Some Inclusion and Radius Problems of Certain Subclasses of Analytic Functions

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Abstract

This article presents the study of certain subclasses of analytic functions defined by using the Hadamard product. We derive certain inclusion results and discuss the applications of multiplier transformation. Several radius problems are also investigated.

1 Introduction

Let \( \mathcal{A} \) denote the class of normalized analytic functions in \( E = \{ z \in \mathbb{C} : |z| < 1 \} \), of the series representation

\[
 f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}
\]

The class \( \mathcal{S} \subset \mathcal{A} \), represents the class of univalent functions in \( E \). We denote \( \mathcal{S}^* \) and \( \mathcal{C} \) be the classes of starlike and convex univalent functions in \( E \), respectively.

Let \( f \) and \( g \) be the analytic functions in \( E \), we say \( f \) is subordinate to \( g \) (written

Received: June 14, 2020; Accepted: July 18, 2020

2010 Mathematics Subject Classification: 30C45, 30C55.

Keywords and phrases: analytic functions, Janowski functions, conic region, multiplier transformation.

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as \( f \prec g \) if and only if there exists a Schwartz function \( w(z) \) (that is, \( w(0) = 0 \) and \( |w(z)| < 1 \)) in \( E \) such that

\[
f(z) = g(w(z)).
\]

Particularly, if \( g(z) \) is univalent function in \( E \), then \( f \prec g \) is equivalent to

\[
f(0) = g(0) \text{ and } f(E) \subset g(E).
\]

The convolution of two power series \( f \) and \( g \) in \( E \) denoted by \(*\), and is defined as follows,

\[
(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ (} z \in E \text{)}.
\]

We consider \( \mathcal{S} \) be the class of analytic univalent functions \( h(z) \) in \( E \) with \( h(0) = 1 \) and \( Re \{h(z)\} > 0, \ (z \in E) \).

Now, we define the following.

**Definition 1.** Let \( f, g \in A \) with \( (f \ast g)(z) \neq 0 \ (z \in E) \). Then \( f \in \mathcal{S}_g^*(h) \) if and only if

\[
\frac{z(g \ast f)'(z)}{(g \ast f)(z)} \prec h(z), \ z \in E.
\]

Analogously,

\[
\mathcal{C}_g(h) = \{ f \in A : zf' \in \mathcal{S}_g^*(h) \}.
\]

Obviously, for the particular choices of functions \( g \) and \( h \), we have specific subclasses of \( \mathcal{S} \).

1. Let \( h(z) = p_k(z), k \in [0,1] \), where \( p_k(z) \) is convex univalent in \( E \) and has the form

\[
p_k(z) = \begin{cases} \frac{1+z}{1-z}, & k = 0, \\ 1 + \frac{2}{\pi^2} \left( \log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2, & k = 1, \\ 1 + \frac{2}{1-k^2} \left[ \left( \frac{\pi}{2} \arccos k \right) \arctan \sqrt{z} \right], & 0 < k < 1. \end{cases}
\]
We denote
\[ k-ST_g = \left\{ f \in A : \frac{z(g \ast f)'(z)}{(g \ast f)(z)} < p_k(z), \ g \in A, \ z \in E \right\}, \]
\[ k-UCV_g = \left\{ f \in A : \frac{(z(g \ast f)'(z))'}{(g \ast f)'(z)} < p_k(z), \ g \in A, \ z \in E \right\}. \]

Note that if \( g(z) = \frac{z}{1-z} \), \( z \in E \), we obtain the well-known classes \( k-UCV \) and \( k-ST \) of \( k \)-uniformly convex and corresponding starlike functions respectively, introduced and studied by Kanas et al. \[3, 4\]. Also, we refer to \[5, 6\].

2. Let \( h(z) = \left( \frac{1+az}{1+bz} \right)^\beta \), \(-1 \leq b < a \leq 1, 0 < \beta \leq 1\), be convex univalent in \( E \) and the series representation of \( h(z) \) be as follows:
\[ p(z) = 1 + \beta (a-b) z + \left[ -\beta (a-b)^2 + \frac{1}{2} \beta (\beta - 1) (a-b)^2 \right] z^2 + \ldots. \]

We denote
\[ S^*_g(a, b; \beta) = \left\{ f \in A : \frac{z(g \ast f)'(z)}{(g \ast f)(z)} < \left( \frac{1+az}{1+bz} \right)^\beta, \ g \in A, \ z \in E \right\}, \]
\[ C_g(a, b; \beta) = \left\{ f \in A : \frac{(z(g \ast f)'(z))'}{(g \ast f)'(z)} < \left( \frac{1+az}{1+bz} \right)^\beta, \ g \in A, \ z \in E \right\}. \]

It is noted that, if \( g(z) = \frac{z}{1-z} \), \( z \in E \), we have \( S^*(a, b; \beta) \) and \( C(a, b; \beta) \), respectively. Moreover, if \( \beta = 1 \), then it reduces to the well-known classes \( S^* [a, b] \) and \( C [a, b] \), respectively, we refer to \[2, 10, 12\]. Furthermore, if \( a = 1 - 2\alpha \) and \( b = -1 \), we get \( S^*(\alpha) \) and \( C(\alpha) \), (see \[13\]), respectively. For \( \alpha = 0 \), we have \( S^* \) and \( C \).

The multiplier transformation \( I_{\lambda,s} : A \rightarrow A \) is defined as follows \[1\]:
\[ I_{\lambda,s} f(z) = z + \sum_{n=2}^{\infty} \left( \frac{n+\lambda}{1+\lambda} \right)^s a_n z^n \quad (\lambda > -1, s \in \mathbb{R}). \]
Clearly, $I_{\lambda,s} (I_{\lambda,t} f(z)) = I_{\lambda,s+t} f(z)$, for $(s,t \in \mathbb{R})$. For different values of $s$ and $\lambda$, the operator $I_{\lambda,s}$ has been studied by several authors [7, 8, 11, 15].

From equation (1.3), we can easily have the following identity,

$$z(I_{\lambda,s} f(z))' = (\lambda + 1) I_{\lambda,s+1} f(z) - \lambda I_{\lambda,s} f(z). \quad (1.4)$$

We now define the following by taking the value of $g = I_{\lambda,s}$ in the Definition 1.

$$S^x_{\lambda,s}(h) = \left\{ f \in A : \frac{z(I_{\lambda,s} f(z))'}{I_{\lambda,s} f(z)} \prec h(z), \ z \in E \right\},$$

$$C_{\lambda,s}(h) = \left\{ f \in A : \frac{\left(\frac{z(I_{\lambda,s} f(z))'}{I_{\lambda,s} f(z)}\right)'}{I_{\lambda,s} f(z)} \prec h(z), \ z \in E \right\}.$$

Particularly, for $h(z) = p_k(z)$ given by (1.2), we have

$$k-\mathcal{ST}_g = \left\{ f \in A : \frac{z(I_{\lambda,s} f(z))'}{I_{\lambda,s} f(z)} \prec p_k(z), \ g \in A, \ z \in E \right\},$$

$$k-\mathcal{UCV}_g = \left\{ f \in A : \frac{\left(\frac{z(I_{\lambda,s} f(z))'}{I_{\lambda,s} f(z)}\right)'}{I_{\lambda,s} f(z)} \prec p_k(z), \ g \in A, \ z \in E \right\}.$$

And, for $h(z) = \left(\frac{1+az}{1+bz}\right)^\beta, -1 \leq b < a \leq 1, 0 < \beta \leq 1$, we define

$$S^x_{\lambda,s}(a, b; \beta) = \left\{ f \in A : \frac{z(I_{\lambda,s} f(z))'}{I_{\lambda,s} f(z)} \prec \left(\frac{1+az}{1+bz}\right)^\beta, \ z \in E \right\},$$

$$C_{\lambda,s}(a, b; \beta) = \left\{ f \in A : \frac{\left(\frac{z(I_{\lambda,s} f(z))'}{I_{\lambda,s} f(z)}\right)'}{I_{\lambda,s} f(z)} \prec \left(\frac{1+az}{1+bz}\right)^\beta, \ z \in E \right\}.$$
2 Basic Results

Lemma 1. [9] Let \( h(z) \) be analytic univalent in \( E \) with \( h(0) = 1 \) and
\[
\Re \{ \beta h(z) + \gamma \} > 0, \quad (\beta, \gamma \in \mathbb{C}).
\]
If \( p(z) \) is analytic in \( E \) with \( p(0) = 1 \), then
\[
p(z) + \frac{zp'(z)}{\beta h(z) + \gamma} < h(z), \quad z \in E,
\]
implies that \( p(z) < h(z), \ z \in E \).

Lemma 2. [14] If \( \varphi \in C, \ f \in S^* \) and \( p \) is analytic in \( E \) with \( p(0) = 1 \), then
\[
\left( \frac{\varphi * pf}{\varphi * f} \right) (z) \subset \overline{CO} p(E), \quad \text{(2.1)}
\]
where \( \overline{CO} \) is the closed convex hull.

3 Main Results

We take \( k \in [0, 1], \ -1 \leq b < a \leq 1, \ \beta \in (0, 1], \ \lambda > -1 \) and \( s \in \mathbb{R} \) throughout the paper unless stated otherwise.

Theorem 1. Let \( f \in C_g(h) \). Then \( f \in S^*_g(h) \), where \( h \in H \) and \( g \in \mathcal{A} \).

Proof. Let \( f \in C_g(h) \) and we set
\[
\frac{z (g * f)' (z)}{(g * f) (z)} = p(z). \quad \text{(3.1)}
\]
We note that \( p(0) = 1 \).

Now, by logarithmic differentiation and simple computation, we have
\[
\frac{\left( z (g * f)' (z) \right)}{(g * f)' (z)} = p(z) + \frac{zp'(z)}{p(z)}.
\]
By using Lemma 1, we obtain,
\[
p(z) < h(z), \quad z \in E. \quad \text{(3.3)}
\]
Consequently, \( f \in S^*_g(h), \ z \in E \).
When we take $h(z) = \left(\frac{1+az}{1+bz}\right)^\beta$ in Theorem 1, then we have

**Corollary 1.** Let $g \in \mathcal{A}$. Then $C_g(a, b; \beta) \subset S_g^*(a, b; \beta)$.

Note that, for $g(z) = \frac{z}{1-z}$, $z \in E$, we get $C(a, b; \beta) \subset S^*(a, b; \beta)$. Moreover, if $a = 1-2\alpha$, $b = -1$, then this inclusion reduces to $C(\alpha) \subset S^*(\alpha)$. Furthermore, when we take $\alpha = 0$, we imply $C \subset S^*$.

Again, if we take $h(z) = p_k(z)$ given by (1.2) in Theorem 1, then we have

**Corollary 2.** Let $g \in \mathcal{A}$. Then $k-UCV_g \subset k-ST_g$.

**Theorem 2.** Let $f \in S_g^*(h)$ and $\varphi$ be convex univalent in $E$. Then for $g \in \mathcal{A}$, $h \in \mathcal{F}$, $\varphi \ast f \in S_g^*(h)$.

**Proof.** Consider

$$
\frac{z (g \ast (\varphi \ast f))'(z)}{(g \ast (\varphi \ast f))(z)} = \frac{(g \ast (\varphi \ast f))(z)'}{(\varphi \ast (g \ast f))(z)} = \frac{(\varphi \ast g \ast z')(z)}{(\varphi \ast (g \ast f))(z)} = \frac{(\varphi \ast (g \ast f)'(z))}{(\varphi \ast (g \ast f))(z)} = \frac{\left(\varphi \ast \frac{z(g \ast f)'}{(g \ast f)}\ast (g \ast f)\right)(z)}{(\varphi \ast (g \ast f))(z)} = \frac{(\varphi \ast p(g \ast f))(z)}{(\varphi \ast (g \ast f))(z)}.
$$

Since $f \in S_g^*(h)$, we have $g \ast f \in S_g^*(h)$ implies $p(z) = \frac{z(g \ast f)'}{g \ast f} \prec h(z)$. Therefore, by using Lemma 2, we conclude

$$
\frac{z (g \ast (\varphi \ast f))'(z)}{(g \ast (\varphi \ast f))'(z)} \prec h(z).
$$

Consequently, $(\varphi \ast f) \in S_g^*(h)$, for $z \in E$. \qed

When we choose $h(z) = \left(\frac{1+az}{1+bz}\right)^\beta$, we get the following.

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**Corollary 3.** Let \( f \in S^*_g(a, b; \beta) \) and \( \varphi \) be convex univalent in \( E \). Then \( \varphi \ast f \in S^*_g(a, b; \beta) \), for \( g \in \mathcal{A} \).

**Remark 1.** For \( g(z) = \frac{z}{1-z} \), we can deduce that \( S^*[a, b], S^*(\alpha), \alpha \in [0, 1) \), and \( S^* \) are also closed under the convex convolution.

If we take \( h(z) = p_k(z) \) given by (1.2) in Theorem 2, we get the following.

**Corollary 4.** Let \( f \in k-ST_g \) and \( \varphi \) be convex univalent in \( E \). Then, for \( g \in \mathcal{A} \), \( \varphi \ast f \in k-ST_g \).

**Theorem 3.** Let \( h \in \mathcal{H} \). Then \( S^*_{\lambda,s+1}(h) \subset S^*_{\lambda,s}(h) \).

**Proof.** Suppose \( f \in S^*_{\lambda,s+1}(h) \).

We consider

\[
\frac{z(I_{\lambda,s}f(z))'}{I_{\lambda,s}f(z)} = p(z),
\]

where \( p(0) - 1 = 0 \).

By applying (1.4) and (3.4), we have

\[
(\lambda + 1) \frac{I_{\lambda,s+1}f(z)}{I_{\lambda,s}f(z)} = p(z) + \lambda.
\]

On logarithmic differentiation of (3.5), we get

\[
\frac{z(I_{\lambda,s+1}f(z))}{I_{\lambda,s+1}f(z)} = \frac{z(I_{\lambda,s}f(z))'}{I_{\lambda,s}f(z)} + \frac{zp'(z)}{p(z) + \lambda},
\]

\[
= p(z) + \frac{zp'(z)}{p(z) + \lambda}.
\]

Since \( h(z) \in \mathcal{H} \), and \( f \in S^*_{\lambda,s+1} \) for \( z \in E \), we see that

\[
\text{Re} \{h(z) + \lambda\} > 0, \quad z \in E,
\]
and

\[ p(z) + \frac{z p'(z)}{p(z) + \lambda} < h(z), \quad z \in E. \quad (3.6) \]

Thus, by Lemma 1, we conclude \( p(z) < h(z), z \in E. \)

For \( h(z) = \left( \frac{1+a}{1+bz} \right)^\beta \) in Theorem 3, we have

**Corollary 5.** Let \( f \in S_{\lambda,s+1}^* (a, b; \beta). \) Then \( f \in S_{\lambda,s}^* (a, b; \beta). \)

Note that for \( \beta = 1, \) we have \( S_{\lambda,s+1}^* [a, b] \subset S_{\lambda,s}^* [a, b]. \) Moreover, for \( a = 1 - 2\alpha, \ b = -1, \) we get \( S_{\lambda,s+1}^* (\alpha) \subset S_{\lambda,s}^* (\alpha), \ \alpha \in [0, 1], \) and if \( \alpha = 0, \) then \( S_{\lambda,s+1}^* \subset S_{\lambda,s}^*. \)

Now, for \( h(z) = p_k(z) \) given by (1.2) in Theorem 3, we have

**Corollary 6.** Let \( f \in k - ST_{\lambda,s+1}. \) Then \( f \in k - ST_{\lambda,s}. \)

**Theorem 4.** Let \( h \in S_1. \) Then \( C_{\lambda,s+1}(h) \subset C_{\lambda,s}(h). \)

**Proof.** The proof is immediate. In fact

\[ f \in C_{\lambda,s+1}(h) \iff zf' \in S_{\lambda,s+1}^* (h) \iff zf' \in S_{\lambda,s}^* (h) \iff f \in C_{\lambda,s}(h). \]

On similar arguments as used before we have some special cases as corollaries by choosing \( h(z) = \left( \frac{1+a}{1+bz} \right)^\beta \) and \( h(z) = p_k(z) \) in Theorem 4.

**Corollary 7.** Let \( f \in C_{\lambda,s+1} (a, b; \beta). \) Then \( f \in C_{\lambda,s} (a, b; \beta). \)

**Corollary 8.** Let \( f \in k - UCV_{\lambda,s+1}. \) Then \( f \in k - UCV_{\lambda,s}. \)

### 3.1 Radius Problems

**Theorem 5.** Let \( f \in S_g^* (a, b; \beta). \) Then \( f \in C_g (1, -1; 1) \) for \( |z| < r_o, \) where \( r_o \) is the positive root in \((0, 1)\) of the following equation

\[ (1-ar)^\beta+1 - \beta (a-b) (1-br)^\beta-1 = 0. \quad (3.7) \]
Proof. Since \( f \in S^*_g (a, b; \beta) \) implies that

\[
\frac{z \left( (g * f)' (z) \right)}{(g * f) (z)} = p (z) \prec \left( \frac{1 + az}{1 + bz} \right)^\beta, \text{ for } z \in E. \tag{3.8}
\]

By logarithmic differentiation and simple computation, we have

\[
\frac{\left( z \left( (g * f)' (z) \right) \right)'}{(g * f)' (z)} = p (z) + \frac{zp' (z) \cdot p (z)}{p (z)}. \tag{3.9}
\]

For \( p \in P (a, b; \beta) \), we can easily write

\[
\left( \frac{1 - ar}{1 - br} \right)^\beta \leq \text{Re} \left\{ p (z) \right\} \leq |p (z)| \leq \left( \frac{1 + ar}{1 + br} \right)^\beta, \tag{3.10}
\]

\[
\text{Re} \left\{ \frac{zp' (z)}{p (z)} \right\} \leq \frac{\beta (a - b) r}{(1 - ar)(1 - br)}. \tag{3.11}
\]

From equation (3.9)-(3.11), we get

\[
\text{Re} \left\{ \frac{\left( z \left( (g * f)' (z) \right) \right)'}{(g * f)' (z)} \right\} \geq \left( \frac{1 - ar}{1 - br} \right)^\beta - \frac{\beta (a - b) r}{(1 - ar)(1 - br)}
\]

\[
= \frac{(1 - ar)^{\beta + 1} - \beta (a - b)(1 - br)^{\beta - 1} r}{(1 - ar)(1 - br)^\beta}. \tag{3.12}
\]

The right hand side is positive if and only if

\[
\frac{(1 - ar)^{\beta + 1} - \beta (a - b)(1 - br)^{\beta - 1} r}{(1 - ar)(1 - br)^\beta} \geq 0.
\]

Taking \( T (r) = (1 - ar)^{\beta + 1} - \beta (a - b)(1 - br)^{\beta - 1} \). Here, \( T (0) > 0 \) and \( T (1) < 0 \), there exists \( r_o \in (0, 1) \) is least root of the equation given by (3.7). \( \square \)

Remark 2. When we take \( g (z) = \frac{z}{1 - z}, z \in E, a = 1, b = -1 \) and \( \beta = 1 \). Then, we have well-known result \( S^*_r \subset C \) for \( |z| < r_o = 2 - \sqrt{3}. \)

Theorem 6. Let \( f \in S^*_g \lambda, s (a, b; \beta) \). Then \( f \in S^* \lambda, s+1 (1, -1; 1) \) for \( |z| < r_o \), where \( r_o \) is positive root in \( (0, 1) \) of the equation

\[
\lambda (1 - ar)(1 - br)^\beta + (1 - ar)^2 - \beta (a - b)(1 - br)^{\beta - 1} r = 0. \tag{3.13}
\]
**Proof.** Suppose \( f \in S^*_\lambda,\beta \). Then, we write

\[
\frac{z (I_{\lambda,\beta} f (z))'}{I_{\lambda,\beta} f (z)} = p (z) < \left( \frac{1 + a z}{1 + b z} \right)^\beta. \tag{3.14}
\]

On making use of (1.4) and (3.14), we have

\[
(\lambda + 1) \frac{I_{\lambda,\beta+1} f (z)}{I_{\lambda,\beta} f (z)} = p (z) + \lambda. \tag{3.15}
\]

The logarithmic differentiation and simple calculation yield.

\[
\frac{z (I_{\lambda,\beta+1} f (z))'}{I_{\lambda,\beta+1} f (z)} = p (z) + \frac{zp' (z)}{p (z) + \lambda}. \tag{3.16}
\]

From (3.10), (3.11) and (3.16), we obtain

\[
\text{Re} \frac{z (I_{\lambda,\beta+1} f (z))'}{I_{\lambda,\beta+1} f (z)} \geq \text{Re} (p (z)) \left( \frac{\lambda (1 - ar) (1 - br)^\beta + (1 - ar)^2 - \beta (a - b) (1 - br)^{\beta-1} r}{\lambda (1 - ar) (1 - br)^\beta + (1 - ar)^{\beta+1}} \right). \tag{3.17}
\]

The right hand side is positive if and only if

\[
\left( \frac{\lambda (1 - ar) (1 - br)^\beta + (1 - ar)^2 - \beta (a - b) (1 - ar)^{\beta-1} r}{\lambda (1 - ar) (1 - br)^\beta + (1 - ar)^{\beta+1}} \right) \geq 0.
\]

Taking \( T (r) = \lambda (1 - ar) (1 - br)^\beta + (1 - ar)^{\beta+1} - \beta (a - b) (1 - br)^{\beta-1} r \). Here \( T (0) > 0 \) and \( T (1) < 0 \), then there exists \( r_o \in (0, 1) \) is the least root of the equation given by (3.13).

**Theorem 7.** Let \( f \in C_{\lambda,\beta} (a, b; \beta) \). Then \( f \in C_{\lambda,\beta+1} (1, -1; 1) \) for \( |z| < r_o \), where \( r_o \) is least positive root of the equation (3.13).

**Proof.** Assume that
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\[ f \in C_{\lambda,s}(a,b;\beta), \quad z \in E. \]
\[ \iff \quad I_{\lambda,s}f \in C(a,b;\beta) \]
\[ \iff \quad z (I_{\lambda,s}f)' \in S^*(a,b;\beta) \]
\[ \iff \quad I_{\lambda,s}(zf') \in S^*(a,b;\beta) \]
\[ \iff \quad zf' \in S_{\lambda,s}^*(a,b;\beta) \]
\[ \iff \quad zf' \in S_{\lambda,s}^*(1,-1;1) \text{ in } |z| < r_o \]
\[ \iff \quad I_{\lambda,s+1}(zf') \in S^*(1,-1;1) \]
\[ \iff \quad z (I_{\lambda,s+1}f)' \in S^*(1,-1;1) \]
\[ \iff \quad I_{\lambda,s+1}f \in \mathcal{C}(1,-1;1) \]
\[ \iff \quad f \in C_{\lambda,s+1}(1,-1;1) \text{ in } |z| < r_o. \]

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Earthline J. Math. Sci. Vol. 5 No. 1 (2021), 75-86


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