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Connectivity and Distance Properties of Zero-Divisor Graphs of Finite Commutative Semilocal Rings

Presley Kiplagat

Department of Pure and Applied Sciences, Kirinyaga University, P. O. Box 143-10300, Kerugoya, Kenya
e-mail: presleykiplagat@gmail.com

Abstract

The zero-divisor graph of a commutative ring provides a natural connection between algebraic and graph-theoretic structures. Although extensive research has been conducted on the algebraic and combinatorial properties of zero-divisor graphs, their connectivity and metric properties over finite semilocal rings remain comparatively less explored. In this paper, we investigate the zero-divisor graph associated with the finite semilocal ring $R = \mathbb{Z}_{p^n q^m}$, where p and q are distinct primes and $n, m \geq 2$. Using a valuation-theoretic approach, we establish a natural decomposition of the vertex set into valuation layers. This decomposition enables us to determine the connectivity, shortest-path structure, eccentricities of valuation layers, diameter, radius, centre, and peripheral vertices of the graph. The results demonstrate that the valuation-layer decomposition completely governs the structural, metric, and symmetry properties of the zero-divisor graph, providing a unified framework for the study of finite semilocal rings and extending several known results on zero-divisor graphs.

1 Introduction

The study of zero-divisor graphs has become an important area of interaction between algebra and graph theory. The concept was first introduced by Beck [5] and later refined by Anderson and Livingston [2]. Since then, zero-divisor graphs have been extensively investigated from algebraic, combinatorial, and graph-theoretic viewpoints [3, 6]. Since its introduction, the zero-divisor graph has served as an effective tool for investigating the structural properties of finite and infinite commutative rings. Numerous graph-theoretic invariants, including connectedness, diameter, girth, clique number, chromatic number, planarity, and automorphism groups, have been shown to reflect important algebraic properties of the underlying ring. Comprehensive surveys by DeMeyer and Schneider [6], together with subsequent contributions by Anderson and Badawi [3], Lucas [9], Mulay [10], Akbari *et al.* [1], Axtell and Stickles [4], and Spiroff and Wickham [12], have established zero-divisor graphs as powerful tools for understanding the interplay between algebraic and combinatorial structures. Among these graph invariants, connectivity and distance-related properties play a fundamental role because they describe the global metric structure of the graph. In particular, quantities such as shortest paths, eccentricity, diameter, radius, centre, and peripheral

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vertices reveal how information propagates through a graph and provide insight into its overall organisation. Although these invariants have been investigated for several classes of commutative rings, relatively few explicit descriptions are available for finite semilocal rings of the form $R = \mathbb{Z}_{p^n q^m}$, where p and q are distinct primes and $n, m \geq 2$. Such rings possess a natural two-dimensional valuation structure arising from the p- and q-adic valuations of their nonzero zero-divisors. This valuation structure induces a decomposition of the vertex set into disjoint valuation layers. The resulting valuation-layer decomposition provides a natural framework for analysing adjacency and distance relations, allowing graph-theoretic properties to be expressed directly in terms of valuation coordinates. It also complements recent investigations on the automorphism groups of zero-divisor graphs over finite rings and nilradicals [7, 8, 11]. The principal contribution of this paper is the application of a valuation-layer approach to the study of the zero-divisor graph of $\mathbb{Z}_{p^n q^m}$. Using this framework, we establish an explicit valuation criterion for adjacency and show that the valuation-layer decomposition completely determines both the structural and metric properties of the graph. In particular, we prove that the graph is connected, characterize shortest paths between valuation layers, determine the eccentricities of valuation layers, diameter, radius, centre, and peripheral vertices. These results provide a unified valuation-based framework for studying the structural, metric, and symmetry properties of zero-divisor graphs over finite semilocal rings.

2 Preliminaries

The metric properties of the zero-divisor graph are governed by the adjacency relations between valuation layers, an approach that complements earlier investigations based on graph-theoretic constructions and realizability [4, 12]. Since every vertex belongs to exactly one valuation layer, it is sufficient to determine when two such layers are adjacent. This valuation-based description considerably simplifies the analysis of connectivity, distances, diameter, and girth by reducing the problem to comparisons between valuation pairs [7].

Definition 2.0.1. *Let $R = \mathbb{Z}_{p^n q^m}$, where p and q are distinct primes. For $0 \leq r \leq n$ and $0 \leq s \leq m$, the valuation layer $S_{p^r q^s}$ is defined by $S_{p^r q^s} = \{x \in Z^*(R) : v_p(x) = r,\ v_q(x) = s\}$, where $Z^*(R)$ denotes the set of all nonzero zero-divisors of R, and $v_p(x)$ and $v_q(x)$ denote the p-adic and q-adic valuations of x, respectively.*

Theorem 2.0.2. *Let $R = \mathbb{Z}_{p^n q^m}$, where p and q are distinct primes and $n, m \geq 1$. Let $x \in S_{p^r q^s}$ and $y \in S_{p^t q^u}$, where $0 \leq r, t \leq n, 0 \leq s, u \leq m$, with $(r, s) \neq (0, 0), (t, u) \neq (0, 0)$, and neither x nor y is the zero element of R. Then $x \sim y$ in $\Gamma(R)$ if and only if $r + t \geq n$ and $s + u \geq m$.*

Proof. Let $x \in S_{p^r q^s}$ and $y \in S_{p^t q^u}$. By the definition of the valuation layers, we may write $x = p^r q^s a$ and $y = p^t q^u b$, where the remaining factors are relatively prime to the corresponding primes. Hence $xy = p^{r+t} q^{s+u} ab$. Since $R = \mathbb{Z}_{p^n q^m}$, the product xy is congruent to zero modulo $p^n q^m$ if and only if it is divisible by both p^n and q^m. Therefore, $xy = 0$ in R if and only if $r + t \geq n$ and $s + u \geq m$. Because adjacency in the zero-divisor graph is defined by $x \sim y \Longleftrightarrow x \neq y,\ xy = 0$, the stated criterion follows: $x \sim y \Longleftrightarrow r + t \geq n$ and $s + u \geq m$. $\square$

Theorem 2.0.2 shows that adjacency depends solely on the valuation pairs of the vertices and is independent of the particular unit factors in their representations. Consequently, vertices belonging to the same valuation layer have identical adjacency patterns with respect to every other valuation layer. In particular, any permutation of the vertices within a fixed valuation layer preserves all adjacency relations between that layer and the remaining layers.

Example 2.0.3. *Consider the ring $R = \mathbb{Z}_{36} = \mathbb{Z}_{2^2 3^2}$. The nonzero zero-divisors of R are*

$$Z^*(R) = \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 26, 27, 28, 30, 32, 33, 34\}.$$

Using the valuation-layer definition, these vertices are partitioned according to their 2-adic and 3-adic valuations. The resulting valuation layers are

$$\begin{aligned} S_2 &= \{2, 10, 14, 22, 26, 34\}, & S_3 &= \{3, 15, 21, 33\},\\ S_4 &= \{4, 8, 16, 20, 28, 32\}, & S_6 &= \{6, 30\},\\ S_9 &= \{9, 27\}, & S_{12} &= \{12, 24\},\\ S_{18} &= \{18\}. \end{aligned}$$

Thus, $Z^*(R) = S_2 \cup S_3 \cup S_4 \cup S_6 \cup S_9 \cup S_{12} \cup S_{18}$. The corresponding layer cardinalities are

$$\begin{aligned} &|S_2| = 6, \quad |S_3| = 4, \quad |S_4| = 6, \quad |S_6| = 2,\\ &|S_9| = 2, \quad |S_{12}| = 2, \quad |S_{18}| = 1. \end{aligned}$$

For example, if $x = 8$, then $v_2(8) = 2$ and $v_3(8) = 0$, so $8 \in S_4$. Similarly, if $x = 12$, then $v_2(12) = 2$ and $v_3(12) = 1$, so $12 \in S_{12}$. By Theorem 2.0.2, if $x \in S_{2^r 3^s}$ and $y \in S_{2^t 3^u}$, then $x \sim y$ if and only if $r + t \geq 2$ and $s + u \geq 2$. For instance, if $x \in S_2$ and $y \in S_{12}$, then their valuation vectors are $(1, 0)$ and $(2, 1)$, respectively. Since $1 + 2 \geq 2$ but $0 + 1 < 2$, we have $x \nsim y$. On the other hand, if $x \in S_2$ and $y \in S_{18}$, then their valuation vectors are $(1, 0)$ and $(1, 2)$, respectively. Hence $1 + 1 \geq 2$ and $0 + 2 \geq 2$, so $x \sim y$. This example illustrates how the valuation vectors determine the adjacency relations between the vertices of $\Gamma(R)$.

Corollary 2.0.4. *Let $S_{p^r q^s}$ and $S_{p^t q^u}$ be two valuation layers of $\Gamma(R)$. Then every vertex of $S_{p^r q^s}$ is adjacent to every vertex of $S_{p^t q^u}$ if and only if $r + t \geq n$ and $s + u \geq m$.*

The adjacency criterion illustrated in this section forms the foundation for the distance analysis carried out in the following sections. In particular, it enables the shortest paths between vertices to be determined using only valuation pairs, thereby reducing metric computations to simple arithmetic conditions on the corresponding valuations.

Definition 2.0.5. *Let G be a connected graph and let $v \in V(G)$. The* eccentricity *of v is $e(v) = \max_{u \in V(G)} d(u, v)$.*

Definition 2.0.6. *For a valuation layer $S_{p^r q^s}$, its eccentricity is defined by $e(S_{p^r q^s}) = e(x)$, where $x \in S_{p^r q^s}$. This definition is well-defined by Lemma 3.4.1.*

3 Main Results

The valuation-layer decomposition allows the connectivity of $\Gamma(R)$ to be analysed through the interaction of valuation pairs. Rather than considering individual vertices, it is sufficient to investigate how the valuation layers are linked. This approach not only establishes the connectedness of the graph but also provides explicit descriptions of shortest paths between arbitrary vertices.

Lemma 3.0.1. *Let $x \in S_{p^r}$, where $1 \leq r < n$. Then every vertex of $S_{p^{n-r}q^m}$ is adjacent to x. Similarly, if $x \in S_{q^s}$, where $1 \leq s < m$, then every vertex of $S_{p^n q^{m-s}}$ is adjacent to x.*

Proof. Let $x = p^r a$, where $\gcd(a, pq) = 1$. Then $v(x) = (r, 0)$. Choose $y = p^{n-r}q^m b$, where $\gcd(b, pq) = 1$. Then $v(y) = (n-r, m)$. Hence $r+(n-r) = n$ and $0+m = m$. Therefore, $r+(n-r) \geq n$ and $0+m \geq m$. By Theorem 2.0.2, $x \sim y$. The proof of the second assertion is analogous. If $x \in S_{q^s}$, then $v(x) = (0, s)$. Choose $y \in S_{p^n q^{m-s}}$, whose valuation vector is $(n, m-s)$. Since $0 + n = n$ and $s + (m-s) = m$, Theorem 2.0.2 again implies that $x \sim y$. $\square$

Theorem 3.0.2. *$\Gamma(\mathbb{Z}_{p^n q^m})$ is connected.*

Proof. Let x, y be arbitrary vertices. There are three possibilities.

Case 1.

x, y belong to valuation layers having positive p- and q-valuations. By Lemma 3.0.1, each vertex is adjacent to every vertex of $S_{p^{n-1}q^{m-1}}$. Hence $x \longrightarrow S_{p^{n-1}q^{m-1}} \longrightarrow y$ is a path.

Case 2.

Exactly one vertex belongs to a layer of the form S_{p^r} or S_{q^s}. By Lemma 3.0.1, this vertex is adjacent to a vertex whose valuation has both coordinates positive. Applying Lemma 3.0.1 then yields a path to the second vertex.

Case 3.

Both vertices belong to one-coordinate valuation layers. Each can first be connected to a valuation layer having positive coordinates by Lemma 3.0.1. The resulting vertices are then connected through the hub layer $S_{p^{n-1}q^{m-1}}$. Thus every pair of vertices is joined by a path. Therefore $\Gamma(\mathbb{Z}_{p^n q^m})$ is connected.

$\square$

3.1 Distance Between Valuation Layers

Since adjacency depends only on valuation pairs, the distance between two vertices is completely determined by the valuation layers to which they belong. Consequently, it is sufficient to study distances between valuation layers rather than individual vertices.

Remark 3.1.1. *For distinct vertices $x \in S_{p^r q^s}$ and $y \in S_{p^t q^u}$, Theorem 2.0.2 immediately gives $d(x, y) = 1 \iff r + t \geq n \quad and \quad s + u \geq m$. Thus, distance one is completely characterized by the valuation-sum conditions in Theorem 2.0.2.*

Theorem 3.1.2. *Let $x \in S_{p^r q^s}, y \in S_{p^t q^u}$, and suppose $r + t < n$ or $s + u < m$. Then $d(x, y) = 2$ if and only if there exists a valuation pair (a, b) satisfying $r + a \geq n$, $s + b \geq m$, $a + t \geq n$, and $b + u \geq m$.*

Proof. Assume that $d(x, y) = 2$. Then there exists a vertex z such that $x \sim z$ and $z \sim y$. Suppose $v(z) = (a, b)$. Applying the adjacency criterion twice yields $r+a \geq n$, $s+b \geq m$, $a+t \geq n$, and $b+u \geq m$. Conversely, suppose there exists a valuation pair (a, b) satisfying these four inequalities. Choose any vertex $z \in S_{p^a q^b}$. Then $x \sim z$ and $z \sim y$ by Theorem 2.0.2. Since $x \nsim y$, the shortest path has length two. Therefore, $d(x, y) = 2$. □

Lemma 3.1.3. *Let $x \in S_{p^r q^s}, y \in S_{p^t q^u}$. If $r + t \geq n$ or $s + u \geq m$, then every common neighbour of x and y belongs to a valuation layer $S_{p^a q^b}$ satisfying $a \geq \max\{n - r, n - t\}$, and $b \geq \max\{m - s, m - u\}$.*

Proof. Suppose $z \in S_{p^a q^b}$ is incident to both x and y. Since $x \sim z$, Theorem 2.0.2 gives $r + a \geq n$, and $s + b \geq m$. Similarly, $y \sim z$ implies $t + a \geq n$, and $u + b \geq m$. Hence $a \geq n - r, a \geq n - t$, which gives $a \geq \max\{n - r, n - t\}$. Likewise, $b \geq \max\{m - s, m - u\}$. □

3.2 Diameter of the Zero-Divisor Graph

The characterization of the shortest paths obtained in the preceding section allows the diameter of $\Gamma(\mathbb{Z}_{p^n q^m})$ to be determined through the valuation pairs of its vertices.

Lemma 3.2.1. *Let $x \in S_{p^r q^s}, y \in S_{p^t q^u}$. Suppose $r+t < n$ or $s+u < m$. If there does not exist a valuation pair (a, b) satisfying $r + a \geq n, s + b \geq m$, and $a + t \geq n, b + u \geq m$, then $d(x, y) \geq 3$.*

Proof. Suppose, to the contrary, $d(x, y) \leq 2$. Since $x \nsim y$, $d(x, y) = 2$. Hence there exists a vertex z such that $x \sim z$ and $z \sim y$. Writing $v(z) = (a, b)$, Theorem 2.0.2 yields $r + a \geq n, s + b \geq m$, and $a + t \geq n, b + u \geq m$. This contradicts the assumption that no such valuation pair exists. Therefore, $d(x, y) \geq 3$. □

Theorem 3.2.2. *For any pair $x, y \in V(\Gamma(\mathbb{Z}_{p^n q^m}))$, we have $d(x, y) \leq 3$.*

Proof. If $x \sim y$, then $d(x,y) = 1$. Suppose that $x \nsim y$. If a common neighbour exists, then by Theorem 3.1.2, $d(x,y) = 2$. Assume therefore that no common neighbour exists. Since $\Gamma(\mathbb{Z}_{p^n q^m})$ is connected, there exists a path joining x and y. Choose such a path having minimum length, $x = x_0, x_1, \ldots, x_k = y$. Since neither $k = 1$ nor $k = 2$ is possible, we obtain $k \geq 3$. On the other hand, Lemmas 3.0.1 and 3.0.1 show that every vertex can be connected to a valuation layer having positive valuations in both coordinates, and every such layer is adjacent to the hub layer $S_{p^{n-1}q^{m-1}}$. Consequently every pair of vertices admits a path of the form $x \longrightarrow x' \longrightarrow S_{p^{n-1}q^{m-1}} \longrightarrow y$, where x' is omitted whenever x already belongs to a two-dimensional valuation layer. Hence $d(x,y) \leq 3$. □

Corollary 3.2.3. $\operatorname{diam}(\Gamma(\mathbb{Z}_{p^n q^m})) \leq 3$.

Proof. $\operatorname{diam}(G) = \max\{d(u,v) : u,v \in V(G)\}$. By Theorem 3.2.2, every pair of distinct vertices in $\Gamma(\mathbb{Z}_{p^n q^m})$ can be joined by a path of length at most three. Consequently, $d(u,v) \leq 3$, for all $u,v \in V(\Gamma(\mathbb{Z}_{p^n q^m}))$. Since the diameter is the maximum of all such pairwise distances, no distance between two vertices can exceed three. Therefore, $\operatorname{diam}(\Gamma(\mathbb{Z}_{p^n q^m})) \leq 3$.

□

3.3 Exact Diameter of $\Gamma(\mathbb{Z}_{p^n q^m})$

The previous section established that every pair of vertices of $\Gamma(\mathbb{Z}_{p^n q^m})$ is connected by a path of length at most three. We now show that this upper bound is attained, thereby determining the exact diameter of the graph.

Lemma 3.3.1. *Let $x \in S_p, y \in S_q$. Then $d(x,y) = 3$.*

Proof. The valuation pairs of x and y are $v(x) = (1,0), v(y) = (0,1)$. Since $1 + 0 < n, 0 + 1 < m$, the vertices are not adjacent. Next suppose that x and y possess a common neighbour $z \in S_{p^a q^b}$. Since $x \sim z$, Theorem 2.0.2 gives $1 + a \geq n, b \geq m$. Similarly, $y \sim z$ implies $a \geq n, 1 + b \geq m$. These inequalities require simultaneously $a \geq n, b \geq m$. However, $(a,b) = (n,m)$ corresponds only to the zero element of the ring, which is not a vertex of $\Gamma(R)$. Hence no common neighbour exists. Therefore, $d(x,y) \neq 2$. Since the graph is connected and every pair of vertices has distance at most three, $d(x,y) = 3$. □

Theorem 3.3.2. *Let $R = \mathbb{Z}_{p^n q^m}$, where $n, m \geq 2$. Then* $\operatorname{diam}\big(\Gamma(R)\big) = 3$.

Proof. By Theorem 3.2.2, $\operatorname{diam}(\Gamma(R)) \leq 3$. On the other hand, Lemma 3.3.1 exhibits two vertices whose distance is exactly three. Hence $\operatorname{diam}(\Gamma(R)) \geq 3$. Therefore, $\operatorname{diam}(\Gamma(R)) = 3$. □

3.4 Eccentricity, Radius and Centre

Vertices belonging to the same valuation layer have identical neighbourhoods, and their metric properties coincide. Consequently, it is sufficient to determine these invariants for the valuation layers. Since graph automorphisms preserve distances, they also preserve eccentricities. Hence, the eccentricity is constant on each valuation layer.

Lemma 3.4.1. *Let $x, y \in S_{p^r q^s}$. Then $e(x) = e(y)$.*

Proof. By Corollary 2.0.4, every vertex of $S_{p^r q^s}$ has the same adjacency relations with every valuation layer. Consequently, the distance from any vertex of $S_{p^r q^s}$ to an arbitrary valuation layer is independent of the chosen vertex. Therefore, for every valuation layer $S_{p^t q^u}$, the distance from x to that layer equals the distance from y to the same layer. It follows that the maximum of these distances is the same for both vertices. Hence $e(x) = e(y)$.

□

Theorem 3.4.2. *Every vertex belonging to the valuation layer $S_{p^{n-1} q^{m-1}}$ has eccentricity* 2.

Proof. Let $x \in S_{p^{n-1} q^{m-1}}$. By Theorem 2.0.2, every valuation layer having positive p- and q-valuations is adjacent to $S_{p^{n-1} q^{m-1}}$. Hence every such vertex is at distance one from x. Consider now a valuation layer of the form S_{p^r} or S_{q^s}. By Lemma 3.0.1, every such layer is adjacent to some valuation layer having positive valuations in both coordinates. Consequently every vertex in these layers is at distance at most two from x. Therefore, $e(x) \leq 2$. Since vertices belonging to one-coordinate valuation layers are not adjacent to x, the eccentricity cannot equal one. Hence $e(x) = 2$. □

Corollary 3.4.3. *The radius of* $\Gamma(\mathbb{Z}_{p^n q^m})$ *is* 2.

Proof. The radius of a connected graph is the minimum eccentricity among all its vertices, $\mathrm{rad}(G) = \min_{v \in V(G)} e(v)$. The previous theorem shows that vertices of $S_{p^{n-1} q^{m-1}}$ have eccentricity two. Since every connected graph satisfies $\mathrm{rad}(G) \leq \mathrm{diam}(G)$, and $\mathrm{diam}(\Gamma(\mathbb{Z}_{p^n q^m})) = 3$, the minimum eccentricity is precisely two. Therefore, $\mathrm{rad}(\Gamma(\mathbb{Z}_{p^n q^m})) = 2$. □

Theorem 3.4.4. *The centre of* $\Gamma(\mathbb{Z}_{p^n q^m})$ *contains the valuation layer* $S_{p^{n-1} q^{m-1}}$.

Proof. By the previous theorem, $e(x) = 2$ for every $x \in S_{p^{n-1} q^{m-1}}$. Since the radius equals two, every such vertex has minimum eccentricity. Hence every vertex of $S_{p^{n-1} q^{m-1}}$ belongs to the centre of the graph. □

3.5 Peripheral Vertices and the Centre

The valuation-layer decomposition provides a convenient framework for studying these metric properties, since vertices belonging to the same valuation layer possess identical neighbourhoods and consequently have the same eccentricity.

Theorem 3.5.1. *Every vertex belonging to either valuation layer S_p or S_q is peripheral.*

Proof. Let $x \in S_p$. By Lemma 3.3.1, there exists a vertex $y \in S_q$ such that $d(x, y) = 3$. Therefore, $e(x) \geq 3$. Since $\operatorname{diam}\big(\Gamma(\mathbb{Z}_{p^n q^m})\big) = 3$, no vertex can have eccentricity exceeding three. Hence $e(x) = 3$. Thus every vertex of S_p is peripheral. The proof for S_q is identical. □

Theorem 3.5.2. *The centre of $\Gamma(\mathbb{Z}_{p^n q^m})$ is precisely $S_{p^{n-1}q^{m-1}}$.*

Proof. From the previous section, every vertex of $S_{p^{n-1}q^{m-1}}$ has eccentricity 2. Hence $S_{p^{n-1}q^{m-1}} \subseteq C(\Gamma(R))$, where $C(\Gamma(R))$ denotes the centre of the graph. Now let $x \notin S_{p^{n-1}q^{m-1}}$. Then at least one coordinate of the valuation of x is strictly less than its maximum possible value. Without loss of generality, suppose $v(x) = (r, s)$, where $r < n - 1$. Choose $y \in S_q$. Since $r + 0 < n$, the vertices are not adjacent. Moreover, by Lemma 3.3.1, no common neighbour joins x and y. Therefore, $d(x, y) = 3$. Consequently, $e(x) = 3$. Thus every vertex outside $S_{p^{n-1}q^{m-1}}$ has eccentricity three. Hence no other valuation layer belongs to the centre. Therefore, $C(\Gamma(R)) = S_{p^{n-1}q^{m-1}}$. □

The preceding results completely determine the principal metric characteristics of the graph.

Corollary 3.5.3. *For $R = \mathbb{Z}_{p^n q^m}, n, m \geq 2$, the following hold.*

(i) $\operatorname{diam}(\Gamma(R)) = 3$.

(ii) $\operatorname{rad}(\Gamma(R)) = 2$.

(iii) The centre is $S_{p^{n-1}q^{m-1}}$.

(iv) The peripheral vertices are precisely those belonging to $S_p \cup S_q$.

Proof. By Theorem 3.2.2, every pair of vertices in $\Gamma(R)$ is joined by a path of length at most three, and there exist pairs of vertices whose distance is exactly three. Consequently, $\operatorname{diam}(\Gamma(R)) = 3$, establishing statement (i). Statement (ii) follows from Theorem 3.4.2. Hence $\operatorname{rad}(\Gamma(R)) = 2$. Statement (iii) is precisely Theorem 3.5.2, which identifies the centre of $\Gamma(R)$ as the valuation layer $S_{p^{n-1}q^{m-1}}$. Finally, Theorem 3.5.1 characterizes all vertices having maximum eccentricity. Therefore, the peripheral vertices are exactly those belonging to $S_p \cup S_q$, establishing statement (iv). □

3.6 Bounds on the Eccentricity of Valuation Layers

The valuation layer decomposition also provides useful information about the eccentricities of the vertices of $\Gamma(\mathbb{Z}_{p^n q^m})$. Since every vertex in a fixed valuation layer has the same neighbourhood, their eccentricities coincide. Thus it is sufficient to determine the eccentricity of each valuation layer.

Theorem 3.6.1. *Let $x \in S_{p^r q^s}$. Then $2 \leq e(x) \leq 3$.*

Proof. Since $\Gamma(\mathbb{Z}_{p^n q^m})$ is connected, $e(x) \geq 1$. Moreover, by Theorem 3.2.2, $d(x, y) \leq 3$ for every vertex $y \in V(\Gamma(\mathbb{Z}_{p^n q^m}))$. Therefore, $e(x) \leq 3$. Suppose that $e(x) = 1$. Then every other vertex of the graph would be adjacent to x. However, if $y \in S_p$ or $y \in S_q$, the adjacency criterion shows that at least one valuation inequality fails unless $v(x) = (n, m)$, which corresponds to the zero element of the ring and is not a vertex of the graph. Hence no vertex is adjacent to every other vertex. Therefore, $e(x) \neq 1$, and consequently $2 \leq e(x) \leq 3$. □

Example 3.6.2. *Let $R = \mathbb{Z}_{p^2 q^3}$, where p and q are distinct primes. The nonzero zero-divisors of R are partitioned into the valuation layers S_p, S_q, S_{pq}, S_{pq^2}, S_{p^2}, S_{q^2}, S_{q^3}, $S_{p^2 q}$, $S_{p^2 q^2}$, S_{pq^3}. The corresponding layer cardinalities are*

$$
\begin{aligned}
|S_p| &= q^2(p-1), & |S_q| &= pq(q-1), \\
|S_{pq}| &= q(p-1)(q-1), & |S_{pq^2}| &= (p-1)(q-1), \\
|S_{p^2}| &= q^2(q-1), & |S_{q^2}| &= p(q-1), \\
|S_{q^3}| &= p-1, & |S_{p^2 q}| &= q(q-1), \\
|S_{p^2 q^2}| &= q-1, & |S_{pq^3}| &= p-1.
\end{aligned}
$$

By Theorem 2.0.2, two distinct vertices $x \in S_{p^r q^s}$ and $y \in S_{p^t q^u}$ are adjacent if and only if $r + t \geq 2$ and $s + u \geq 3$. In particular, a valuation layer $S_{p^r q^s}$ induces a clique whenever $2r \geq 2$ and $2s \geq 3$. Thus, the layers $S_{pq^2}, S_{p^2 q^2}, S_{pq^3}$ are clique-forming. For example, if $x \in S_{pq^2}$ and $y \in S_{p^2 q}$, then $1 + 2 \geq 2$ and $2 + 1 \geq 3$. Hence $x \sim y$ and therefore $d(x, y) = 1$. As another example, let $x \in S_p$ and $y \in S_{q^2}$. Their valuation pairs are $(1, 0)$ and $(0, 2)$, respectively. Since $1 + 0 < 2$ and $0 + 2 < 3$, the vertices x and y are not adjacent. Moreover, they have no common neighbour. Indeed, a common neighbour z of x and y would have to satisfy $v_p(z) \geq 2$ to be adjacent to y, and $v_q(z) \geq 3$ to be adjacent to x. This would require $v(z) = (2, 3)$, which corresponds to the zero element of R and hence is not a vertex of the zero-divisor graph. Consequently, $d(x, y) = 3$. Similarly, if $x \in S_p$ and $y \in S_q$, then their valuation pairs are $(1, 0)$ and $(0, 1)$, respectively. They are not adjacent, and a common neighbour would have to satisfy $v_p(z) \geq 2$ and $v_q(z) \geq 3$, again forcing $v(z) = (2, 3)$, which represents the zero element. Hence $d(x, y) = 3$. Therefore, $\operatorname{diam}(\Gamma(R)) = 3$. In particular, the graph is connected. The valuation layer $S_{p^2 q^2} = S_{p^{n-1} q^{m-1}}$ for $n = 2$ and $m = 3$ plays a central role in the metric structure. Let $x \in S_{p^2 q^2}$. If $y \in S_{p^r q^s}$ has positive valuations in both coordinates, then $2 + r \geq 2$ and $2 + s \geq 3$, so $x \sim y$. If y belongs to a one-coordinate layer, then x is either adjacent to y or has a common neighbour with y. Hence every vertex is at distance at most two from x. Since vertices in S_p are not adjacent to x, it follows that $e(x) = 2$. Consequently, $\operatorname{rad}(\Gamma(R)) = 2$

and $C(\Gamma(R)) = S_{p^2q^2}$. On the other hand, for every $x \in S_p$ there exists $y \in S_q$ with $d(x,y) = 3$. Thus $e(x) = 3$. Similarly, every vertex of S_q has eccentricity 3. Hence $S_p \cup S_q$ consists of peripheral vertices.

4 Conclusion

Using a valuation-layer decomposition based on the p- and q-adic valuations of the nonzero zero-divisors, we partitioned the vertex set into valuation layers whose vertices possess identical valuation vectors and neighbourhood structures. This approach builds upon the classical theory of zero-divisor graphs initiated by Beck and subsequently developed by Anderson and Livingston, DeMeyer and Schneider, and Anderson and Badawi [2,3,5,6]. The valuation-layer decomposition was shown to determine the adjacency relation completely, thereby providing a unified description of the structural and metric properties of the graph. In particular, we established an explicit valuation criterion for adjacency, proved that the graph is connected, characterized shortest paths between valuation layers, determined the diameter, eccentricities of the valuation layers, radius, centre, and peripheral vertices, and obtained bounds for the eccentricities of all valuation layers. These results demonstrate that the valuation-layer decomposition provides a natural and effective framework for analysing zero-divisor graphs of finite semilocal rings. By reducing graph-theoretic questions to valuation data, a direct correspondence is established between the arithmetic structure of the underlying ring and the combinatorial, metric, and symmetry properties of its associated zero-divisor graph. Consequently, the paper extends several known results on zero-divisor graphs by providing a systematic valuation-based characterization of adjacency, connectivity, and metric invariants, complementing earlier studies on planarity, graph realizability, equivalence-class zero-divisor graphs, and graph symmetries [1, 4, 7, 8, 11, 12]. The valuation-layer framework developed in this paper provides a foundation for further investigations of zero-divisor graphs over more general classes of finite rings. Future work may extend these methods to finite semilocal rings with three or more maximal ideals. In such cases, the valuation vector would involve three or more coordinates rather than the two-dimensional vector $(v_p(x), v_q(x))$, resulting in a higher-dimensional layer structure. Moreover, the adjacency condition would require simultaneous valuation inequalities in all coordinates, while the number of valuation layers and the possible shortest-path configurations would increase substantially. Thus, the extension is not a direct repetition of the two-prime case and may lead to richer and more intricate connectivity, metric, and symmetry properties.

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