right). \end{aligned}$$

Therefore, we derive

$$\phi \left(\int_a^b f_i(x)^p g_{i+1}(x)^q dx \right) \leq \phi \left(\left(\int_a^b f_i(x)^{p+q} dx \right)^{p/(p+q)} \right) \phi \left(\left(\int_a^b f_{i+1}(x)^{p+q} dx \right)^{q/(p+q)} \right).$$

Taking the cyclic product both sides, we find

$$\begin{aligned} & \prod_{i=1}^n \phi \left(\int_a^b f_i(x)^p g_{i+1}(x)^q dx \right) \\ & \leq \prod_{i=1}^n \phi \left(\left(\int_a^b f_i(x)^{p+q} dx \right)^{p/(p+q)} \right) \phi \left(\left(\int_a^b f_{i+1}(x)^{p+q} dx \right)^{q/(p+q)} \right) \\ & = \left(\prod_{i=1}^n \phi \left(\left(\int_a^b f_i(x)^{p+q} dx \right)^{p/(p+q)} \right) \right) \left(\prod_{i=1}^n \phi \left(\left(\int_a^b f_{i+1}(x)^{p+q} dx \right)^{q/(p+q)} \right) \right). \end{aligned} \quad (3.21)$$

Due to the cyclic symmetry, we get

$$\prod_{i=1}^n \phi \left(\left(\int_a^b f_{i+1}(x)^{p+q} dx \right)^{q/(p+q)} \right) = \prod_{i=1}^n \phi \left(\left(\int_a^b f_i(x)^{p+q} dx \right)^{q/(p+q)} \right).$$

Therefore, we have

$$\begin{aligned} & \left(\prod_{i=1}^n \phi \left(\left(\int_a^b f_i(x)^{p+q} dx \right)^{p/(p+q)} \right) \right) \left(\prod_{i=1}^n \phi \left(\left(\int_a^b f_{i+1}(x)^{p+q} dx \right)^{q/(p+q)} \right) \right) \\ & = \left(\prod_{i=1}^n \phi \left(\left(\int_a^b f_i(x)^{p+q} dx \right)^{p/(p+q)} \right) \right) \left(\prod_{i=1}^n \phi \left(\left(\int_a^b f_i(x)^{p+q} dx \right)^{q/(p+q)} \right) \right) \\ & = \prod_{i=1}^n \phi \left(\left(\int_a^b f_i(x)^{p+q} dx \right)^{p/(p+q)} \right) \phi \left(\left(\int_a^b f_i(x)^{p+q} dx \right)^{q/(p+q)} \right). \end{aligned} \quad (3.22)$$

It follows from Equations (3.21) and (3.22) that

$$\begin{aligned} & \prod_{i=1}^n \phi \left(\int_a^b f_i(x)^p g_{i+1}(x)^q dx \right) \\ & \leq \prod_{i=1}^n \phi \left(\left(\int_a^b f_i(x)^{p+q} dx \right)^{p/(p+q)} \right) \phi \left(\left(\int_a^b f_i(x)^{p+q} dx \right)^{q/(p+q)} \right). \end{aligned}$$

This concludes the proof. $\square$

In particular, choosing $\phi(x) = x^s$ with $s > 0$, we have

$$\prod_{i=1}^n \left(\int_a^b f_i(x)^p g_{i+1}(x)^q dx \right)^s \leq \prod_{i=1}^n \left(\int_a^b f_i(x)^{p+q} dx \right)^s.$$

3.10 Tenth theorem

The tenth theorem is stated below.

Theorem 3.10. Let $n \in \mathbb{N} \setminus \{0, 1\}$, and $f_1, f_2, \dots, f_n : (0, \infty) \rightarrow (0, \infty)$ and $g_1, g_2, \dots, g_n : (0, \infty) \rightarrow (0, \infty)$ be two cyclic sequences of measurable functions. We suppose that, for any $i = 1, 2, \dots, n$, f_i^2 and g_i^2 are integrable, and

$$\int_0^\infty g_i(x)^2 dx = \int_0^\infty f_i(x)^2 dx.$$

Then we have

$$\sum_{i=1}^n \int_0^\infty \left(\frac{1}{x} \int_0^x f_i(t) dt \right) g_{i+1}(x) dx \leq 2 \sum_{i=1}^n \int_0^\infty f_i(x)^2 dx.$$

Proof. Applying the Cauchy-Schwarz inequality and using the equality $\int_0^\infty g_{i+1}(x)^2 dx = \int_0^\infty f_{i+1}(x)^2 dx$, we obtain

$$\begin{aligned} \int_0^\infty H(f_i)(x) g_{i+1}(x) dx &\leq \left(\int_0^\infty H(f_i)(x)^2 dx \right)^{1/2} \left(\int_0^\infty g_{i+1}(x)^2 dx \right)^{1/2} \\ &= \left(\int_0^\infty H(f_i)(x)^2 dx \right)^{1/2} \left(\int_0^\infty f_{i+1}(x)^2 dx \right)^{1/2}. \end{aligned}$$

The Hardy inequality applied with the exponent 2 yields

$$\int_0^\infty H(f_i)(x)^2 dx \leq 4 \int_0^\infty f_i(x)^2 dx.$$

Therefore, we have

$$\left(\int_0^\infty H(f_i)(x)^2 dx \right)^{1/2} \left(\int_0^\infty f_{i+1}(x)^2 dx \right)^{1/2} \leq 2 \left(\int_0^\infty f_i(x)^2 dx \right)^{1/2} \left(\int_0^\infty f_{i+1}(x)^2 dx \right)^{1/2}.$$

By the arithmetic mean-geometric mean inequality, we obtain

$$2 \left(\int_0^\infty f_i(x)^2 dx \right)^{1/2} \left(\int_0^\infty f_{i+1}(x)^2 dx \right)^{1/2} \leq \int_0^\infty f_i(x)^2 dx + \int_0^\infty f_{i+1}(x)^2 dx.$$

Therefore, we derive

$$\int_0^\infty H(f_i)(x) g_{i+1}(x) dx \leq \int_0^\infty f_i(x)^2 dx + \int_0^\infty f_{i+1}(x)^2 dx.$$

Summing cyclically both sides, we get

$$\sum_{i=1}^n \int_0^\infty H(f_i)(x) g_{i+1}(x) dx \leq \sum_{i=1}^n \int_0^\infty f_i(x)^2 dx + \sum_{i=1}^n \int_0^\infty f_{i+1}(x)^2 dx. \quad (3.23)$$

Due to the cyclic symmetry, we have

$$\sum_{i=1}^n \int_0^\infty f_i(x)^2 dx = \sum_{i=1}^n \int_0^\infty f_{i+1}(x)^2 dx.$$

Therefore, we derive

$$\sum_{i=1}^n \int_0^\infty f_i(x)^2 dx + \sum_{i=1}^n \int_0^\infty f_{i+1}(x)^2 dx = 2 \sum_{i=1}^n \int_0^\infty f_i(x)^2 dx. \quad (3.24)$$

It follows from Equations (3.23) and (3.24) that

$$\sum_{i=1}^n \int_0^\infty H(f_i)(x) g_{i+1}(x) dx \leq 2 \sum_{i=1}^n \int_0^\infty f_i(x)^2 dx.$$

This completes the proof. $\square$

3.11 Eleventh theorem

The eleventh theorem is presented below.

Theorem 3.11. *Let $n \in \mathbb{N} \setminus \{0, 1\}$, and $f_1, f_2, \dots, f_n : \mathbb{R} \rightarrow \mathbb{R}$ and $g_1, g_2, \dots, g_n : \mathbb{R} \rightarrow \mathbb{R}$ be two cyclic sequences of continuously differentiable functions. We suppose that, for any $i = 1, 2, \dots, n$, $\lim_{x \rightarrow -\infty} f_i(x) = 0$, $\lim_{x \rightarrow -\infty} g_i(x) = 0$, and f_i^2 , g_i^2 , $(f'_i)^2$ and $(g'_i)^2$ are integrable, and*

$$\int_{-\infty}^\infty g_i(t)^2 dt = \int_{-\infty}^\infty f_i(t)^2 dt$$

and

$$\int_{-\infty}^\infty (g'_i(t))^2 dt = \int_{-\infty}^\infty (f'_i(t))^2 dt.$$

Then we have

$$\sum_{i=1}^n \sup_{x \in \mathbb{R}} |f_i(x) g_{i+1}(x)| \leq \sum_{i=1}^n \int_{-\infty}^\infty (f_i(x)^2 + f'_i(x)^2) dx.$$

Proof. Using $\lim_{x \rightarrow -\infty} f_i(x) = 0$ (or $\lim_{x \rightarrow -\infty} g_{i+1}(x) = 0$), and standard differentiation rules, we obtain

$$\begin{aligned} f_i(x) g_{i+1}(x) &= \int_{-\infty}^x (f_i(t) g_{i+1}(t))' dt + \lim_{t \rightarrow -\infty} (f_i(t) g_{i+1}(t)) \\ &= \int_{-\infty}^x (f'_i(t) g_{i+1}(t) + f_i(t) g'_{i+1}(t)) dt. \end{aligned}$$

Therefore, we have

$$\begin{aligned} |f_i(x) g_{i+1}(x)| &= \left| \int_{-\infty}^x (f'_i(t) g_{i+1}(t) + f_i(t) g'_{i+1}(t)) dt \right| \\ &\leq \int_{-\infty}^x |f'_i(t) g_{i+1}(t) + f_i(t) g'_{i+1}(t)| dt \\ &\leq \int_{-\infty}^\infty |f'_i(t) g_{i+1}(t) + f_i(t) g'_{i+1}(t)| dt, \end{aligned}$$

so that

$$\sup_{x \in \mathbb{R}} |f_i(x)g_{i+1}(x)| \leq \int_{-\infty}^{\infty} |f'_i(t)g_{i+1}(t) + f_i(t)g'_{i+1}(t)| dt.$$

Summing cyclically both sides, we get

$$\sum_{i=1}^n \sup_{x \in \mathbb{R}} |f_i(x)g_{i+1}(x)| \leq \sum_{i=1}^n \int_{-\infty}^{\infty} |f'_i(t)g_{i+1}(t) + f_i(t)g'_{i+1}(t)| dt. \quad (3.25)$$

It follows from the triangle inequality and the arithmetic mean-geometric mean inequality that

$$\begin{aligned} |f'_i(t)g_{i+1}(t) + f_i(t)g'_{i+1}(t)| &\leq |f'_i(t)g_{i+1}(t)| + |f_i(t)g'_{i+1}(t)| \\ &\leq \frac{1}{2} \left((f'_i(t))^2 + g_{i+1}(t)^2 \right) + \frac{1}{2} \left(f_i(t)^2 + (g'_{i+1}(t))^2 \right). \end{aligned}$$

Integrating, summing cyclically both sides and using the equalities $\int_{-\infty}^{\infty} g_{i+1}(t)^2 dt = \int_{-\infty}^{\infty} f_{i+1}(t)^2 dt$ and $\int_{-\infty}^{\infty} (g'_{i+1}(t))^2 dt = \int_{-\infty}^{\infty} (f'_{i+1}(t))^2 dt$, we get

$$\begin{aligned} &\sum_{i=1}^n \int_{-\infty}^{\infty} |f'_i(t)g_{i+1}(t) + f_i(t)g'_{i+1}(t)| dt \\ &\leq \frac{1}{2} \left(\sum_{i=1}^n \int_{-\infty}^{\infty} (f'_i(t))^2 dt + \sum_{i=1}^n \int_{-\infty}^{\infty} g_{i+1}(t)^2 dt \right) \\ &\quad + \frac{1}{2} \left(\sum_{i=1}^n \int_{-\infty}^{\infty} f_i(t)^2 dt + \sum_{i=1}^n \int_{-\infty}^{\infty} (g'_{i+1}(t))^2 dt \right) \\ &= \frac{1}{2} \left(\sum_{i=1}^n \int_{-\infty}^{\infty} (f'_i(t))^2 dt + \sum_{i=1}^n \int_{-\infty}^{\infty} f_{i+1}(t)^2 dt \right) \\ &\quad + \frac{1}{2} \left(\sum_{i=1}^n \int_{-\infty}^{\infty} f_i(t)^2 dt + \sum_{i=1}^n \int_{-\infty}^{\infty} (f'_{i+1}(t))^2 dt \right). \end{aligned} \quad (3.26)$$

Due to the cyclic symmetry, we have

$$\sum_{i=1}^n \int_{-\infty}^{\infty} f_{i+1}(t)^2 dt = \sum_{i=1}^n \int_{-\infty}^{\infty} f_i(t)^2 dt$$

and

$$\sum_{i=1}^n \int_{-\infty}^{\infty} (f'_{i+1}(t))^2 dt = \sum_{i=1}^n \int_{-\infty}^{\infty} (f'_i(t))^2 dt.$$

Therefore, we derive

$$\begin{aligned}
 & \frac{1}{2} \left(\sum_{i=1}^n \int_{-\infty}^{\infty} (f'_i(t))^2 dt + \sum_{i=1}^n \int_{-\infty}^{\infty} f_{i+1}(t)^2 dt \right) \\
 & + \frac{1}{2} \left(\sum_{i=1}^n \int_{-\infty}^{\infty} f_i(t)^2 dt + \sum_{i=1}^n \int_{-\infty}^{\infty} (f'_{i+1}(t))^2 dt \right) \\
 & = \sum_{i=1}^n \int_{-\infty}^{\infty} f_i(t)^2 dt + \sum_{i=1}^n \int_{-\infty}^{\infty} (f'_i(t))^2 dt \\
 & = \sum_{i=1}^n \int_{-\infty}^{\infty} (f_i(x)^2 + f'_i(x)^2) dx.
 \end{aligned} \tag{3.27}$$

It follows from Equations (3.25), (3.26), and (3.27) that

$$\sum_{i=1}^n \sup_{x \in \mathbb{R}} |f_i(x)g_{i+1}(x)| \leq \sum_{i=1}^n \int_{-\infty}^{\infty} (f_i(x)^2 + f'_i(x)^2) dx.$$

This ends the proof. $\square$

3.12 Twelfth theorem

The twelfth theorem is proposed below. It implicitly relies on foundational notions from probability theory, including probability density functions and variance.

Theorem 3.12. *Let $n \in \mathbb{N} \setminus \{0, 1\}$, $a, b \in \mathbb{R}$ with $b > a$, and $f_1, f_2, \dots, f_n : [a, b] \rightarrow (0, \infty)$ be a cyclic sequence of measurable functions. We suppose that, for any $i = 1, 2, \dots, n$, $\int_a^b f_i(x) dx = 1$. We set, for any $i = 1, 2, \dots, n$,*

$$M_i = \int_a^b x f_i(x) dx$$

and

$$\text{Var}(f_i) = \int_a^b x^2 f_i(x) dx - M_i^2.$$

Then we have

$$\sum_{i=1}^n \int_a^b \int_a^b (x - y)^2 f_i(x) f_{i+1}(y) dx dy \geq 2 \sum_{i=1}^n \text{Var}(f_i).$$

Proof. Expanding, applying the Fubini-Tonelli theorem, and using the equalities $\int_a^b f_i(x)dx = 1$ and $\int_a^b f_{i+1}(x)dx = 1$, we obtain

$$\begin{aligned} \int_a^b \int_a^b (x-y)^2 f_i(x) f_{i+1}(y) dx dy &= \int_a^b \int_a^b (x^2 - 2xy + y^2) f_i(x) f_{i+1}(y) dx dy \\ &= \left(\int_a^b x^2 f_i(x) dx \right) \left(\int_a^b f_{i+1}(y) dy \right) - 2 \left(\int_a^b x f_i(x) dx \right) \left(\int_a^b y f_{i+1}(y) dy \right) \\ &\quad + \left(\int_a^b f_i(x) dx \right) \left(\int_a^b y^2 f_{i+1}(y) dy \right) \\ &= \int_a^b x^2 f_i(x) dx - 2M_i M_{i+1} + \int_a^b y^2 f_{i+1}(y) dy. \end{aligned}$$

Summing cyclically both sides, we get

$$\begin{aligned} \sum_{i=1}^n \int_a^b \int_a^b (x-y)^2 f_i(x) f_{i+1}(y) dx dy \\ = \sum_{i=1}^n \int_a^b x^2 f_i(x) dx - 2 \sum_{i=1}^n M_i M_{i+1} + \sum_{i=1}^n \int_a^b y^2 f_{i+1}(y) dy. \end{aligned} \quad (3.28)$$

Due to the cyclic symmetry, we have

$$\sum_{i=1}^n \int_a^b y^2 f_{i+1}(y) dy = \sum_{i=1}^n \int_a^b y^2 f_i(y) dy.$$

Therefore, we derive

$$\begin{aligned} \sum_{i=1}^n \int_a^b x^2 f_i(x) dx - 2 \sum_{i=1}^n M_i M_{i+1} + \sum_{i=1}^n \int_a^b y^2 f_{i+1}(y) dy \\ = 2 \left(\sum_{i=1}^n \int_a^b x^2 f_i(x) dx - \sum_{i=1}^n M_i M_{i+1} \right). \end{aligned} \quad (3.29)$$

Let us now bound the last term. The arithmetic mean-geometric mean inequality gives

$$M_i M_{i+1} \leq \frac{1}{2} (M_i^2 + M_{i+1}^2).$$

Summing cyclically both sides, we get

$$\sum_{i=1}^n M_i M_{i+1} \leq \frac{1}{2} \left(\sum_{i=1}^n M_i^2 + \sum_{i=1}^n M_{i+1}^2 \right). \quad (3.30)$$

Due to the cyclic symmetry, we have

$$\sum_{i=1}^n M_{i+1}^2 = \sum_{i=1}^n M_i^2,$$

so that

$$\frac{1}{2} \left(\sum_{i=1}^n M_i^2 + \sum_{i=1}^n M_{i+1}^2 \right) = \sum_{i=1}^n M_i^2. \quad (3.31)$$

It follows from Equations (3.28), (3.29), (3.30), and (3.31) that

$$\begin{aligned} \sum_{i=1}^n \int_a^b \int_a^b (x-y)^2 f_i(x) f_{i+1}(y) dx dy &\geq 2 \left(\sum_{i=1}^n \int_a^b x^2 f_i(x) dx - \sum_{i=1}^n M_i^2 \right) \\ &= 2 \sum_{i=1}^n \text{Var}(f_i). \end{aligned}$$

This concludes the proof. $\square$

One may mention that, for any $i = 1, 2, \dots, n$, $\text{Var}(f_i) \geq 0$.

Remark 3.3. The normalization condition $\int_a^b f_i(x) dx = 1$, combined with the non-negativity of f_i , allows Theorem 3.12 to be interpreted within the context of probability distribution theory. If the cyclic sequence $f_1, f_2, \dots, f_n$ represents the probability density functions of continuous unit distributions defined on $[0, 1]$ (such as the beta, Kumaraswamy, or trigonometric unit distributions), this inequality provides a global lower bound on the expected squared differences between independent variables drawn from adjacent distributions in the cycle, governed solely by the sum of their individual variances.

3.13 Thirteenth theorem

The thirteenth theorem is stated below. It implicitly relies on foundational notions from operator theory.

Theorem 3.13. Let $n \in \mathbb{N} \setminus \{0, 1\}$, $a, b \in \mathbb{R}$ with $b > a$, $f_1, f_2, \dots, f_n : [a, b] \rightarrow (0, \infty)$ be a cyclic sequence of measurable functions, and $K : [a, b]^2 \rightarrow (0, \infty)$ be a symmetric kernel, i.e., such that, for any $x, y \in [a, b]$, $K(x, y) = K(y, x)$. We suppose that, for any $i = 1, 2, \dots, n$, $K(x, y) f_i(x)^2$ is integrable. Then we have

$$\sum_{i=1}^n \int_a^b \int_a^b K(x, y) f_i(x) f_{i+1}(y) dx dy \leq \sum_{i=1}^n \int_a^b \int_a^b K(x, y) f_i(x)^2 dx dy.$$

Proof. The arithmetic mean-geometric mean inequality gives

$$f_i(x) f_{i+1}(y) \leq \frac{1}{2} (f_i(x)^2 + f_{i+1}(y)^2).$$

Multiplying by the non-negative kernel and integrating, we get

$$\begin{aligned} &\int_a^b \int_a^b K(x, y) f_i(x) f_{i+1}(y) dx dy \\ &\leq \frac{1}{2} \left(\int_a^b \int_a^b K(x, y) f_i(x)^2 dx dy + \int_a^b \int_a^b K(x, y) f_{i+1}(y)^2 dx dy \right). \end{aligned}$$

Summing cyclically both sides, we obtain

$$\begin{aligned} & \sum_{i=1}^n \int_a^b \int_a^b K(x, y) f_i(x) f_{i+1}(y) dx dy \\ & \leq \frac{1}{2} \left(\sum_{i=1}^n \int_a^b \int_a^b K(x, y) f_i(x)^2 dx dy + \sum_{i=1}^n \int_a^b \int_a^b K(x, y) f_{i+1}(y)^2 dx dy \right). \end{aligned} \quad (3.32)$$

Using the fact that K is symmetric and the Fubini-Tonelli theorem, we derive

$$\begin{aligned} \int_a^b \int_a^b K(x, y) f_{i+1}(y)^2 dx dy &= \int_a^b \int_a^b K(y, x) f_{i+1}(y)^2 dx dy \\ &= \int_a^b \int_a^b K(x, y) f_{i+1}(x)^2 dx dy. \end{aligned}$$

Due to the cyclic symmetry, we have

$$\sum_{i=1}^n \int_a^b \int_a^b K(x, y) f_{i+1}(x)^2 dx dy = \sum_{i=1}^n \int_a^b \int_a^b K(x, y) f_i(x)^2 dx dy.$$

Therefore, we get

$$\sum_{i=1}^n \int_a^b \int_a^b K(x, y) f_{i+1}(y)^2 dx dy = \sum_{i=1}^n \int_a^b \int_a^b K(x, y) f_i(x)^2 dx dy,$$

so that

$$\begin{aligned} & \frac{1}{2} \left(\sum_{i=1}^n \int_a^b \int_a^b K(x, y) f_i(x)^2 dx dy + \sum_{i=1}^n \int_a^b \int_a^b K(x, y) f_i(x)^2 dx dy \right) \\ &= \sum_{i=1}^n \int_a^b \int_a^b K(x, y) f_i(x)^2 dx dy. \end{aligned} \quad (3.33)$$

It follows from Equations (3.32) and (3.33) that

$$\sum_{i=1}^n \int_a^b \int_a^b K(x, y) f_i(x) f_{i+1}(y) dx dy \leq \sum_{i=1}^n \int_a^b \int_a^b K(x, y) f_i(x)^2 dx dy.$$

This completes the proof. □

Remark 3.4. While Theorem 3.13 assumes a symmetric kernel, the underlying arithmetic mean-geometric mean structure can be adapted to investigate non-homogeneous Hilbert integral inequalities. By decomposing the kernel via independent parameter weightings, such as incorporating logarithmic-exponential elements or general homogeneous functions, this cyclic inequality can be transformed to bound multi-variable integral operators over $(0, \infty)^2$.

3.14 Fourteenth theorem

The fourteenth theorem is presented below.

Theorem 3.14. *Let $n \in \mathbb{N} \setminus \{0, 1\}$, $a, b \in \mathbb{R}$ with $b > a$, and $f_1, f_2, \dots, f_n : [a, b] \rightarrow (0, \infty)$ and $g_1, g_2, \dots, g_n : [a, b] \rightarrow (0, \infty)$ be two cyclic sequences of measurable functions. We suppose that, for any $i = 1, 2, \dots, n$, f_i and g_i are integrable, and*

$$\int_a^b g_i(x) dx = \int_a^b f_i(x) dx.$$

We set, for any $i = 1, 2, \dots, n$,

$$F_i(x) = \int_a^x f_i(t) dt$$

and

$$G_i(x) = \int_a^x g_i(t) dt.$$

Then we have

$$\sum_{i=1}^n \int_a^b \sqrt{F_i(x)g_{i+1}(x)f_i(x)G_{i+1}(x)} dx \leq \frac{1}{2} \sum_{i=1}^n \left(\int_a^b f_i(x) dx \right)^2.$$

Proof. The arithmetic mean-geometric mean inequality gives

$$\sqrt{F_i(x)g_{i+1}(x)f_i(x)G_{i+1}(x)} \leq \frac{1}{2} (F_i(x)g_{i+1}(x) + f_i(x)G_{i+1}(x)).$$

Let us notice that

$$F_i(x)g_{i+1}(x) + f_i(x)G_{i+1}(x) = (F_i(x)G_{i+1}(x))'.$$

Therefore, we have

$$\sqrt{F_i(x)g_{i+1}(x)f_i(x)G_{i+1}(x)} \leq \frac{1}{2} (F_i(x)G_{i+1}(x))'.$$

Integrating and summing cyclically, we obtain

$$\sum_{i=1}^n \int_a^b \sqrt{F_i(x)g_{i+1}(x)f_i(x)G_{i+1}(x)} dx \leq \frac{1}{2} \sum_{i=1}^n \int_a^b (F_i(x)G_{i+1}(x))' dx. \quad (3.34)$$

Since $F_i(a) = 0$ (or $G_{i+1}(a) = 0$), we have

$$\int_a^b (F_i(x)G_{i+1}(x))' dx = F_i(b)G_{i+1}(b) - F_i(a)G_{i+1}(a) = F_i(b)G_{i+1}(b),$$

so that

$$\sum_{i=1}^n \int_a^b (F_i(x)G_{i+1}(x))' dx = \sum_{i=1}^n F_i(b)G_{i+1}(b). \quad (3.35)$$

Applying the arithmetic mean-geometric mean inequality and the equality $\int_a^b g_{i+1}(x)dx = \int_a^b f_{i+1}(x)dx$, i.e., $G_{i+1}(b) = F_{i+1}(b)$, we get

$$F_i(b)G_{i+1}(b) \leq \frac{1}{2} (F_i(b)^2 + G_{i+1}(b)^2) = \frac{1}{2} (F_i(b)^2 + F_{i+1}(b)^2).$$

Summing cyclically both sides, we obtain

$$\sum_{i=1}^n F_i(b)G_{i+1}(b) \leq \frac{1}{2} \left(\sum_{i=1}^n F_i(b)^2 + \sum_{i=1}^n F_{i+1}(b)^2 \right). \quad (3.36)$$

Due to the cyclic symmetry, we have

$$\sum_{i=1}^n F_{i+1}(b)^2 = \sum_{i=1}^n F_i(b)^2,$$

so that

$$\frac{1}{2} \left(\sum_{i=1}^n F_i(b)^2 + \sum_{i=1}^n F_{i+1}(b)^2 \right) = \sum_{i=1}^n F_i(b)^2 = \sum_{i=1}^n \left(\int_a^b f_i(x)dx \right)^2. \quad (3.37)$$

It follows from Equations (3.34), (3.35), (3.36) and (3.37) that

$$\sum_{i=1}^n \int_a^b \sqrt{F_i(x)g_{i+1}(x)f_i(x)G_{i+1}(x)}dx \leq \frac{1}{2} \sum_{i=1}^n \left(\int_a^b f_i(x)dx \right)^2.$$

This ends the proof. $\square$

3.15 Fifteenth theorem

The fifteenth theorem is established below.

Theorem 3.15. Let $n \in \mathbb{N} \setminus \{0, 1\}$, $a, b \in \mathbb{R}$ with $b > a$, and $f_1, f_2, \dots, f_n : [a, b] \rightarrow (0, \infty)$ and $g_1, g_2, \dots, g_n : [a, b] \rightarrow (0, \infty)$ be two cyclic sequences of measurable functions. We suppose that, for any $i = 1, 2, \dots, n$, f_i and g_i are integrable, and

$$\int_a^b g_i(x)dx = \int_a^b f_i(x)dx.$$

Then we have

$$\prod_{i=1}^n \int_a^b \exp(f_i(x) - g_{i+1}(x))dx \geq (b-a)^n.$$

Proof. By the Jensen inequality applied to the exponential function which is convex, we get

$$\frac{1}{b-a} \int_a^b \exp(f_i(x) - g_{i+1}(x))dx \geq \exp\left(\frac{1}{b-a} \int_a^b (f_i(x) - g_{i+1}(x))dx\right).$$

Taking the cyclic product both sides and using the equality $\int_a^b g_{i+1}(x)dx = \int_a^b f_{i+1}(x)dx$, we obtain

$$\begin{aligned}
 & \prod_{i=1}^n \left(\frac{1}{b-a} \int_a^b \exp(f_i(x) - g_{i+1}(x)) dx \right) \\
 & \geq \prod_{i=1}^n \exp \left(\frac{1}{b-a} \int_a^b (f_i(x) - g_{i+1}(x)) dx \right) \\
 & = \exp \left(\frac{1}{b-a} \sum_{i=1}^n \int_a^b (f_i(x) - g_{i+1}(x)) dx \right) \\
 & = \exp \left(\frac{1}{b-a} \left(\sum_{i=1}^n \int_a^b f_i(x) dx - \sum_{i=1}^n \int_a^b g_{i+1}(x) dx \right) \right) \\
 & = \exp \left(\frac{1}{b-a} \left(\sum_{i=1}^n \int_a^b f_i(x) dx - \sum_{i=1}^n \int_a^b f_{i+1}(x) dx \right) \right).
 \end{aligned} \tag{3.38}$$

Due to the cyclic symmetry, we have

$$\sum_{i=1}^n \int_a^b f_{i+1}(x) dx = \sum_{i=1}^n \int_a^b f_i(x) dx.$$

Therefore, we derive

$$\exp \left(\frac{1}{b-a} \left(\sum_{i=1}^n \int_a^b f_i(x) dx - \sum_{i=1}^n \int_a^b f_{i+1}(x) dx \right) \right) = \exp(0) = 1. \tag{3.39}$$

It follows from Equations (3.38) and (3.39) that

$$\prod_{i=1}^n \left(\frac{1}{b-a} \int_a^b \exp(f_i(x) - g_{i+1}(x)) dx \right) \geq 1,$$

which implies that

$$\prod_{i=1}^n \int_a^b \exp(f_i(x) - g_{i+1}(x)) dx \geq (b-a)^n.$$

This concludes the proof. $\square$

3.16 Sixteenth theorem

The sixteenth theorem is stated below.

Theorem 3.16. *Let $n \in \mathbb{N} \setminus \{0, 1\}$, $a, b \in \mathbb{R}$ with $b > a$, $p > 0$, and $f_1, f_2, \dots, f_n : [a, b] \rightarrow (0, \infty)$ and $g_1, g_2, \dots, g_n : [a, b] \rightarrow (0, \infty)$ be cyclic sequences of measurable functions. We suppose that, for any $i = 1, 2, \dots, n$, $f_i^p \ln(f_i/g_{i+1})$, f_i^p and g_i^p are integrable, and*

$$\int_a^b g_i(x)^p dx = \int_a^b f_i(x)^p dx.$$

Then we have

$$\sum_{i=1}^n \int_a^b f_i(x)^p \ln \left(\frac{f_i(x)}{g_{i+1}(x)} \right) dx \geq 0.$$

Proof. By the elementary logarithmic inequality $\ln(t) \geq 1 - 1/t$ for any $t > 0$, taking $t = f_i(x)^p/g_{i+1}(x)^p$, we obtain

$$\ln\left(\frac{f_i(x)^p}{g_{i+1}(x)^p}\right) \geq 1 - \frac{g_{i+1}(x)^p}{f_i(x)^p},$$

which simplifies to

$$\ln\left(\frac{f_i(x)}{g_{i+1}(x)}\right) \geq \frac{1}{p}\left(1 - \frac{g_{i+1}(x)^p}{f_i(x)^p}\right).$$

Multiplying both sides by $f_i(x)^p$, we get

$$f_i(x)^p \ln\left(\frac{f_i(x)}{g_{i+1}(x)}\right) \geq \frac{1}{p}(f_i(x)^p - g_{i+1}(x)^p).$$

Integrating and summing cyclically both sides, and using the equality $\int_a^b g_{i+1}(x)^p dx = \int_a^b f_{i+1}(x)^p dx$, we obtain

$$\begin{aligned} \sum_{i=1}^n \int_a^b f_i(x)^p \ln\left(\frac{f_i(x)}{g_{i+1}(x)}\right) dx &\geq \frac{1}{p} \left(\sum_{i=1}^n \int_a^b f_i(x)^p dx - \sum_{i=1}^n \int_a^b g_{i+1}(x)^p dx \right) \\ &= \frac{1}{p} \left(\sum_{i=1}^n \int_a^b f_i(x)^p dx - \sum_{i=1}^n \int_a^b f_{i+1}(x)^p dx \right). \end{aligned} \quad (3.40)$$

Due to the cyclic symmetry, we have

$$\sum_{i=1}^n \int_a^b f_{i+1}(x)^p dx = \sum_{i=1}^n \int_a^b f_i(x)^p dx,$$

so that

$$\sum_{i=1}^n \int_a^b f_i(x)^p dx - \sum_{i=1}^n \int_a^b f_{i+1}(x)^p dx = 0. \quad (3.41)$$

It follows from Equations (3.40) and (3.41) that

$$\sum_{i=1}^n \int_a^b f_i(x)^p \ln\left(\frac{f_i(x)}{g_{i+1}(x)}\right) dx \geq 0.$$

This completes the proof. $\square$

Remark 3.5. The integral structure in Theorem 3.16 has a natural connection to information theory. Specifically, when $p = 1$ and the cyclic sequence consists of probability density functions, the integral $\int_a^b f_i(x) \ln(f_i(x)/g_{i+1}(x)) dx$ represents the relative entropy between adjacent distributions. Consequently, this theorem demonstrates that the sum of the cyclic relative entropies in this continuous framework is bounded from below by zero, reinforcing its non-negativity property in a cyclic sequence.

Remark 3.6. Across the sixteen theorems, the necessary and sufficient conditions for equality are governed by the equality cases of the foundational inequalities employed in their proofs (e.g., the Minkowski, Young, Cauchy-Schwarz, Hardy, and Jensen inequalities). Combined with cyclic symmetry and integral mass identities, strict equality typically requires all functions $f_1, f_2, \dots, f_n$ and $g_1, g_2, \dots, g_n$ to equal a common constant almost everywhere on their respective domains.

4 Summary and Conclusion

4.1 Summary of the results

Table 1 summarizes the core inequalities established in the main theorems for ease of reference. Full notation and exact assumptions are provided in their respective statements.

Table 1: Direct presentation of the theorems

Theorem	Core inequality
Theorem 3.1	$\sum_{i=1}^n \left(\int_a^b (f_i(x) + g_{i+1}(x))^p dx \right)^{1/p} \leq 2 \sum_{i=1}^n \left(\int_a^b f_i(x)^p dx \right)^{1/p}$
Theorem 3.2	$\sum_{i=1}^n \int_a^b f_i(x) \prod_{k=1}^m g_{i+k}(x) dx \leq \sum_{i=1}^n \int_a^b f_i(x)^{m+1} dx$
Theorem 3.3	$\sum_{i=1}^n \int_a^b \frac{g_i(x)^2}{\int_a^b f_{i+1}(t) dt} dx \geq \frac{1}{b-a} \sum_{i=1}^n \int_a^b f_i(x) dx$
Theorem 3.4	$\sum_{i=1}^n \int_a^b \frac{f_i(x)^2}{g_{i+1}(x)} dx \geq \sum_{i=1}^n \int_a^b f_i(x) dx$
Theorem 3.5	$\sum_{i=1}^n \int_a^b f_i(x) g'_{i+1}(x) dx \leq \sum_{i=1}^n \left(\int_a^b f'_i(x) dx \right)^2$
Theorem 3.6	$\sum_{i=1}^n \int_a^b \phi(f_i(x)^p g_{i+1}(x)^q) dx \leq \sum_{i=1}^n \int_a^b \phi(f_i(x)^{p+q}) dx$
Theorem 3.7	$\sum_{i=1}^n \phi \left(\frac{1}{b-a} \int_a^b f_i(x)^p g_{i+1}(x)^q dx \right) \leq \frac{1}{b-a} \sum_{i=1}^n \int_a^b \phi(f_i(x)^{p+q}) dx$
Theorem 3.8	$\sum_{i=1}^n \int_a^b \int_a^b \frac{g_i(x)}{f_{i+1}(y)} dx dy \geq n(b-a)^2$
Theorem 3.9	$\prod_{i=1}^n \phi \left(\int_a^b f_i(x)^p g_{i+1}(x)^q dx \right) \leq \prod_{i=1}^n \phi \left(\left(\int_a^b f_i(x)^{p+q} dx \right)^{p/(p+q)} \right) \phi \left(\left(\int_a^b f_i(x)^{p+q} dx \right)^{q/(p+q)} \right)$
Theorem 3.10	$\sum_{i=1}^n \int_0^\infty \left(\frac{1}{x} \int_0^x f_i(t) dt \right) g_{i+1}(x) dx \leq 2 \sum_{i=1}^n \int_0^\infty f_i(x)^2 dx$
Theorem 3.11	$\sum_{i=1}^n \sup_{x \in \mathbb{R}} f_i(x) g_{i+1}(x) \leq \sum_{i=1}^n \int_{-\infty}^\infty (f_i(x)^2 + f'_i(x)^2) dx$
Theorem 3.12	$\sum_{i=1}^n \int_a^b \int_a^b (x-y)^2 f_i(x) f_{i+1}(y) dx dy \geq 2 \sum_{i=1}^n \text{Var}(f_i)$
Theorem 3.13	$\sum_{i=1}^n \int_a^b \int_a^b K(x, y) f_i(x) f_{i+1}(y) dx dy \leq \sum_{i=1}^n \int_a^b \int_a^b K(x, y) f_i(x)^2 dx dy$
Theorem 3.14	$\sum_{i=1}^n \int_a^b \sqrt{F_i(x) g_{i+1}(x) f_i(x) G_{i+1}(x)} dx \leq \frac{1}{2} \sum_{i=1}^n \left(\int_a^b f_i(x) dx \right)^2$
Theorem 3.15	$\prod_{i=1}^n \int_a^b \exp(f_i(x) - g_{i+1}(x)) dx \geq (b-a)^n$
Theorem 3.16	$\sum_{i=1}^n \int_a^b f_i(x)^p \ln \left(\frac{f_i(x)}{g_{i+1}(x)} \right) dx \geq 0$

4.2 Concluding remarks

This paper introduced a continuous framework for cyclic inequalities by extending classical discrete structures to cyclic sequences of measurable functions. By integrating cyclic symmetry with key analytical tools, such as the Minkowski, Young, Cauchy-Schwarz, Hardy, and Jensen inequalities, we derived sixteen general families of cyclic integral inequalities that provide global bounds on adjacent functional terms over real intervals. Several promising perspectives for future research emerge, including extending these cyclic integral bounds to multi-dimensional domains or general measure spaces, establishing sharp constants and precise conditions for equality, and incorporating fractional integral operators or weight functions into the

cyclic setup. Additionally, applying these functional bounds to probability theory, differential equations, and operator theory represents an avenue for further investigation.

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