Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 16, Number 6, November 2026, Pages 1047-1054
https://doi.org/10.34198/ejms.16626.67.10471054

Linear-Interpolative Contraction Mapping Theorem for the Berinde Weak, Kannan, Ciric-Reich-Rus, and Chatterjea Operators in Metric Spaces with Application

Clement Boateng Ampadu

[1] Independent Researcher, USA

e-mail: profampadu@gmail.com; drampadu@hotmail.com

Abstract

Motivated by the definition of interpolative metric space [6], in which the triangle inequality has been modified into an inequality consisting of linear and interpolative terms, we introduce in this paper the notion of linear-interpolative contractions which is analogous to the triangle inequality in the definition of interpolative metric space. These inequalities consist of linear and interpolative terms. Some results related to the linear-interpolative contractions are obtained in the setting of metric spaces with an illustrative example. Finally, we apply our result to the Fredholm integral equation.

1 Preliminaries

Definition 1.1. [1] Let B be a set and n be a metric on B. $C : B \mapsto B$ forms an interpolative Kannan type contraction, assuming there is a constant $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$n(Cg, Ch) \leq \lambda n(g, Cg)^{\alpha} n(h, Ch)^{1-\alpha}$$

in every case $g, h \in B$, $g, h \notin Fix(C)$, where $Fix(C) = \{g \in B : Cg = g\}$.

Theorem 1.2. *[1] Let B be a set and n be a metric on B. Assume C is an interpolative Kannan type contraction. Then C admits one and only one fixed point in B.*

Definition 1.3. [2] Let B be a set and n be a metric on B. $C : B \mapsto B$ forms an interpolative Berinde weak operator, if it holds that

$$n(Cg, Ch) \leq \lambda n(g, h)^{\alpha} n(g, Cg)^{1-\alpha},$$

where $\lambda \in [0, 1)$, in all cases $g, h \in B \backslash Fix(C)$.

Alternatively we have

Received: May 12, 2026; Revised & Accepted: August 31, 2026; Published online: September 7, 2026
2020 Mathematics Subject Classification: 47H10, 54H25.
Keywords and phrases: linear-interpolative contraction, Berinde weak contraction, Kannan contraction, Ciric-Reich-Rus contraction, Chatterjea contraction, Fredholm integral equation.

Definition 1.4. [2] Let B be a set and n be a metric on B. $C : B \mapsto B$ forms an interpolative Berinde weak operator, if it holds that

$$n(Cg, Ch) \leq \lambda n(g, h)^{\frac{1}{2}} n(g, Cg)^{\frac{1}{2}},$$

where $\lambda \in [0, 1)$, in all cases $g, h \in B \backslash Fix(C)$.

Theorem 1.5. *[2] Let B be a set and n be a metric on B. Suppose $C : B \mapsto B$ is an interpolative Berinde weak operator. The completeness of B ensures the existence of a fixed point for C.*

Definition 1.6. [3] Let B be a set and n be a metric on B. $C : B \mapsto B$ forms an interpolative Ciric-Reich-Rus operator, should there be $\lambda \in [0, 1)$ and positive reals α, β with $\alpha + \beta < 1$ such that

$$n(Cg, Ch) \leq \lambda n(g, h)^{\alpha} n(g, Cg)^{\beta} n(h, Ch)^{1-\alpha-\beta}$$

in all cases $g, h \in B \backslash Fix(C)$.

Theorem 1.7. *[3] Let $C : B \mapsto B$ be an interpolative Ciric-Reich-Rus operator on a complete Branciari distance space (B, n). Then C has a fixed point in B.*

Definition 1.8. [4] Let B be a set and n be a metric on B. $C : B \mapsto B$ forms an alternate interpolative Ciric-Reich-Rus operator, should there be $\lambda \in [0, 1)$ such that

$$n(Cg, Ch) \leq \lambda n(g, h)^{\frac{1}{3}} n(g, Cg)^{\frac{1}{3}} n(h, Ch)^{\frac{1}{3}}$$

in all cases $g, h \in B \backslash Fix(C)$.

Theorem 1.9. *[4] Let B be a set and n be a metric on B. Suppose $C : B \mapsto B$ is an alternate interpolative Ciric-Reich-Rus operator. Completeness of (B, n) ensures the existence of a fixed point.*

Definition 1.10. [5] Let B be a set and qb_p be a quasi partial b-metric on B. $C : B \mapsto B$ forms an interpolative Chatterjea type contraction, should there be $\rho \in [0, \frac{1}{s})$, $\alpha \in (0, 1)$ such that

$$qp_b(Cg, Ch) \leq \rho[qp_b(g, Ch)]^{\alpha}[\frac{1}{s^2} qp_b(h, Cg)]^{1-\alpha}$$

for all $g, h \in B \backslash Fix(C)$.

Theorem 1.11. *[5] Let (B, qp_b) be a complete quasi partial b-metric space and C be an interpolative Chatterjea type contraction. Then C has a fixed point in B.*

2 Main Result

Theorem 2.1. *(Linear-Interpolative Berinde Weak Contraction Mapping Theorem) Let B be a set and n be a metric on B. Assume $C : B \mapsto B$ forms a linear-interpolative Berinde weak contraction operator, that is, there is $\lambda \in [0, \frac{1}{3})$ and $\alpha \in (0, 1)$ such that*

$$n(Cg, Ch) \leq \lambda[n(g, h) + n(g, Cg) + n(g, h)^{\alpha} n(g, Cg)^{1-\alpha}]$$

in all cases $g, h \in B \backslash Fix(C)$. If (B, n) is complete, then any linear-interpolative Berinde weak contraction mapping $C : B \mapsto B$ has a fixed point.

Proof. Let $q_0 \in B$ be chosen arbitrarily. Define the sequence $\{q_s\}_{s=0}^{\infty}$ recursively by $q_{s+1} = Cq_s$ for all s. Letting $g = q_{s-1}$ and $h = q_s$ in the contractive definition of the theorem, we have

$$\begin{aligned} n(q_s, q_{s+1}) &\leq \lambda[n(q_{s-1}, q_s) + n(q_{s-1}, q_s) + n(q_{s-1}, q_s)^{\alpha} n(q_{s-1}, q_s)^{1-\alpha}] \\ &= 3\lambda n(q_{s-1}, q_s) \\ &= \gamma n(q_{s-1}, q_s), \end{aligned}$$

where $\gamma = 3\lambda < 1$, since $\lambda < \frac{1}{3}$. By induction, we obtain

$$n(q_s, q_{s+1}) \leq \gamma^s n(q_0, q_1).$$

Thus, making use of the above inequality and the triangle inequality, we obtain that for all $s \geq 0$ and $m \geq 1$ that

$$\begin{aligned} n(q_s, q_{s+m}) &\leq n(q_s, q_{s+1}) + n(q_{s+1}, q_{s+2}) + \cdots + n(q_{s+m-1}, q_{s+m}) \\ &\leq n(q_0, q_1)(\gamma^s + \gamma^{s+1} + \cdots + \gamma^{s+m-1}) \\ &= n(q_0, q_1)\gamma^s \frac{1-\gamma^m}{1-\gamma} \\ &\leq n(q_0, q_1)\frac{\gamma^s}{1-\gamma} \to 0 \ as \ s, m \to \infty. \end{aligned}$$

This shows that $\{q_s\}$ is a Cauchy sequence in the complete metric space (B, n) ensuring its convergence to a point $j^* \in B$. To confirm that j^* is a fixed point, we substitute $g = j^*$ and $h = q_s$ in the contractive definition of the theorem, then we have

$$n(Cj^*, q_{s+1}) \leq \lambda[n(j^*, q_s) + n(j^*, Cj^*) + n(j^*, q_s)^{\alpha} n(j^*, Cj^*)^{1-\alpha}].$$

Letting $s \to \infty$ in the above inequality, we have, $n(Cj^*, j^*) \leq \lambda n(j^*, Cj^*)$. Since $1 - \lambda \neq 0$, we obtain $n(Cj^*, j^*) = 0$, that is, $Cj^* = j^*$, and so j^* is a fixed point of C. This completes the proof. □

Theorem 2.2. *(Linear-Interpolative Kannan Contraction Mapping Theorem) Let B be a set and n be a metric on B. Assume $C : B \mapsto B$ forms a linear-interpolative Kannan contraction operator, that is, there is $\lambda \in [0, \frac{1}{3})$ and $\alpha \in (0, 1)$ such that*

$$n(Cg, Ch) \leq \lambda[n(g, Cg) + n(h, Ch) + n(g, Cg)^{\alpha} n(h, Ch)^{1-\alpha}]$$

in all cases $g, h \in B \backslash Fix(C)$. If (B, n) is complete, then any linear-interpolative Kannan contraction mapping $C : B \mapsto B$ has a fixed point.

Proof. Let $q_0 \in B$ be chosen arbitrarily. Define the sequence $\{q_s\}_{s=0}^{\infty}$ recursively by $q_{s+1} = Cq_s$ for all s. Letting $g = q_{s-1}$ and $h = q_s$ in the contractive definition of the theorem, we have

$$n(q_s, q_{s+1}) \leq \lambda[n(q_{s-1}, q_s) + n(q_s, q_{s+1}) + n(q_{s-1}, q_s)^{\alpha} n(q_s, q_{s+1})^{1-\alpha}].$$

By Weighted AM-GM inequality the above inequality turns into

$$n(q_s, q_{s+1}) \leq \lambda[n(q_{s-1}, q_s) + n(q_s, q_{s+1}) + \alpha n(q_{s-1}, q_s) + (1-\alpha)n(q_s, q_{s+1})]$$

From the above inequality, we have

$$n(q_s, q_{s+1}) \leq \frac{\lambda + \lambda\alpha}{1 - (\lambda + \lambda(1-\alpha))} n(q_{s-1}, q_s) = \gamma n(q_{s-1}, q_s),$$

where $\gamma = \frac{\lambda+\lambda\alpha}{1-(\lambda+\lambda(1-\alpha))} < 1$, since $\lambda < \frac{1}{3}$. By induction, we have $n(q_s, q_{s+1}) \leq \gamma^s n(q_0, q_1)$. Similar to the proof of the previous theorem, we have that $\{q_s\}$ is a Cauchy sequence in the complete metric space (B, n), ensuring its convergence to a point $j^* \in B$. To confirm that j^* is a fixed point, we substitute $g = j^*$ and $h = q_s$ in the contractive definition of the theorem, then we have

$$n(Cj*, q_{s+1}) \leq \lambda[n(j^*, Cj^*) + n(q_s, q_{s+1}) + n(j^*, Cj^*)^{\alpha} n(q_s, q_{s+1})^{1-\alpha}].$$

Letting $s \to \infty$ in the above inequality, we have $n(Cj^*, j^*) \leq \lambda n(j^*, Cj^*)$. Since $1 - \lambda \neq 0$, we have $n(Cj^*, j^*) = 0$, which implies that $Cj^* = j^*$, and so j^* is a fixed point of C. This completes the proof. □

Theorem 2.3. *(Linear-Interpolative Ciric-Reich-Rus Contraction Mapping Theorem) Let B be a set and n be a metric on B. Assume $C : B \mapsto B$ forms a linear-interpolative Ciric-Reich-Rus contraction operator, that is, there is $\lambda \in [0, \frac{1}{4})$, $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$ such that*

$$n(Cg, Ch) \leq \lambda[n(g, h) + n(g, Cg) + n(h, Ch) + n(g, h)^{\alpha} n(g, Cg)^{\beta} n(h, Ch)^{1-\alpha-\beta}]$$

in all cases $g, h \in B \backslash Fix(C)$. If (B, n) is complete, then any linear-interpolative Ciric-Reich-Rus contraction mapping $C : B \mapsto B$ has a fixed point.

Proof. Let $q_0 \in B$ be chosen arbitrarily. Define the sequence $\{q_s\}_{s=0}^{\infty}$ recursively by $q_{s+1} = Cq_s$ for all s. Letting $g = q_{s-1}$ and $h = q_s$ in the contractive definition of the theorem, we have

$$n(q_s, q_{s+1}) \leq \lambda[2n(q_{s-1}, q_s) + n(q_s, q_{s+1}) + n(q_{s-1}, q_s)^{\alpha} n(q_{s-1}, q_s)^{\beta} n(q_s, q_{s+1})^{1-\alpha-\beta}].$$

By Weighted AM-GM inequality the above inequality turns into

$$n(q_s, q_{s+1}) \leq \lambda[(2 + \alpha + \beta)n(q_{s-1}, q_s) + (2 - \alpha - \beta)n(q_s, q_{s+1})].$$

The above inequality implies that

$$n(q_s, q_{s+1}) \leq \frac{\lambda(2 + \alpha + \beta)}{1 - \lambda(2 - \alpha - \beta)} n(q_{s-1}, q_s) = \gamma n(q_{s-1}, q_s),$$

where $\gamma = \frac{\lambda(2+\alpha+\beta)}{1-\lambda(2-\alpha-\beta)} < 1$, since $\lambda < \frac{1}{4}$. By induction, we have $n(q_s, q_{s+1}) \leq \gamma^s n(q_0, q_1)$. Similar to the proof of Theorem 2.1, we have that $\{q_s\}$ is a Cauchy sequence in the complete metric space (B, n), ensuring its convergence to a point $j^* \in B$. To confirm that j^* is a fixed point, we substitute $g = j^*$ and $h = q_s$ in the contractive definition of the theorem, then we have

$$n(Cj^*, q_{s+1}) \leq \lambda[n(j^*, q_s) + n(j^*, Cj^*) + n(q_s, q_{s+1}) + n(j^*, q_s)^{\alpha} n(j^*, Cj^*)^{\beta} n(q_s, q_{s+1})^{1-\alpha-\beta}].$$

Letting $s \to \infty$ in the above inequality, we have $n(Cj^*, j^*) \leq \lambda n(j^*, Cj^*)$. Since $1 - \lambda \neq 0$, we have $n(Cj^*, j^*) = 0$, which implies that $Cj^* = j^*$, and hence j^* is a fixed point of C. This completes the proof. $\square$

Theorem 2.4. *(Linear-Interpolative Chatterjea Contraction Mapping Theorem) Let B be a set and n be a metric on B. Assume $C : B \mapsto B$ forms a linear-interpolative Chatterjea contraction operator, that is, there is $\lambda \in [0, \frac{1}{2})$ and $\alpha \in (0, 1)$ such that*

$$n(Cg, Ch) \leq \lambda[n(g, Ch) + n(h, Cg) + n(g, Ch)^{\alpha} n(h, Cg)^{1-\alpha}]$$

in all cases $g, h \in B \backslash Fix(C)$. If (B, n) is complete, then any linear interpolative Chatterjea contraction mapping $C : B \mapsto B$ has a fixed point.

Proof. Let $q_0 \in B$ be chosen arbitrarily. Define the sequence $\{q_s\}_{s=0}^{\infty}$ recursively by $q_{s+1} = Cq_s$ for all s. Letting $g = q_{s-1}$ and $h = q_s$ in the contractive definition of the theorem, we have

$$\begin{aligned} n(q_s, q_{s+1}) &= n(Cq_{s-1}, Cq_s) \\ &\leq \lambda[n(q_{s-1}, q_{s+1}) + n(q_s, q_s) + n(q_{s-1}, q_{s+1})^{\alpha} n(q_s, q_s)^{1-\alpha}] \\ &= \lambda n(q_{s-1}, q_{s+1}) \\ &\leq \lambda[n(q_{s-1}, q_s) + n(q_s, q_{s+1})] \end{aligned}$$

The above inequality implies that $n(q_s, q_{s+1}) \leq \gamma n(q_{s-1}, q_s)$, where $\gamma = \frac{\lambda}{1-\lambda} < 1$, since $\lambda < \frac{1}{2}$. By induction, we have $n(q_s, q_{s+1}) \leq \gamma^s n(q_0, q_1)$. Similar to the proof of Theorem 2.1, we find that $\{q_s\}$ is a Cauchy sequence in the complete metric space (B, n), ensuring its convergence to a point $j^* \in B$. To confirm that j^* is fixed point, we substitute $g = j^*$ and $h = q_s$ in the contractive definition of the theorem, then, we have

$$\begin{aligned} n(Cj^*, q_{s+1}) &= n(Cj^*, Cq_s) \\ &\leq \lambda[n(j^*, q_{s+1}) + n(q_s, Cj^*) + n(j^*, q_{s+1})^{\alpha} n(q_s, Cj^*)^{1-\alpha}]. \end{aligned}$$

Letting $s \to \infty$ in the above inequality, we have $n(Cj^*, j^*) \leq \lambda n(j^*, Cj^*)$. This inequality implies that $(1 - \lambda) n(Cj^*, j^*) \leq 0$. Since $1 - \lambda \neq 0$, we have $n(Cj^*, j^*) = 0$ or $Cj^* = j^*$. So the fixed point exists. This completes the proof. $\square$

Example 2.5. We consider (B, n) a metric space with $B = [0, 1]$ and the standard Euclidean metric $n(g, h) = |g - h|$. Define $C : B \mapsto B$ by $Cg = \frac{g}{3}$. We admit C is a linear-interpolative Berinde weak contraction with $\lambda = \frac{1}{6}$, and $\alpha = \frac{1}{2}$. To verify the contractive condition, we must establish that for some $\lambda \in [0, \frac{1}{3})$ and $\alpha \in (0, 1)$ the inequality below holds for all $g, h \in B \backslash Fix(C)$:

$$n(Cg, Ch) \leq \lambda[n(g, h) + n(g, Cg) + n(g, h)^{\alpha} n(g, Cg)^{1-\alpha}].$$

Figure 1 shows that the "BlueGreenYellow" surface (RHS of the contraction inequality) dominates the "LakeColors" surface (LHS of the contraction inequality). This demonstrates that the inequality is satisfied for all $g, h \in (0, 1]$ with $\lambda = \frac{1}{6}$ and $\alpha = \frac{1}{2}$.

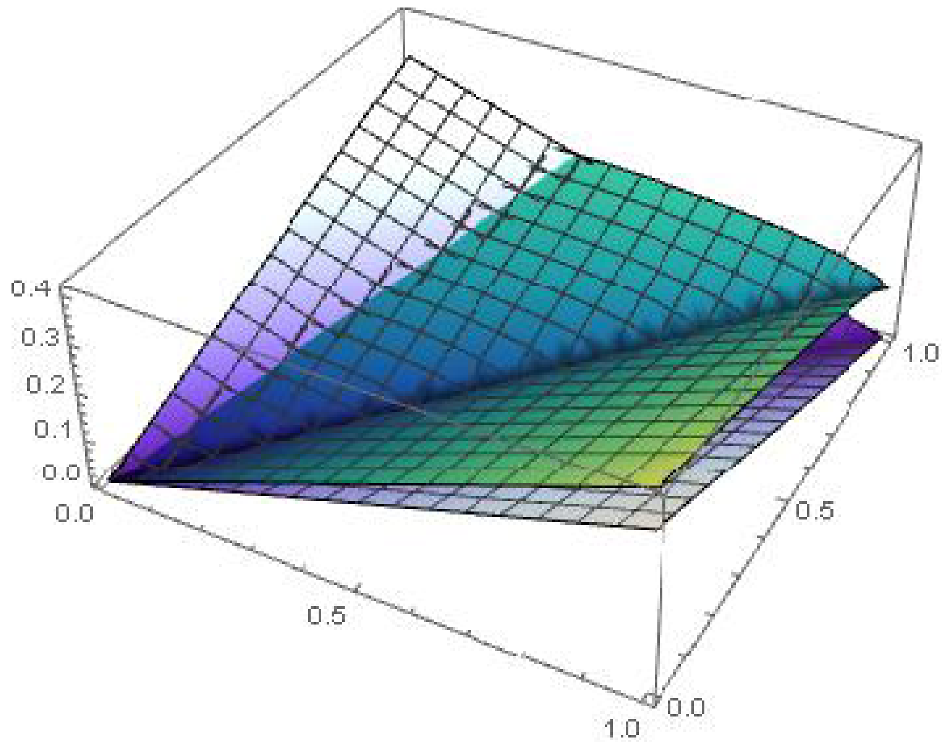

Figure 1: Comparison of both sides of the linear-interpolative Berinde weak contraction inequality in the above example over $(0, 1]$.

3 Application

We apply our result to establish an existence theorem for non-linear Fredholm integral equation. Let $Y = C[0,1]$ be a set of all real continuous functions on $[0,1]$ equipped with the metric $p(b,c) = |b-c| = \max_{t\in[0,1]} |b(t)-c(t)|$, for all $b, c \in C[0,1]$. Then (Y,p) is a complete metric space. Now we consider the non-linear Fredholm integral equation

$$b(t) = c(t) + \int_0^1 K(t,s,b(s))ds, \tag{3.1}$$

where $s, t \in [0,1]$. Assume that $K : [0,1]\times[0,1]\times Y \mapsto \mathbb{R}$ and $c : [0,1] \mapsto \mathbb{R}$ are continuous, where $c(t)$ is a given function in Y.

Theorem 3.1. *Suppose (Y,p) is a metric space equipped with the metric $p(b,c) = |b-c| = \max_{t\in[0,1]} |b(t)-c(t)|$ for all $b, c \in Y$, and $F : Y \mapsto Y$ be an operator on Y defined by*

$$Fb(t) = c(t) + \int_0^1 K(t,s,b(s))ds. \tag{3.2}$$

If there exists $\lambda \in [0, \frac{1}{3})$, and $\alpha \in (0,1)$, such that for all $b, c \in Y\backslash Fix(F)$, $s, t \in [0,1]$, satisfying the following

$$|K(t,s,b(s)) - K(t,s,c(s))| \leq \lambda M(b(s),c(s)),$$

where

$$M(b(s), c(s)) = |b(s) - c(s)| + |b(s) - Fb(s)| + |b(s) - c(s)|^{\alpha}|b(s) - Fb(s)|^{1-\alpha},$$

then the integral equation (3.2) has a solution in Y*.*

Proof. From (3.1) and (3.2), we obtain

$$\begin{aligned}|Fb(t) - Fc(t)| &= \left|\int_0^1 K(t,s,b(s))ds - \int_0^1 K(t,s,c(s))ds\right| \\ &\le \int_0^1 |K(t,s,b(s)) - K(t,s,c(s))|ds \\ &\le \int_0^1 \lambda(|b(s) - c(s)| + |b(s) - Fb(s)| + |b(s) - c(s)|^{\alpha}|b(s) - Fb(s)|^{1-\alpha})ds.\end{aligned}$$

Taking maximum on both sides for all $t \in [0,1]$, we conclude that

$$p(Fb, Fc) \le \lambda(p(b,c) + p(b,Fb) + p(b,c)^{\alpha}p(b,Fb)^{1-\alpha})$$

which shows that F satisfies the condition of a linear-interpolative Berinde weak contraction. Since $Y = C[0,1]$ is a complete metric space, we find that all the conditions of Theorem 2.1 are satisfied, and hence the integral equation (3.2) has a solution in Y. □

Acknowledgement

The author sincerely thanks the anonymous reviewer(s) for their constructive comments and suggestions, which greatly improved this manuscript.

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