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Some Inclusion and Radius Problems of Certain Subclasses of Analytic Functions

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Abstract

This article presents the study of certain subclasses of analytic functions defined by using the Hadamard product. We derive certain inclusion results and discuss the applications of multiplier transformation. Several radius problems are also investigated.

1 Introduction

Let $\mathcal{A}$ denote the class of normalized analytic functions in $E = \{z \in \mathbb{C} : |z| < 1\}$, of the series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

The class $\mathcal{S} \subset \mathcal{A}$, represents the class of univalent functions in E. We denote $\mathcal{S}^*$ and $\mathcal{C}$ be the classes of starlike and convex univalent functions in E, respectively. Let f and g be the analytic functions in E, we say f is subordinate to g (written

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as $f \prec g$) if and only if there exists a Schwartz function $w(z)$ (that is, $w(0) = 0$ and $|w(z)| < 1$) in E such that

$$f(z) = g\left(w\left(z\right)\right).$$

Particularly, if $g(z)$ is univalent function in E, then $f \prec g$ is equivalent to

$$f(0) = g(0) \text{ and } f(E) \subset g(E).$$

The convolution of two power series f and g in E denoted by $*$, and is defined as follows,

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n z^n \ (z \in E).$$

We consider $\mathfrak{H}$ be the class of analytic univalent functions $h(z)$ in E with $h(0) = 1$ and $Re\left\{h(z)\right\} > 0$, $(z \in E)$.

Now, we define the following.

Definition 1. Let $f, g \in \mathcal{A}$ with $(f * g)(z) \neq 0$ $(z \in E)$. Then $f \in \mathcal{S}_g^*(h)$ if and only if

$$\frac{z\left(g * f\right)^{'}(z)}{(g * f)(z)} \prec h(z), \ z \in E.$$

Analogously,

$$\mathcal{C}_g(h) = \left\{f \in \mathcal{A} : zf' \in \mathcal{S}_g^*(h)\right\}.$$

Obviously, for the particular choices of functions g and h, we have specific subclasses of $\mathcal{S}$.

1. Let $h\left(z\right) = p_k\left(z\right)$, $k \in [0, 1]$, where $p_k\left(z\right)$ is convex univalent in E and has the form

$$p_k\left(z\right) = \begin{cases} \frac{1+z}{1-z}, & k = 0, \\ 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k = 1, \\ 1 + \frac{2}{1-k^2}\left[\left(\frac{2}{\pi}\arccos k\right)\arctan h\sqrt{z}\right], & 0 < k < 1. \end{cases} \tag{1.2}$$

We denote

$$k-\mathcal{ST}_g=\left\{f\in\mathcal{A}:\frac{z\left(g*f\right)'(z)}{(g*f)(z)}\prec p_k(z),\ g\in\mathcal{A},\ z\in E\right\},$$

$$k-\mathcal{UCV}_g=\left\{f\in\mathcal{A}:\frac{\left(z\left(g*f\right)'(z)\right)'}{(g*f)'(z)}\prec p_k(z),\ g\in\mathcal{A},\ z\in E\right\}.$$

Note that if $g(z)=\frac{z}{1-z}$, $z\in E$, we obtain the well-known classes $k-\mathcal{UCV}$ and $k-\mathcal{ST}$ of k-uniformly convex and corresponding starlike functions respectively, introduced and studied by Kanas et al. [3, 4]. Also, we refer to [5, 6].

2. Let $h(z)=\left(\frac{1+\mathfrak{a}z}{1+\mathfrak{b}z}\right)^{\beta}$, $-1\le\mathfrak{b}<\mathfrak{a}\le 1$, $0<\beta\le 1$, be convex univalent in E and the series representation of $h(z)$ be as follows:

$$p(z)=1+\beta(\mathfrak{a}-\mathfrak{b})z+\left[-\beta(\mathfrak{a}-\mathfrak{b})\mathfrak{b}+\frac{1}{2}\beta(\beta-1)(\mathfrak{a}-\mathfrak{b})^2\right]z^2+\ldots.$$

We denote

$$S_g^*(\mathfrak{a},\mathfrak{b};\beta)=\left\{f\in\mathcal{A}:\frac{z\left(g*f\right)'(z)}{(g*f)(z)}\prec\left(\frac{1+\mathfrak{a}z}{1+\mathfrak{b}z}\right)^{\beta},g\in\mathcal{A},\ z\in E\right\},$$

$$C_g(\mathfrak{a},\mathfrak{b};\beta)=\left\{f\in\mathcal{A}:\frac{\left(z\left(g*f\right)'(z)\right)'}{(g*f)'(z)}\prec\left(\frac{1+\mathfrak{a}z}{1+\mathfrak{b}z}\right)^{\beta},g\in\mathcal{A},\ z\in E\right\}.$$

It is noted that, if $g(z)=\frac{z}{1-z}$, $z\in E$, we have $\mathcal{S}^*(\mathfrak{a},\mathfrak{b};\beta)$ and $\mathcal{C}(\mathfrak{a},\mathfrak{b};\beta)$, respectively. Moreover, if $\beta=1$, then it reduces to the well-known classes $S^*[\mathfrak{a},\mathfrak{b}]$ and $C[\mathfrak{a},\mathfrak{b}]$, respectively, we refer to [2, 10, 12]. Furthermore, if $\mathfrak{a}=1-2\alpha$ and $\mathfrak{b}=-1$, we get $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$, (see [13]), respectively. For $\alpha=0$, we have S^* and C.

The multiplier transformation $I_{\lambda,s}:\mathcal{A}\longrightarrow\mathcal{A}$ is defined as follows [1]:

$$I_{\lambda,s}f(z)=z+\sum_{n=2}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^{s}a_nz^n\qquad(\lambda>-1,s\in\mathbb{R}).\tag{1.3}$$

Clearly, $I_{\lambda,s}\left(I_{\lambda,t}f\left(z\right)\right) = I_{\lambda,s+t}f\left(z\right)$, for $(s, t \in \mathbb{R})$. For different values of s and λ, the operator $I_{\lambda,s}$ has been studied by several authors [7, 8, 11, 15].

From equation (1.3), we can easily have the following identity,

$$z\left(I_{\lambda,s}f\left(z\right)\right)' = \left(\lambda+1\right)I_{\lambda,s+1}f\left(z\right) - \lambda I_{\lambda,s}f\left(z\right). \tag{1.4}$$

We now define the following by taking the value of $g = I_{\lambda,s}$ in the Definition 1.

$$\mathcal{S}^*_{\lambda,s}(h) = \left\{f \in \mathcal{A} : \frac{z\left(I_{\lambda,s}f(z)\right)'}{I_{\lambda,s}f(z)} \prec h\left(z\right),\ z \in E\right\},$$

$$\mathcal{C}_{\lambda,s}(h) = \left\{f \in \mathcal{A} : \frac{\left(z\left(I_{\lambda,s}f\right)'(z)\right)'}{\left(I_{\lambda,s}f(z)\right)'} \prec h\left(z\right),\ z \in E\right\}.$$

Particularly, for $h\left(z\right) = p_k\left(z\right)$ given by (1.2), we have

$$k\text{–}\mathcal{ST}_g = \left\{f \in \mathcal{A} : \frac{z\left(I_{\lambda,s}f(z)\right)'}{I_{\lambda,s}f(z)} \prec p_k(z),\ g \in \mathcal{A},\ z \in E\right\},$$

$$k\text{–}\mathcal{UCV}_g = \left\{f \in \mathcal{A} : \frac{\left(z\left(I_{\lambda,s}f\right)'(z)\right)'}{\left(I_{\lambda,s}f(z)\right)'} \prec p_k(z),\ g \in \mathcal{A},\ z \in E\right\}.$$

And, for $h\left(z\right) = \left(\frac{1+\mathfrak{a}z}{1+\mathfrak{b}z}\right)^{\beta}, -1 \leq \mathfrak{b} < \mathfrak{a} \leq 1, 0 < \beta \leq 1$, we define

$$S^*_{\lambda,s}\left(\mathfrak{a}, \mathfrak{b}; \beta\right) = \left\{f \in \mathcal{A} : \frac{z\left(I_{\lambda,s}f(z)\right)'}{\left(I_{\lambda,s}f(z)\right)} \prec \left(\frac{1+\mathfrak{a}z}{1+\mathfrak{b}z}\right)^{\beta},\ z \in E\right\},$$

$$C_{\lambda,s}\left(\mathfrak{a}, \mathfrak{b}; \beta\right) = \left\{f \in \mathcal{A} : \frac{\left(z\left(I_{\lambda,s}f(z)\right)'\right)'}{\left(I_{\lambda,s}f(z)\right)'} \prec \left(\frac{1+\mathfrak{a}z}{1+\mathfrak{b}z}\right)^{\beta},\ z \in E\right\}.$$

2 Basic Results

Lemma 1. [9] *Let $h(z)$ be analytic univalent in E with $h(0) = 1$ and*

$$Re\{\beta h(z) + \gamma\} > 0, \ \ (\beta, \gamma \in \mathbb{C}).$$

If $p(z)$ is analytic in E with $p(0) = 1$, then

$$p(z) + \frac{zp'(z)}{\beta h(z) + \gamma} \prec h(z)\ , \qquad z \in E,$$

implies that $p(z) \prec h(z)$, $z \in E$.

Lemma 2. [14] *If $\varphi \in C$, $f \in \mathcal{S}^*$ and p is analytic in E with $p(0) = 1$, then*

$$\frac{(\varphi * pf)(z)}{\varphi * f} \subset \overline{CO}p(E), \tag{2.1}$$

where $\overline{CO}$ is the closed convex hull.

3 Main Results

We take $k \in [0, 1]$, $-1 \leq \mathfrak{b} < \mathfrak{a} \leq 1$, $\beta \in (0, 1]$, $\lambda > -1$ and $s \in \mathbb{R}$ throughout the paper unless stated otherwise.

Theorem 1. *Let $f \in \mathcal{C}_g(h)$. Then $f \in \mathcal{S}^*_g(h)$, where $h \in \mathfrak{H}$ and $g \in \mathcal{A}$.*

Proof. Let $f \in \mathcal{C}_g(h)$ and we set

$$\frac{z(g * f)'(z)}{(g * f)(z)} = p(z). \tag{3.1}$$

We note that $p(0) = 1$.

Now, by logarithmic differentiation and simple computation, we have

$$\frac{\left(z(g * f)'(z)\right)}{(g * f)'(z)} = p(z) + \frac{zp'(z)}{p(z)}.$$

By using Lemma 1, we obtain,

$$p(z) \prec h(z), \qquad z \in E. \tag{3.3}$$

Consequently, $f \in \mathcal{S}^*_g(h)$, $z \in E$. □

When we take $h(z)=\left(\frac{1+\mathfrak{a}z}{1+\mathfrak{b}z}\right)^{\beta}$ in Theorem 1, then we have

Corollary 1. *Let $g\in\mathcal{A}$. Then $\mathcal{C}_g(\mathfrak{a},\mathfrak{b};\beta)\subset S_g^*(\mathfrak{a},\mathfrak{b};\beta)$.*

Note that, for $g(z)=\frac{z}{1-z},\ z\in E,$ we get $C(\mathfrak{a},\mathfrak{b};\beta)\subset\mathcal{S}^*(\mathfrak{a},\mathfrak{b};\beta)$. Moreover, if $\mathfrak{a}=1-2\alpha,\ \mathfrak{b}=-1$, then this inclusion reduces to $\mathcal{C}(\alpha)\subset\mathcal{S}^*(\alpha)$. Furthermore, when we take $\alpha=0$, we imply $\mathcal{C}\subset\mathcal{S}^*$.

Again, if we take $h(z)=p_k(z)$ given by (1.2) in Theorem 1, then we have

Corollary 2. *Let $g\in\mathcal{A}$. Then $k-\mathcal{UCV}_g\subset k-\mathcal{ST}_g$.*

Theorem 2. *Let $f\in\mathcal{S}_g^*(h)$ and φ be convex univalent in E. Then for $g\in\mathcal{A}$, $h\in\mathfrak{H}$, $\varphi*f\in\mathcal{S}_g^*(h)$.*

Proof. Consider

$$\begin{aligned}
\frac{z\left(g*(\varphi*f)\right)'(z)}{\left(g*(\varphi*f)\right)(z)} &= \frac{\left(g*z(\varphi*f)'\right)(z)}{\left(\varphi*(g*f)\right)(z)}\\
&= \frac{\left(\varphi*g*zf'\right)(z)}{\left(\varphi*(g*f)\right)(z)}\\
&= \frac{\left(\varphi*z(g*f)'\right)(z)}{\left(\varphi*(g*f)\right)(z)}\\
&= \frac{\left(\varphi*\frac{z(g*f)'}{(g*f)}(g*f)\right)(z)}{\left(\varphi*(g*f)\right)(z)}\\
&= \frac{\left(\varphi*p(g*f)\right)(z)}{\left(\varphi*(g*f)\right)(z)}.
\end{aligned}$$

Since $f\in\mathcal{S}_g^*(h)$, we have $g*f\in\mathcal{S}_g^*(h)$ implies $p(z)=\frac{z(g*f)'}{g*f}\prec h(z)$. Therefore, by using Lemma 2, we conclude

$$\frac{z\left(g*(\varphi*f)'\right)(z)}{\left(g*(\varphi*f)'\right)(z)}\prec h(z).$$

Consequently, $(\varphi*f)\in\mathcal{S}_g^*(h)$, for $z\in E$. □

When we choose $h(z)=\left(\frac{1+\mathfrak{a}z}{1+\mathfrak{b}z}\right)^{\beta}$, we get the following.

Corollary 3. *Let $f \in \mathcal{S}_g^*(\mathfrak{a}, \mathfrak{b}; \beta)$ and φ be convex univalent in E. Then $\varphi * f \in \mathcal{S}_g^*(\mathfrak{a}, \mathfrak{b}; \beta)$, for $g \in \mathcal{A}$.*

Remark 1. For $g(z) = \frac{z}{1-z}$, we can deduce that $S^*[\mathfrak{a}, \mathfrak{b}]$, $\mathcal{S}^*(\alpha)$, $\alpha \in [0, 1)$, and $\mathcal{S}^*$ are also closed under the convex convolution.

If we take $h(z) = p_k(z)$ given by (1.2) in Theorem 2, we get the following.

Corollary 4. *Let $f \in k - \mathcal{ST}_g$ and φ be convex univalent in E. Then, for $g \in \mathcal{A}$, $\varphi * f \in k - \mathcal{ST}_g$.*

Theorem 3. *Let $h \in \mathfrak{H}$. Then $\mathcal{S}^*_{\lambda,s+1}(h) \subset \mathcal{S}^*_{\lambda,s}(h)$.*

Proof. Suppose $f \in \mathcal{S}^*_{\lambda,s+1}(h)$.

We consider

$$\frac{z(I_{\lambda,s}f(z))'}{I_{\lambda,s}f(z)} = p(z), \tag{3.4}$$

where $p(0) - 1 = 0$.

By applying (1.4) and (3.4), we have

$$(\lambda + 1)\frac{I_{\lambda,s+1}f(z)}{I_{\lambda,s}f(z)} = p(z) + \lambda. \tag{3.5}$$

On logarithmic differentiation of (3.5), we get

$$\begin{aligned} \frac{z(I_{\lambda,s+1}f(z))}{I_{\lambda,s+1}f(z)} &= \frac{z(I_{\lambda,s}f(z))'}{I_{\lambda,s}f(z)} + \frac{zp'(z)}{p(z)+\lambda}, \\ &= p(z) + \frac{zp'(z)}{p(z)+\lambda}. \end{aligned}$$

Since $h(z) \in \mathfrak{H}$, and $f \in \mathcal{S}^*_{\lambda,s+1}$ for $z \in E$, we see that

$$Re\{h(z) + \lambda\} > 0, \qquad z \in E,$$

and

$$p(z)+\frac{zp'(z)}{p(z)+\lambda}\prec h(z), \quad z\in E. \tag{3.6}$$

Thus, by Lemma 1, we conclude $p(z)\prec h(z)$, $z\in E$. □

For $h(z)=\left(\frac{1+\mathfrak{a}z}{1+\mathfrak{b}z}\right)^{\beta}$ in Theorem 3, we have

Corollary 5. *Let $f\in\mathcal{S}^*_{\lambda,s+1}(\mathfrak{a},\mathfrak{b};\beta)$. Then $f\in\mathcal{S}^*_{\lambda,s}(\mathfrak{a},\mathfrak{b};\beta)$.*

Note that for $\beta=1$, we have $\mathcal{S}^*_{\lambda,s+1}[\mathfrak{a},\mathfrak{b}]\subset\mathcal{S}^*_{\lambda,s}[\mathfrak{a},\mathfrak{b}]$. Moreover, for $\mathfrak{a}=1-2\alpha$, $\mathfrak{b}=-1$, we get $\mathcal{S}^*_{\lambda,s+1}(\alpha)\subset\mathcal{S}^*_{\lambda,s}(\alpha)$, $\alpha\in[0,1)$, and if $\alpha=0$, then $\mathcal{S}^*_{\lambda,s+1}\subset\mathcal{S}^*_{\lambda,s}$.

Now, for $h(z)=p_k(z)$ given by (1.2) in Theorem 3, we have

Corollary 6. *Let $f\in k-\mathcal{ST}_{\lambda,s+1}$. Then $f\in k-\mathcal{ST}_{\lambda,s}$.*

Theorem 4. *Let $h\in\mathfrak{H}$. Then $\mathcal{C}_{\lambda,s+1}(h)\subset\mathcal{C}_{\lambda,s}(h)$.*

Proof. The proof is immediate. In fact

$$f\in\mathcal{C}_{\lambda,s+1}(h)\Longleftrightarrow zf'\in\mathcal{S}^*_{\lambda,s+1}(h)\Longrightarrow zf'\in\mathcal{S}^*_{\lambda,s}(h)\Longleftrightarrow f\in\mathcal{C}_{\lambda,s}(h).$$

□

On similar arguments as used before we have some special cases as corollaries by choosing $h(z)=\left(\frac{1+\mathfrak{a}z}{1+\mathfrak{b}z}\right)^{\beta}$ and $h(z)=p_k(z)$ in Theorem 4.

Corollary 7. *Let $f\in\mathcal{C}_{\lambda,s+1}(\mathfrak{a},\mathfrak{b};\beta)$. Then $f\in\mathcal{C}_{\lambda,s}(\mathfrak{a},\mathfrak{b};\beta)$.*

Corollary 8. *Let $f\in k-\mathcal{UCV}_{\lambda,s+1}$. Then $f\in k-\mathcal{UCV}_{\lambda,s}$.*

3.1 Radius Problems

Theorem 5. *Let $f\in\mathcal{S}^*_g(\mathfrak{a},\mathfrak{b};\beta)$. Then $f\in\mathcal{C}_g(1,-1;1)$ for $|z|<r_o$, where r_o is the positive root in $(0,1)$ of the following equation*

$$(1-\mathfrak{a}r)^{\beta+1}-\beta(\mathfrak{a}-\mathfrak{b})(1-\mathfrak{b}r)^{\beta-1}=0. \tag{3.7}$$

Proof. Since $f \in \mathcal{S}_g^*(\mathfrak{a}, \mathfrak{b}; \beta)$ implies that

$$\frac{z(g*f)'(z)}{(g*f)(z)} = p(z) \prec \left(\frac{1+\mathfrak{a}z}{1+\mathfrak{b}z}\right)^{\beta}, \text{ for } z \in E. \tag{3.8}$$

By logarithmic differentiation and simple computation, we have

$$\frac{\left(z(g*f)'(z)\right)'}{(g*f)'(z)} = p(z) + \frac{zp'(z)}{p(z)}. \tag{3.9}$$

For $p \in P(\mathfrak{a}, \mathfrak{b}; \beta)$, we can easily write

$$\left(\frac{1-\mathfrak{a}r}{1-\mathfrak{b}r}\right)^{\beta} \leq Re\{p(z)\} \leq |p(z)| \leq \left(\frac{1+\mathfrak{a}r}{1+\mathfrak{b}r}\right)^{\beta}, \tag{3.10}$$

$$Re\left\{\frac{zp'(z)}{p(z)}\right\} \leq \frac{\beta(\mathfrak{a}-\mathfrak{b})r}{(1-\mathfrak{a}r)(1-\mathfrak{b}r)}. \tag{3.11}$$

From equation (3.9)-(3.11), we get

$$\begin{aligned} Re\left\{\frac{\left(z(g*f)'(z)\right)'}{(g*f)'(z)}\right\} &\geq \left(\frac{1-\mathfrak{a}r}{1-\mathfrak{b}r}\right)^{\beta} - \frac{\beta(\mathfrak{a}-\mathfrak{b})r}{(1-\mathfrak{a}r)(1-\mathfrak{b}r)} \\ &= \frac{(1-\mathfrak{a}r)^{\beta+1} - \beta(\mathfrak{a}-\mathfrak{b})(1-\mathfrak{b}r)^{\beta-1}r}{(1-\mathfrak{a}r)(1-\mathfrak{b}r)^{\beta}}. \end{aligned} \tag{3.12}$$

The right hand side is positive if and only if

$$\frac{(1-\mathfrak{a}r)^{\beta+1} - \beta(\mathfrak{a}-\mathfrak{b})(1-\mathfrak{b}r)^{\beta-1}r}{(1-\mathfrak{a}r)(1-\mathfrak{b}r)^{\beta}} \geq 0.$$

Taking $T(r) = (1-\mathfrak{a}r)^{\beta+1} - \beta(\mathfrak{a}-\mathfrak{b})(1-\mathfrak{b}r)^{\beta-1}$. Here, $T(0) > 0$ and $T(1) < 0$, there exists $r_o \in (0,1)$ is least root of the equation given by (3.7). □

Remark 2. When we take $g(z) = \frac{z}{1-z}$, $z \in E$, $\mathfrak{a} = 1$, $\mathfrak{b} = -1$ and $\beta = 1$. Then, we have well-known result $\mathcal{S}^* \subset \mathcal{C}$ for $|z| < r_o = 2 - \sqrt{3}$.

Theorem 6. *Let $f \in \mathcal{S}_{\lambda,s}^*(\mathfrak{a}, \mathfrak{b}; \beta)$. Then $f \in \mathcal{S}_{\lambda,s+1}^*(1, -1; 1)$ for $|z| < r_o$, where r_o is positive root in $(0,1)$ of the equation*

$$\lambda(1-\mathfrak{a}r)(1-\mathfrak{b}r)^{\beta} + (1-\mathfrak{a}r)^2 - \beta(\mathfrak{a}-\mathfrak{b})(1-\mathfrak{b}r)^{\beta-1}r = 0. \tag{3.13}$$

Proof. Suppose $f \in S^{*}_{\lambda,s}(\mathfrak{a},\mathfrak{b};\beta)$. Then, we write

$$\frac{z\left(I_{\lambda,s}f(z)\right)'}{I_{\lambda,s}f(z)} = p(z) \prec \left(\frac{1+\mathfrak{a}z}{1+\mathfrak{b}z}\right)^{\beta}. \tag{3.14}$$

On making use of (1.4) and (3.14), we have

$$(\lambda+1)\frac{I_{\lambda,s+1}f(z)}{I_{\lambda,s}f(z)} = p(z)+\lambda. \tag{3.15}$$

The logarithmic differentiation and simple calculation yield.

$$\frac{z\left(I_{\lambda,s+1}f(z)\right)'}{I_{\lambda,s+1}f(z)} = p(z)+\frac{zp'(z)}{p(z)+\lambda}. \tag{3.16}$$

From (3.10), (3.11) and (3.16), we obtain

$$Re\frac{z\left(I_{\lambda,s+1}f(z)\right)'}{I_{\lambda,s+1}f(z)} \geq (Rep(z))\left(\frac{\lambda(1-\mathfrak{a}r)(1-\mathfrak{b}r)^{\beta}+(1-\mathfrak{a}r)^{2}-\beta(\mathfrak{a}-\mathfrak{b})(1-\mathfrak{b}r)^{\beta-1}r}{\lambda(1-\mathfrak{a}r)(1-\mathfrak{b}r)^{\beta}+(1-\mathfrak{a}r)^{\beta+1}}\right). \tag{3.17}$$

The right hand side is positive if and only if

$$\left(\frac{\lambda(1-\mathfrak{a}r)(1-\mathfrak{b}r)^{\beta}+(1-\mathfrak{a}r)^{2}-\beta(\mathfrak{a}-\mathfrak{b})(1-\mathfrak{a}r)^{\beta-1}r}{\lambda(1-\mathfrak{a}r)(1-\mathfrak{b}r)^{\beta}+(1-\mathfrak{a}r)^{\beta+1}}\right) \geq 0.$$

Taking $T(r) = \lambda(1-\mathfrak{a}r)(1-\mathfrak{b}r)^{\beta}+(1-\mathfrak{a}r)^{\beta+1}-\beta(\mathfrak{a}-\mathfrak{b})(1-\mathfrak{b}r)^{\beta-1}r$. Here $T(0)>0$ and $T(1)<0$, then there exists $r_o \in (0,1)$ is the least root of the equation given by (3.13). □

Theorem 7. *Let $f \in \mathcal{C}_{\lambda,s}(\mathfrak{a},\mathfrak{b};\beta)$. Then $f \in \mathcal{C}_{\lambda,s+1}(1,-1;1)$ for $|z|<r_o$, where r_o is least positive root of the equation* (3.13).

Proof. Assume that

$$\begin{aligned}
f &\in \mathcal{C}_{\lambda,s}\left(\mathfrak{a},\mathfrak{b};\beta\right), \qquad z\in E. \\
&\Longleftrightarrow I_{\lambda,s}f\in\mathcal{C}\left(\mathfrak{a},\mathfrak{b};\beta\right) \\
&\Longleftrightarrow z\left(I_{\lambda,s}f\right)'\in\mathcal{S}^*\left(\mathfrak{a},\mathfrak{b};\beta\right) \\
&\Longleftrightarrow I_{\lambda,s}\left(zf'\right)\in\mathcal{S}^*\left(\mathfrak{a},\mathfrak{b};\beta\right) \\
&\Longleftrightarrow zf'\in\mathcal{S}^*_{\lambda,s}\left(\mathfrak{a},\mathfrak{b};\beta\right) \\
&\Longrightarrow zf'\in\mathcal{S}^*_{\lambda,s+1}\left(1,-1;1\right)\text{ in } |z|<r_o \\
&\Longleftrightarrow I_{\lambda,s+1}\left(zf'\right)\in\mathcal{S}^*\left(1,-1;1\right) \\
&\Longleftrightarrow z\left(I_{\lambda,s+1}f\right)'\in\mathcal{S}^*\left(1,-1;1\right) \\
&\Longleftrightarrow I_{\lambda,s+1}f\in\mathcal{C}\left(1,-1;1\right) \\
&\Longleftrightarrow f\in\mathcal{C}_{\lambda,s+1}\left(1,-1;1\right)\text{ in } |z|<r_o.
\end{aligned}$$

□

References

[1] N. E. Cho and J. A. Kim, Inclusion properties of certain subclasses of analytic functions defined by a multiplier transformation, *Comput. Math. Appl.* 52 (2006), 323-330. `https://doi.org/10.1016/j.camwa.2006.08.022`

[2] W. Janowski, Some extremal problems for certain families of analytic functions I, *Ann. Polon. Math.* 28 (1973), 297-326.
`https://doi.org/10.4064/ap-28-3-297-326`

[3] S. Kanas, Alternative characterization of the class *k-UCV* and related classes of univalent functions, *Serdica Math. J.* 25 (1999), 341-350.

[4] S. Kanas and H. M. Srivastava, Linear operators associated with k-uniformly convex functions, *Integral Transforms Spec. Funct.* 9 (2000), 121-132.
`https://doi.org/10.1080/10652460008819249`

[5] S. Kanas and W. Wisniowska, Conic regions and k-uniform convexity, *J. Comput. Appl. Math.* 105 (1999), 327-336.
`https://doi.org/10.1016/S0377-0427(99)00018-7`

[6] S. Kanas and W. Wisniowska, Conic domains and k-starlike functions, *Rev. Roumaine Math. Pures Appl.* 45 (2000), 647-657.

[7] J.-L. Liu, The Noor integral and strongly starlike functions, *J. Math. Anal. Appl.* 261 (2001), 441-447. https://doi.org/10.1006/jmaa.2001.7489

[8] J.-L. Liu and K. I. Noor, Some properties of Noor integral operator, *J. Nat. Geomet.* 21 (2002), 81-90.

[9] S. S. Miller and P. T. Mocanu, Differential subordination. Theory and applications, *Monographs and Textbooks in Pure and Applied Mathematics*, 225, Marcel Dekker Inc., New York, Basel, 2000. https://doi.org/10.1201/9781482289817

[10] K. I. Noor and S. N Malik, On a subclass of starlike univalent functions, *Middle-East J. Sci. Research* 7 (2011), 769-777.

[11] K. I. Noor and M. A. Noor, On integral operators, *J. Math. Anal. Appl.* 238 (1999), 341-352. https://doi.org/10.1006/jmaa.1999.6501

[12] Y. Polatoğlu, M. Bolcal, A. Şen and E. Yavuz, A study on the generalization of Janowski functions in the unit disc, *Acta Math. Acad. Paedagog. Nyházi. (N.S.)* 22 (2006), 27-31.

[13] M. S. Robertson, On the theory of univalent functions, *Ann. Math.* 37 (1936), 374-408. https://doi.org/10.2307/1968451

[14] S. Ruscheweyh and T. Sheil-small, Hadamard products of Schlicht functions and the Pólya-Schoenberg conjecture, *Comment. Math. Helv.* 48 (1973), 119-135. https://doi.org/10.1007/BF02566116

[15] D. Raducanu and H. M. Srivastava, A new class of analytic functions defined by means of convolution operator involving the Hurwitz-Lerch Zeta function, *Integral Transforms Spec. Funct.* 18 (2007), 933-943. https://doi.org/10.1080/10652460701542074