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$(ƒ^* - Ꝏ)$ Quasi Binormal Operator of Order ŋ

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Abstract

In this paper, we introduce a new class of operators on Hilbert space called $(ƒ^* - Ꝏ)$ quasi binormal operator of order ŋ. We study this operator and give some of its properties.

Introduction

Consider $B(H)$ the algebra of all bounded linear operators on Hilbert space H. An operator Ч is called normal if $Ч^*Ч = ЧЧ^*$. Quasi normal operator was introduced by Brown in 1953 [1]. In [3] Campbell introduced the class of binormal operators which is defined as $Ч^*ЧЧЧ^* = ЧЧ^*Ч^*Ч$. In [5] Kuffi and Satar generalized a normal operator.

In this paper, we define a new class of operators on Hilbert space as $(Ч^*)^{ƒ}(Ч^*ЧЧЧ^*)^{ŋ} = Ꝏ(Ч^*ЧЧЧ^*)^{ŋ}(Ч^*)^{ƒ}$ called $(ƒ^* - Ꝏ)$ quasi binormal operator of order ŋ, where Ꝏ is bounded operator, ƒ and ŋ are nonnegative integers, and study some of its properties.

1. Main Results

Definition 1.1. Let Ч be a bounded operator on Hilbert space H. Then Ч is called $(ƒ^* - Ꝏ)$ *quasi binormal operator of order* ŋ if and only if $(Ч^*)^{ƒ}(Ч^*ЧЧЧ^*)^{ŋ} = Ꝏ(Ч^*ЧЧЧ^*)^{ŋ}(Ч^*)^{ƒ}$, where Ꝏ is bounded operator, ƒ and ŋ are nonnegative integers.

Theorem 1.2. *Let Ч be a $(ƒ^* - Ꝏ)$ quasi binormal operator of order ŋ on Hilbert space H. Then $Ч^k$ is a $(ƒ^* - Ꝏ)$ quasi binormal operator of order ŋ.*

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Proof. Let Ч be a $(\mathfrak{f}^* - \mathbb{Q})$ quasi binormal operator of order $\mathfrak{y}$.

By mathematical induction: Since Ч be a $(\mathfrak{f}^* - \mathbb{Q})$ quasi binormal operator of order $\mathfrak{y}$.

The result is true for $k = 1$

$$(Ч^*)^{\mathfrak{f}}(Ч^*ЧЧЧ^*)^{\mathfrak{y}} = \mathbb{Q}(Ч^*ЧЧЧ^*)^{\mathfrak{y}}(Ч^*)^{\mathfrak{f}}. \quad (1)$$

Assume that the result is true when $(k = z)$

$$[(Ч^*)^{\mathfrak{f}}(Ч^*ЧЧЧ^*)^{\mathfrak{y}}]^z = [\mathbb{Q}(Ч^*ЧЧЧ^*)^{\mathfrak{y}}(Ч^*)^{\mathfrak{f}}\,]^z. \quad (2)$$

We prove the result for $k = z + 1$

$$[(Ч^*)^{\mathfrak{f}}(Ч^*ЧЧЧ^*)^{\mathfrak{y}}]^{z+1} = \quad [(Ч^*)^{\mathfrak{f}}(Ч^*ЧЧЧ^*)^{\mathfrak{y}}]^z \quad [(Ч^*)^{\mathfrak{f}}(Ч^*ЧЧЧ^*)^{\mathfrak{y}}]$$

$$= [\mathbb{Q}(Ч^*ЧЧЧ^*)^{\mathfrak{y}}(Ч^*)^{\mathfrak{f}}\,]^z \quad [\mathbb{Q}(Ч^*ЧЧЧ^*)^{\mathfrak{y}}(Ч^*)^{\mathfrak{f}}\,]$$

$$= [\mathbb{Q}(Ч^*ЧЧЧ^*)^{\mathfrak{y}}(Ч^*)^{\mathfrak{f}}\,]^{z+1}.$$

Then the result is true for $k = z + 1$.

Hence, $Ч^k$ is a $(\mathfrak{f}^* - \mathbb{Q})$ quasi binormal operator of order $\mathfrak{y}$.

Theorem 1.3. *Let* Ч *and* $\mathfrak{F}$ *be two* $(\mathfrak{f}^* - \mathbb{Q})$ *binormal operators of order* $\mathfrak{y}$ *on Hilbert space H such that* $Ч^*\mathfrak{F} = Ч\mathfrak{F}^* = Ч^*\mathfrak{F}^* = Ч\mathfrak{F} = 0$. *Then* $Ч + \mathfrak{F}$ *is a* $(\mathfrak{f}^* - \mathbb{Q})$ *quasi binormal operator of order* $\mathfrak{y}$.

Proof. Let Ч and $\mathfrak{F}$ be two $(\mathfrak{f}^* - \mathbb{Q})$ quasi binormal operators of order $\mathfrak{y}$. Then

$$((Ч + \mathfrak{F})^*)^{\mathfrak{f}}[((Ч + \mathfrak{F})^*(Ч + \mathfrak{F})(Ч + \mathfrak{F})(Ч + \mathfrak{F})^*)]^{\mathfrak{y}}$$

$$= [(Ч^*)^{\mathfrak{f}} + \mathfrak{f}(Ч^*)^{\mathfrak{f}-1}\mathfrak{F}^* + \cdots + (\mathfrak{F}^*)^{\mathfrak{f}}]\,[((Ч + \mathfrak{F})^*)^{\mathfrak{y}}(Ч + \mathfrak{F})^{\mathfrak{y}}(Ч + \mathfrak{F})^{\mathfrak{y}}((Ч + \mathfrak{F})^*)^{\mathfrak{y}}]$$

$$= [(Ч^*)^{\mathfrak{f}} + (\mathfrak{F}^*)^{\mathfrak{f}}]\,[((Ч^*)^{\mathfrak{y}} + (\mathfrak{F}^*)^{\mathfrak{y}})(Ч^{\mathfrak{y}} + \mathfrak{F}^{\mathfrak{y}}) \quad (Ч^{\mathfrak{y}} + \mathfrak{F}^{\mathfrak{y}}) \quad ((Ч^*)^{\mathfrak{y}} + (\mathfrak{F}^*)^{\mathfrak{y}})]$$

$$= [(Ч^*)^{\mathfrak{f}} + (\mathfrak{F}^*)^{\mathfrak{f}}]\,[[(Ч^*)^{\mathfrak{y}}Ч^{\mathfrak{y}} + (Ч^*)^{\mathfrak{y}}\mathfrak{F}^{\mathfrak{y}} + (\mathfrak{F}^*)^{\mathfrak{y}}Ч^{\mathfrak{y}} + (\mathfrak{F}^*)^{\mathfrak{y}}\mathfrak{F}^{\mathfrak{y}})][(Ч^{\mathfrak{y}}(Ч^*)^{\mathfrak{y}} + Ч^{\mathfrak{y}}(\mathfrak{F}^*)^{\mathfrak{y}} + \mathfrak{F}^{\mathfrak{y}}(Ч^*)^{\mathfrak{y}} + \mathfrak{F}^{\mathfrak{y}}(\mathfrak{F}^*)^{\mathfrak{y}})]$$

$$= [(Ч^*)^{\mathfrak{f}} + (\mathfrak{F}^*)^{\mathfrak{f}}]\,[[(Ч^*)^{\mathfrak{y}}Ч^{\mathfrak{y}} + (\mathfrak{F}^*)^{\mathfrak{y}}\mathfrak{F}^{\mathfrak{y}})][(Ч^{\mathfrak{y}}(Ч^*)^{\mathfrak{y}} + \mathfrak{F}^{\mathfrak{y}}(\mathfrak{F}^*)^{\mathfrak{y}})]$$

$$= [(Ч^*)^{\mathfrak{f}} + (\mathfrak{F}^*)^{\mathfrak{f}}]\,[[(Ч^*)^{\mathfrak{y}}Ч^{\mathfrak{y}}Ч^{\mathfrak{y}}(Ч^*)^{\mathfrak{y}} + ((\mathfrak{F}^*)^{\mathfrak{y}}\mathfrak{F}^{\mathfrak{y}}\mathfrak{F}^{\mathfrak{y}}(\mathfrak{F}^*)^{\mathfrak{y}})]$$

$$= [(Ч^*)^{\mathfrak{f}}[(Ч^*)^{\mathfrak{y}}Ч^{\mathfrak{y}}Ч^{\mathfrak{y}}(Ч^*)^{\mathfrak{y}} + (\mathfrak{F}^*)^{\mathfrak{f}}((\mathfrak{F}^*)^{\mathfrak{y}}\mathfrak{F}^{\mathfrak{y}}\mathfrak{F}^{\mathfrak{y}}(\mathfrak{F}^*)^{\mathfrak{y}})]$$

$$= [\mathbb{Q}(Ч^*ЧЧЧ^*)^{\mathfrak{y}}(Ч^*)^{\mathfrak{f}}] + [\mathbb{Q}(\mathfrak{F}^*\mathfrak{F}\mathfrak{F}\mathfrak{F}^*)^{\mathfrak{y}}(\mathfrak{F}^*)^{\mathfrak{f}}]$$

$$= \mathfrak{Q}[(\text{Ч}^*\text{Ч}\text{Ч}\text{Ч}^*)^{\mathfrak{y}}(\text{Ч}^*)^{\mathfrak{f}}+(\mathfrak{F}^*\mathfrak{F}\mathfrak{F}\mathfrak{F}^*)^{\mathfrak{y}}(\mathfrak{F}^*)^{\mathfrak{f}}]$$

$$= \mathfrak{Q}[(\text{Ч}^*)^{\mathfrak{y}}\text{Ч}^{\mathfrak{y}}\text{Ч}^{\mathfrak{y}}(\text{Ч}^*)^{\mathfrak{y}}(\text{Ч}^*)^{\mathfrak{f}} + ((\mathfrak{F}^*)^{\mathfrak{y}}\mathfrak{F}^{\mathfrak{y}}\mathfrak{F}^{\mathfrak{y}}(\mathfrak{F}^*)^{\mathfrak{y}})(\mathfrak{F}^*)^{\mathfrak{f}}]$$

$$= \mathfrak{Q}[(\text{Ч}^*)^{\mathfrak{y}}\text{Ч}^{\mathfrak{y}}\text{Ч}^{\mathfrak{y}}(\text{Ч}^*)^{\mathfrak{y}} + ((\mathfrak{F}^*)^{\mathfrak{y}}\mathfrak{F}^{\mathfrak{y}}\mathfrak{F}^{\mathfrak{y}}(\mathfrak{F}^*)^{\mathfrak{y}})] \;\; [(\text{Ч}^*)^{\mathfrak{f}} + (\mathfrak{F}^*)^{\mathfrak{f}}]$$

$$= \mathfrak{Q}[[(\text{Ч}^*)^{\mathfrak{y}}\text{Ч}^{\mathfrak{y}} + (\mathfrak{F}^*)^{\mathfrak{y}}\mathfrak{F}^{\mathfrak{y}})][(\text{Ч}^{\mathfrak{y}}(\text{Ч}^*)^{\mathfrak{y}} + \mathfrak{F}^{\mathfrak{y}}(\mathfrak{F}^*)^{\mathfrak{y}})] \;\; [(\text{Ч}^*)^{\mathfrak{f}} + (\mathfrak{F}^*)^{\mathfrak{f}}]$$

$$= \mathfrak{Q}[(\text{Ч}^*)^{\mathfrak{f}} + (\mathfrak{F}^*)^{\mathfrak{f}}] \, [[(\text{Ч}^*)^{\mathfrak{y}}\text{Ч}^{\mathfrak{y}} + (\text{Ч}^*)^{\mathfrak{y}}\mathfrak{F}^{\mathfrak{y}} + (\mathfrak{F}^*)^{\mathfrak{y}}\text{Ч}^{\mathfrak{y}} + (\mathfrak{F}^*)^{\mathfrak{y}}\mathfrak{F}^{\mathfrak{y}})][(\text{Ч}^{\mathfrak{y}}(\text{Ч}^*)^{\mathfrak{y}} + \text{Ч}^{\mathfrak{y}}(\mathfrak{F}^*)^{\mathfrak{y}} + \mathfrak{F}^{\mathfrak{y}}(\text{Ч}^*)^{\mathfrak{y}} + \mathfrak{F}^{\mathfrak{y}}(\mathfrak{F}^*)^{\mathfrak{y}})]$$

$$[(\text{Ч}^*)^{\mathfrak{f}} + (\mathfrak{F}^*)^{\mathfrak{f}}]$$

$$= \mathfrak{Q}[((\text{Ч} + \mathfrak{F})^*(\text{Ч} + \mathfrak{F})(\text{Ч} + \mathfrak{F})(\text{Ч} + \mathfrak{F})^*)]^{\mathfrak{y}}((\text{Ч} + \mathfrak{F})^*)^{\mathfrak{f}}$$

$$= \mathfrak{Q}\,[((\text{Ч} + \mathfrak{F})^*)^{\mathfrak{y}}(\text{Ч} + \mathfrak{F})^{\mathfrak{y}}(\text{Ч} + \mathfrak{F})^{\mathfrak{y}}((\text{Ч} + \mathfrak{F})^*)^{\mathfrak{y}}] \; [(\text{Ч}^*)^{\mathfrak{f}} + \mathfrak{f}(\text{Ч}^*)^{\mathfrak{f}-1}\mathfrak{F}^* + \cdots + (\mathfrak{F}^*)^{\mathfrak{f}}]$$

$$= \mathfrak{Q}[((\text{Ч}^*)^{\mathfrak{y}} + (\mathfrak{F}^*)^{\mathfrak{y}})(\text{Ч}^{\mathfrak{y}} + \mathfrak{F}^{\mathfrak{y}}) \;\; (\text{Ч}^{\mathfrak{y}} + \mathfrak{F}^{\mathfrak{y}}) \;\; ((\text{Ч}^*)^{\mathfrak{y}} + (\mathfrak{F}^*)^{\mathfrak{y}})] \; [(\text{Ч}^*)^{\mathfrak{f}} + (\mathfrak{F}^*)^{\mathfrak{f}}]$$

$$= \mathfrak{Q}[((\text{Ч}^*)^{\mathfrak{y}} + (\mathfrak{F}^*)^{\mathfrak{y}})(\text{Ч}^{\mathfrak{y}} + \mathfrak{F}^{\mathfrak{y}}) \;\; (\text{Ч}^{\mathfrak{y}} + \mathfrak{F}^{\mathfrak{y}}) \;\; ((\text{Ч}^*)^{\mathfrak{y}} + (\mathfrak{F}^*)^{\mathfrak{y}})] \; [(\text{Ч}^*)^{\mathfrak{f}} + (\mathfrak{F}^*)^{\mathfrak{f}}]$$

$$= \mathfrak{Q}[((\text{Ч} + \mathfrak{F})^*(\text{Ч} + \mathfrak{F})(\text{Ч} + \mathfrak{F})(\text{Ч} + \mathfrak{F})^*)]^{\mathfrak{y}}((\text{Ч} + \mathfrak{F})^*)^{\mathfrak{f}}$$

Hence, $\text{Ч} + \mathfrak{F}$ is a $(\mathfrak{f}^* - \mathfrak{Q})$ quasi binormal operator of order $\mathfrak{y}$.

Theorem 1.4. *Let* Ч *be a* $(\mathfrak{f}^* - \mathfrak{Q})$ *quasi binormal operator of order* $\mathfrak{y}$ *and* $\mathfrak{Y}$ *is a* $(\mathfrak{f}^*) -$ *quasi binormal operator of order* $\mathfrak{y}$ *on Hilbert space H such that* $\text{Ч}^*\mathfrak{Y} = \mathfrak{Y}\,\text{Ч}^*$ *and* $\text{Ч}\mathfrak{Y} = \mathfrak{Y}\text{Ч}$. *Then* $\text{Ч}\mathfrak{Y}$ *is a* $(\mathfrak{f}^* - \mathfrak{Q})$ *quasi binormal operator of order* $\mathfrak{y}$.

Proof. Let Ч be a $(\mathfrak{f}^* - \mathfrak{Q})$ quasi binormal operator of order $\mathfrak{y}$ and $\mathfrak{Y}$ be a $(\mathfrak{f}^*) -$ quasi binormal operator of order $\mathfrak{y}$. Then

$$((\text{Ч}\,\mathfrak{Y})^*)^{\mathfrak{f}}[((\text{Ч}\,\mathfrak{Y})^*(\text{Ч}\,\mathfrak{Y})(\text{Ч}\,\mathfrak{Y})(\text{Ч}\,\mathfrak{Y})^*)]^{\mathfrak{y}}$$

$$= (\text{Ч}^*)^{\mathfrak{f}}\;(\mathfrak{Y}^*)^{\mathfrak{f}}[(\text{Ч}^*)^{\mathfrak{y}}(\mathfrak{Y}^*)^{\mathfrak{y}}(\text{Ч})^{\mathfrak{y}}(\mathfrak{Y})^{\mathfrak{y}}(\text{Ч})^{\mathfrak{y}}(\mathfrak{Y})^{\mathfrak{y}}(\text{Ч}^*)^{\mathfrak{y}}(\mathfrak{Y}^*)^{\mathfrak{y}}]$$

$$= (\text{Ч}^*)^{\mathfrak{f}}(\text{Ч}^*)^{\mathfrak{y}}(\mathfrak{N}^*)^{\mathfrak{f}}(\mathfrak{N}^*)^{\mathfrak{y}}(\text{Ч})^{\mathfrak{y}}(\mathfrak{N})^{\mathfrak{y}}(\text{Ч})^{\mathfrak{y}}(\text{Ч}^*)^{\mathfrak{y}}(\mathfrak{N})^{\mathfrak{y}}(\mathfrak{N}^*)^{\mathfrak{y}}$$

$$= (\text{Ч}^*)^{\mathfrak{f}}(\text{Ч}^*)^{\mathfrak{y}}(\text{Ч})^{\mathfrak{y}}(\mathfrak{Y}^*)^{\mathfrak{f}}(\mathfrak{Y}^*)^{\mathfrak{y}}(\mathfrak{Y})^{\mathfrak{y}}(\text{Ч})^{\mathfrak{y}}(\text{Ч}^*)^{\mathfrak{y}}(\mathfrak{Y})^{\mathfrak{y}}(\mathfrak{Y}^*)^{\mathfrak{y}}$$

$$\vdots$$

$$= [\,(\text{Ч}^*)^{\mathfrak{f}}(\text{Ч}^*)^{\mathfrak{y}}(\text{Ч})^{\mathfrak{y}}(\text{Ч})^{\mathfrak{y}}(\text{Ч}^*)^{\mathfrak{y}}][(\mathfrak{Y}^*)^{\mathfrak{f}}(\mathfrak{Y}^*)^{\mathfrak{y}}(\mathfrak{Y})^{\mathfrak{y}}(\mathfrak{Y})^{\mathfrak{y}}(\mathfrak{Y}^*)^{\mathfrak{y}}]\;,$$

$$= [(\text{Ч}^*)^{\mathfrak{f}}[(\text{Ч}^*)^{\mathfrak{y}}\text{Ч}^{\mathfrak{y}}\text{Ч}^{\mathfrak{y}}(\text{Ч}^*)^{\mathfrak{y}}]][(\mathfrak{Y}^*)^{\mathfrak{f}}((\mathfrak{Y}^*)^{\mathfrak{y}}\mathfrak{Y}^{\mathfrak{y}}\mathfrak{Y}^{\mathfrak{y}}(\mathfrak{Y}^*)^{\mathfrak{y}})]$$

$= [ɷ(Ч^*ЧЧЧ^*)^{\mathfrak{n}}(Ч^*)^{\mathfrak{f}}][(\mathfrak{Y}^*\mathfrak{Y}\mathfrak{Y}\mathfrak{Y}^*)^{\mathfrak{n}}(\mathfrak{Y}^*)^{\mathfrak{f}}]$

$= [ɷ(Ч^*)^{\mathfrak{n}}(Ч)^{\mathfrak{n}}(Ч)^{\mathfrak{n}}(Ч^*)^{\mathfrak{n}}(Ч^*)^{\mathfrak{f}}][(\mathfrak{Y}^*)^{\mathfrak{n}}(\mathfrak{Y})^{\mathfrak{n}}(\mathfrak{Y})^{\mathfrak{n}}(\mathfrak{Y}^*)^{\mathfrak{n}}(\mathfrak{Y}^*)^{\mathfrak{f}}]$

$= ɷ(Ч^*)^{\mathfrak{n}}(Ч)^{\mathfrak{n}}(Ч)^{\mathfrak{n}}(Ч^*)^{\mathfrak{n}}(\mathfrak{Y}^*)^{\mathfrak{n}}(Ч^*)^{\mathfrak{f}}(\mathfrak{Y})^{\mathfrak{n}}(\mathfrak{Y})^{\mathfrak{n}}(\mathfrak{Y}^*)^{\mathfrak{n}}(\mathfrak{Y}^*)^{\mathfrak{f}}$

$= ɷ(Ч^*)^{\mathfrak{n}}(Ч)^{\mathfrak{n}}(Ч)^{\mathfrak{n}}(\mathfrak{Y}^*)^{\mathfrak{n}}(Ч^*)^{\mathfrak{n}}(Ч^*)^{\mathfrak{f}}(\mathfrak{Y})^{\mathfrak{n}}(\mathfrak{Y})^{\mathfrak{n}}(\mathfrak{Y}^*)^{\mathfrak{n}}(\mathfrak{Y}^*)^{\mathfrak{f}}$

$\vdots$

$= ɷ[((Ч\mathfrak{Y})^*(Ч\mathfrak{Y})(Ч\mathfrak{Y})(Ч\mathfrak{Y})^*)]^{\mathfrak{n}}\ ((Ч\mathfrak{Y})^*)^{\mathfrak{f}}.$

Hence, $Ч\mathfrak{Y}$ is a $(\mathfrak{f}^* - ɷ)$ quasi binormal operator of order $\mathfrak{y}$.

Theorem 1.5. *The set of all* $(\mathfrak{f}^* - ɷ)$ *quasi binormal operators of order* $\mathfrak{y}$ *on Hilbert space H is a closed subset of $B(H)$ (the algebra of all bounded linear operators on Hilbert space H) under scalar multiplication.*

Proof. Suppose

$Ƭ(\mathrm{H}) = \{Ч \in B(H): \mathfrak{A}$ is a $(\mathfrak{f}^* - ɷ)$ quasi binormal operator of order $\mathfrak{y}$ on H for some nonnegative integer $\mathfrak{f}\}$.

Let $Ч \in Ƭ(\mathrm{H})$, then we have $Ч$ is a $(\mathfrak{f}^* - ɷ)$ quasi binormal operator of order $\mathfrak{y}$ and $(Ч^*)^{\mathfrak{f}}(Ч^*ЧЧЧ^*)^{\mathfrak{n}} = ɷ(Ч^*ЧЧЧ^*)^{\mathfrak{n}}(Ч^*)^{\mathfrak{f}}$, where $ɷ$ is bounded operator, $\mathfrak{f}$ and $\mathfrak{y}$ are nonnegative integers.

Let Θ be a scalar. Then

$((\Theta Ч)^*)^{\mathfrak{f}}[(\Theta Ч)^*(\Theta Ч)(\Theta Ч)(\Theta Ч)^*]^{\mathfrak{n}}$

$= (\bar{\Theta})^{\mathfrak{f}}\ (Ч^*)^{\mathfrak{f}}[(\bar{\Theta})^{\mathfrak{n}}(Ч^*)^{\mathfrak{n}}(\Theta)^{\mathfrak{n}}(Ч)^{\mathfrak{n}}(\Theta)^{\mathfrak{n}}(Ч)^{\mathfrak{n}}\overline{(\Theta)}^{\mathfrak{n}}(Ч^*)^{\mathfrak{n}}]$

$= (\bar{\Theta})^{\mathfrak{f}}(\bar{\Theta})^{\mathfrak{n}}(\Theta)^{\mathfrak{n}}(\Theta)^{\mathfrak{n}}\overline{(\Theta)}^{\mathfrak{n}}\ (Ч^*)^{\mathfrak{f}}[\ (Ч^*)^{\mathfrak{n}}(Ч)^{\mathfrak{n}}(Ч)^{\mathfrak{n}}(Ч^*)^{\mathfrak{n}}]$

$= (\bar{\Theta})^{\mathfrak{f}}(\bar{\Theta})^{\mathfrak{n}}(\Theta)^{\mathfrak{n}}(\Theta)^{\mathfrak{n}}\overline{(\Theta)}^{\mathfrak{n}}\ (Ч^*)^{\mathfrak{f}}[(Ч^*ЧЧЧ^*)^{\mathfrak{n}}]$

$= (\bar{\Theta})^{\mathfrak{f}}(\bar{\Theta})^{\mathfrak{n}}(\Theta)^{\mathfrak{n}}(\Theta)^{\mathfrak{n}}\overline{(\Theta)}^{\mathfrak{n}}[ɷ(Ч^*ЧЧЧ^*)^{\mathfrak{n}}(Ч^*)^{\mathfrak{f}}]$

$= (\bar{\Theta})^{\mathfrak{f}}(\bar{\Theta})^{\mathfrak{n}}(\Theta)^{\mathfrak{n}}(\Theta)^{\mathfrak{n}}\overline{(\Theta)}^{\mathfrak{n}}\ [ɷ[\ (Ч^*)^{\mathfrak{n}}(Ч)^{\mathfrak{n}}(Ч)^{\mathfrak{n}}(Ч^*)^{\mathfrak{n}}](Ч^*)^{\mathfrak{f}}]$

$= ɷ[(\bar{\Theta})^{\mathfrak{n}}(Ч^*)^{\mathfrak{n}}(\Theta)^{\mathfrak{n}}(Ч)^{\mathfrak{n}}(\Theta)^{\mathfrak{n}}(Ч)^{\mathfrak{n}}\overline{(\Theta)}^{\mathfrak{n}}(Ч^*)^{\mathfrak{n}}](\bar{\Theta})^{\mathfrak{f}}\ (Ч^*)^{\mathfrak{f}}$

$= ɷ[(\Theta Ч)^*(\Theta Ч)(\Theta Ч)(\Theta Ч)^*]^{\mathfrak{n}}\ ((\Theta Ч)^*)^{\mathfrak{f}}.$

Hence, $\Theta Ч \in Ƭ(\mathrm{H})$.

Let $Ч_x$ be a sequence in $Ţ(H)$ and converge to $Ч$. Then we can prove that

$$\|[(Ч^*)^ƒ(Ч^*ЧЧЧ^*)^ŋ] - [Ѡ(Ч^*ЧЧЧ^*)^ŋ(Ч^*)^ƒ]\|$$

$$= \big\|[(Ч^*)^ƒ(Ч^*ЧЧЧ^*)^ŋ] - [(Ч_x{}^*)^ƒ(Ч_x{}^*Ч_xЧ_xЧ_x{}^*)^ŋ] + [Ѡ(Ч_x{}^*Ч_xЧ_xЧ_x{}^*)^ŋ(Ч_x{}^*)^ƒ] - [Ѡ(Ч^*ЧЧЧ^*)^ŋ(Ч^*)^ƒ]\big\|$$

$$\le \big\|[(Ч^*)^ƒ(Ч^*ЧЧЧ^*)^ŋ] - [(Ч_x{}^*)^ƒ(Ч_x{}^*Ч_xЧ_xЧ_x{}^*)^ŋ]\big\| + \big\|[Ѡ(Ч_x{}^*Ч_xЧ_xЧ_x{}^*)^ŋ(Ч_x{}^*)^ƒ] - [Ѡ(Ч^*ЧЧЧ^*)^ŋ(Ч^*)^ƒ]\big\| \to 0 \text{ as } x \to \infty.$$

Therefore, $Ч \in Ţ(H)$. Then, $Ţ(H)$ is a closed subset.

Proposition 1.6. *If* $Ч^{-1}$ *exists and* $Ч$ *is a* $(ƒ^* - Ѡ)$ *quasi binormal operator of order* $ŋ$ *on Hilbert space H and* $(Ч^*ЧЧЧ^*)^ŋ$ *commutes with* $(Ч^*)^ƒ$*, then* $Ч^{-1}$ *is a* $(ƒ^* - Ѡ)$ *quasi binormal operator of order* $ŋ$.

Proof. Since $Ч$ is a $(ƒ^* - Ѡ)$ quasi binormal operator of order $ŋ$, we have $(Ч^*)^ƒ(Ч^*ЧЧЧ^*)^ŋ = Ѡ(Ч^*ЧЧЧ^*)^ŋ(Ч^*)^ƒ$.

Then

$$\begin{aligned}
((Ч^{-1})^*)^ƒ((Ч^{-1})^*(Ч^{-1})(Ч^{-1})(Ч^{-1})^*)^ŋ &= ((Ч^*)^ƒ)^{-1}((Ч^*)^{-1}(Ч^{-1})(Ч^{-1})(Ч^*)^{-1})^ŋ \\
&= ((Ч^*)^ƒ)^{-1}((ЧЧ^*)^{-1}(Ч^*Ч)^{-1})^ŋ \\
&= ((Ч^*)^ƒ)^{-1}((Ч^*ЧЧЧ^*)^{-1})^ŋ \\
&= ((Ч^*)^ƒ)^{-1}((Ч^*ЧЧЧ^*)^ŋ)^{-1} \\
&= \left((Ч^*ЧЧЧ^*)^ŋ(Ч^*)^ƒ\right)^{-1} \\
&= \left((Ч^*)^ƒ(Ч^*ЧЧЧ^*)^ŋ\right)^{-1} \\
&= [Ѡ(Ч^*ЧЧЧ^*)^ŋ(Ч^*)^ƒ]^{-1} \\
&= [(Ч^*)^ƒѠ(Ч^*ЧЧЧ^*)^ŋ]^{-1} \\
&= Ѡ[(Ч^*ЧЧЧ^*)^ŋ]^{-1}\,((Ч^*)^ƒ)^{-1} \\
&= Ѡ[(Ч^*ЧЧЧ^*)^{-1}]^ŋ\,((Ч^{-1})^*)^ƒ \\
&= Ѡ((Ч^{-1})^*(Ч^{-1})(Ч^{-1})(Ч^{-1})^*)^ŋ((Ч^{-1})^*)^ƒ
\end{aligned}$$

Hence, $Ч^{-1}$ is $(ƒ^* - Ѡ)$ quasi binormal operator of order $ŋ$ on Hilbert space H.

Definition 1.7 [4]**.** If A, B are bounded operators on Hilbert space H. Then A, B are *unitary equivalent* if there is an isomorphism $U: H \to H$ such that $B = UAU^*$.

Theorem 1.8. *Let* Ч *be a* $(\mathfrak{f}^* - \mathfrak{w})$ *quasi binormal operator of order* $\mathfrak{y}$ *on Hilbert space H. Then the operator* $\varrho\text{Ч}$ *is a* $(\mathfrak{f}^* - \mathfrak{w})$ *quasi binormal operator of order* $\mathfrak{y}$ *for every real scalar* ϱ*.*

Proof. Let Ч be a $(\mathfrak{f}^* - \mathfrak{w})$ quasi binormal operator of order $\mathfrak{y}$ and let ϱ be a scalar. Then

$$((\varrho\text{Ч})^*)^{\mathfrak{f}}[(\varrho\text{Ч})^*(\varrho\text{Ч})(\varrho\text{Ч})(\varrho\text{Ч})^*]^{\mathfrak{y}}$$

$$= (\bar{\varrho})^{\mathfrak{f}}\ (\text{Ч}^*)^{\mathfrak{f}}[(\bar{\varrho})^{\mathfrak{y}}(\text{Ч}^*)^{\mathfrak{y}}(\varrho)^{\mathfrak{y}}(\text{Ч})^{\mathfrak{y}}(\varrho)^{\mathfrak{y}}(\text{Ч})^{\mathfrak{y}}\varrho\overline{(\varrho)}^{\mathfrak{y}}(\text{Ч}^*)^{\mathfrak{y}}]$$

$$= (\bar{\varrho})^{\mathfrak{f}}(\varrho)^{\mathfrak{y}}(\varrho)^{\mathfrak{y}}(\varrho)^{\mathfrak{y}}\overline{(\varrho)}^{\mathfrak{y}}\ (\text{Ч}^*)^{\mathfrak{f}}[(\text{Ч}^*)^{\mathfrak{y}}(\text{Ч})^{\mathfrak{y}}(\text{Ч})^{\mathfrak{y}}(\text{Ч}^*)^{\mathfrak{y}}]$$

$$= (\bar{\varrho})^{\mathfrak{f}}(\varrho)^{\mathfrak{y}}(\varrho)^{\mathfrak{y}}(\varrho)^{\mathfrak{y}}\overline{(\varrho)}^{\mathfrak{y}}\ (\text{Ч}^*)^{\mathfrak{f}}[(\text{Ч}^*\text{Ч}\text{Ч}\text{Ч}^*)^{\mathfrak{y}}]$$

$$= (\bar{\varrho})^{\mathfrak{f}}(\varrho)^{\mathfrak{y}}(\varrho)^{\mathfrak{y}}(\varrho)^{\mathfrak{y}}\overline{(\varrho)}^{\mathfrak{y}}\ [\mathfrak{w}(\text{Ч}^*\text{Ч}\text{Ч}\text{Ч}^*)^{\mathfrak{y}}(\text{Ч}^*)^{\mathfrak{f}}]$$

$$= (\bar{\varrho})^{\mathfrak{f}}(\varrho)^{\mathfrak{y}}(\varrho)^{\mathfrak{y}}(\varrho)^{\mathfrak{y}}\overline{(\varrho)}^{\mathfrak{y}}\ [\mathfrak{w}[(\text{Ч}^*)^{\mathfrak{y}}(\text{Ч})^{\mathfrak{y}}(\text{Ч})^{\mathfrak{y}}(\text{Ч}^*)^{\mathfrak{y}}](\text{Ч}^*)^{\mathfrak{f}}]$$

$$= \mathfrak{w}[(\bar{\varrho})^{\mathfrak{y}}(\text{Ч}^*)^{\mathfrak{y}}(\varrho)^{\mathfrak{y}}(\text{Ч})^{\mathfrak{y}}(\varrho)^{\mathfrak{y}}(\text{Ч})^{\mathfrak{y}}\overline{(\varrho)}^{\mathfrak{y}}(\text{Ч}^*)^{\mathfrak{y}}](\bar{\varrho})^{\mathfrak{f}}\ (\text{Ч}^*)^{\mathfrak{f}}$$

$$= \mathfrak{w}[(\varrho\text{Ч})^*(\varrho\text{Ч})(\varrho\text{Ч})(\varrho\text{Ч})^*]^{\mathfrak{y}}\ ((\varrho\text{Ч})^*)^{\mathfrak{f}}$$

So, $\varrho\text{Ч}$ is a $(\mathfrak{f}^* - \mathfrak{w})$ quasi binormal operator of order $\mathfrak{y}$ on Hilbert space H.

Theorem 1.9. *Let* Ч *be a* $(\mathfrak{f}^* - \mathfrak{w})$ *quasi binormal operator of order* $\mathfrak{y}$ *on Hilbert space H. If* $\mathfrak{M} \in B(H)$ *is unitary equivalent to* Ч*, then* $\mathfrak{M}$ *is a* $(\mathfrak{f}^* - \mathfrak{w})$ *quasi binormal operator of order* $\mathfrak{y}$*.*

Proof. Since $\mathfrak{M}$ is unitary equivalent to Ч, we have $\mathfrak{M} = U\text{Ч}U^*$, $(U\text{Ч}U^*)^n = U\text{Ч}^nU^*$ and since Ч is a $(\mathfrak{f}^* - \mathfrak{w})$ quasi binormal operator of order $\mathfrak{y}$, we have $(\text{Ч}^*)^{\mathfrak{f}}(\text{Ч}^*\text{Ч}\text{Ч}\text{Ч}^*)^{\mathfrak{y}} = \mathfrak{w}(\text{Ч}^*\text{Ч}\text{Ч}\text{Ч}^*)^{\mathfrak{y}}(\text{Ч}^*)^{\mathfrak{f}}$.

Then

$$(\mathfrak{M}^*)^{\mathfrak{f}}(\mathfrak{M}^*\mathfrak{M}\mathfrak{M}\mathfrak{M}^*)^{\mathfrak{y}}$$

$$= ((U\text{Ч}U^*)^*)^{\mathfrak{f}}((U\text{Ч}U^*)^*(U\text{Ч}U^*)(U\text{Ч}U^*)(U\text{Ч}U^*)^*)^{\mathfrak{y}}$$

$$= (U(\text{Ч}^*)^{\mathfrak{f}}U^*)\big((U(\text{Ч}^*)^{\mathfrak{y}}U^*)(U\text{Ч}^{\mathfrak{y}}U^*)(U\text{Ч}^{\mathfrak{y}}U^*)(U(\text{Ч}^*)^{\mathfrak{y}}U^*)\big)$$

$$= (U(\text{Ч}^*)^{\mathfrak{f}}U^*)(U(\text{Ч}^*)^{\mathfrak{y}}(\text{Ч}^{\mathfrak{y}})(\text{Ч}^{\mathfrak{y}})(\text{Ч}^*)^{\mathfrak{y}}U^*)$$

$$= \big(U(\text{Ч}^*)^{\mathfrak{f}}U^*\big)(U(\text{Ч}^*\text{Ч}\text{Ч}\text{Ч}^*)^{\mathfrak{y}}U^*)$$

$$= \big(U[(\text{Ч}^*)^{\mathfrak{f}}(\text{Ч}^*\text{Ч}\text{Ч}\text{Ч}^*)^{\mathfrak{y}}]U^*\big)$$

$$= \big(U[\mathfrak{Q}(\text{Ч}^*\text{Ч}\text{Ч}\text{Ч}^*)^{\mathfrak{y}}(\text{Ч}^*)^{\mathfrak{f}}]U^*\big)$$

$$= \mathfrak{Q}(U(\text{Ч}^*\text{Ч}\text{Ч}\text{Ч}^*)^{\mathfrak{y}}U^*)\ \ \big(U(\text{Ч}^*)^{\mathfrak{f}}U^*\big)$$

$$= \mathfrak{Q}(U(\text{Ч}^*)^{\mathfrak{y}}(\text{Ч}^{\mathfrak{y}})(\text{Ч}^{\mathfrak{y}})(\text{Ч}^*)^{\mathfrak{y}}U^*)\ \ (U(\text{Ч}^*)^{\mathfrak{f}}U^*)$$

$$= \big(\mathfrak{Q}(U(\text{Ч}^*)^{\mathfrak{y}}U^*)(U\text{Ч}^{\mathfrak{y}}U^*)(U\text{Ч}^{\mathfrak{y}}U^*)(U(\text{Ч}^*)^{\mathfrak{y}}U^*)(U(\text{Ч}^*)^{\mathfrak{f}}U^*)\big)$$

$$= \mathfrak{Q}((U\text{Ч}U^*)^*(U\text{Ч}U^*)(U\text{Ч}U^*)(U\text{Ч}U^*)^*)^{\mathfrak{y}}\ ((U\text{Ч}U^*)^*)^{\mathfrak{f}}$$

$$= \mathfrak{Q}(\mathfrak{M}^*\mathfrak{M}\mathfrak{M}\mathfrak{M}^*)^{\mathfrak{y}}\ (\mathfrak{M}^*)^{\mathfrak{f}}.$$

Hence, $\mathfrak{M}$ is $(\mathfrak{f}^* - \mathfrak{Q})$ quasi binormal operator of order $\mathfrak{y}$.

Theorem 1.10. *If* Ч *is a* $(\mathfrak{f}^* - \mathfrak{Q})$ *quasi binormal operator of order* $\mathfrak{y}$ *and* $(\text{Ч}^*\text{Ч}\text{Ч}\text{Ч}^*)^{\mathfrak{y}}$ *commutes with* $(\text{Ч}^*)^{\mathfrak{f}}$*, then* Ч^* *is a* $(\mathfrak{f}^* - \mathfrak{Q})$ *quasi binormal operator of order* $\mathfrak{y}$.

Proof. Since Ч is a $(\mathfrak{f}^* - \mathfrak{Q})$ quasi binormal operator of order $\mathfrak{y}$, we have $(\text{Ч}^*)^{\mathfrak{f}}(\text{Ч}^*\text{Ч}\text{Ч}\text{Ч}^*)^{\mathfrak{y}} = \mathfrak{Q}(\text{Ч}^*\text{Ч}\text{Ч}\text{Ч}^*)^{\mathfrak{y}}(\text{Ч}^*)^{\mathfrak{f}}$.

Then

$$\begin{aligned}
((\text{Ч}^*)^*)^{\mathfrak{f}}((\text{Ч}^*)^*(\text{Ч}^*)(\text{Ч}^*)(\text{Ч}^*)^*)^{\mathfrak{y}} &= ((\text{Ч}^*)^{\mathfrak{f}})^*((\text{Ч}^*)^*(\text{Ч}^*)(\text{Ч}^*)(\text{Ч}^*)^*)^{\mathfrak{y}} \\
&= ((\text{Ч}^*)^{\mathfrak{f}})^*((\text{Ч}\text{Ч}^*)^*(\text{Ч}^*\text{Ч})^*)^{\mathfrak{y}} \\
&= ((\text{Ч}^*)^{\mathfrak{f}})^*\big((\text{Ч}^*\text{Ч}\text{Ч}\text{Ч}^*)^*\big)^{\mathfrak{y}} \\
&= ((\text{Ч}^*)^{\mathfrak{f}})^*\big((\text{Ч}^*\text{Ч}\text{Ч}\text{Ч}^*)^{\mathfrak{y}}\big)^* \\
&= \big((\text{Ч}^*\text{Ч}\text{Ч}\text{Ч}^*)^{\mathfrak{y}}(\text{Ч}^*)^{\mathfrak{f}}\big)^* \\
&= \big((\text{Ч}^*)^{\mathfrak{f}}(\text{Ч}^*\text{Ч}\text{Ч}\text{Ч}^*)^{\mathfrak{y}}\big)^* \\
&= [\mathfrak{Q}(\text{Ч}^*\text{Ч}\text{Ч}\text{Ч}^*)^{\mathfrak{y}}(\text{Ч}^*)^{\mathfrak{f}}]^*
\end{aligned}$$

$$= [(\text{Ч}^*)^{\mathfrak{f}}\mathfrak{Q}(\text{Ч}^*\text{Ч}\text{Ч}\text{Ч}^*)^{\mathfrak{y}}]^*$$

$$= \mathfrak{Q}[(\text{Ч}^*\text{Ч}\text{Ч}\text{Ч}^*)^{\mathfrak{y}}]^*((\text{Ч}^*)^{\mathfrak{f}})^*$$

$$= \mathfrak{Q}[(\text{Ч}^*\text{Ч}\text{Ч}\text{Ч}^*)^*]^{\mathfrak{y}}((\text{Ч}^*)^*)^{\mathfrak{f}}$$

$$= \mathfrak{Q}((\text{Ч}^*)^*(\text{Ч}^*)(\text{Ч}^*)(\text{Ч}^*)^*)^{\mathfrak{y}}((\text{Ч}^*)^*)^{\mathfrak{f}}.$$

Hence, Ч^* is a $(\mathfrak{f}^* - \mathfrak{Q})$ quasi binormal operator of order $\mathfrak{y}$.

Theorem 1.11. *Consider that* $\text{Ч}_1, \text{Ч}_2, \dots, \text{Ч}_k$ *are* $(\mathfrak{f}^* - \mathfrak{Q})$ *quasi binormal operators of order* $\mathfrak{y}$. *Then the direct sum* $(\text{Ч}_1 \oplus \text{Ч}_2 \oplus \dots \oplus \text{Ч}_k)$ *is a* $(\mathfrak{f}^* - \mathfrak{Q})$ *quasi binormal operator of order* $\mathfrak{y}$.

Proof. Since every operator of $\text{Ч}_1, \text{Ч}_2, \dots, \text{Ч}_k$ is $(\mathfrak{f}^* - \mathfrak{Q})$ quasi binormal operator of order $\mathfrak{y}$, we have

$$({\text{Ч}_j}^*)^{\mathfrak{f}}\left({\text{Ч}_j}^*\text{Ч}_j\text{Ч}_j{\text{Ч}_j}^*\right)^{\mathfrak{y}} = \mathfrak{Q}\left({\text{Ч}_j}^*\text{Ч}_j\text{Ч}_j{\text{Ч}_j}^*\right)^{\mathfrak{y}}({\text{Ч}_j}^*)^{\mathfrak{f}}, \text{ for all } j = 1,2,\dots,k.$$

Then

$$((\text{Ч}_1\oplus\text{Ч}_2\oplus\dots\oplus\text{Ч}_k)^*)^{\mathfrak{f}}((\text{Ч}_1\oplus\text{Ч}_2\oplus\dots\oplus\text{Ч}_k)^*(\text{Ч}_1\oplus\text{Ч}_2\oplus\dots\oplus\text{Ч}_k)(\text{Ч}_1\oplus\text{Ч}_2\oplus\dots\oplus\text{Ч}_k)(\text{Ч}_1\oplus\text{Ч}_2\oplus\dots\oplus\text{Ч}_k)^*)^{\mathfrak{y}}$$

$$= [({\mathfrak{A}_1}^*)^{\mathfrak{f}}\oplus ({\mathfrak{A}_2}^*)^{\mathfrak{f}}\oplus\dots\oplus ({\mathfrak{A}_n}^*)^{\mathfrak{f}}][((\text{Ч}_1\oplus\text{Ч}_2\oplus\dots\oplus\text{Ч}_k)^*]^{\mathfrak{y}}(\text{Ч}_1\oplus\text{Ч}_2\oplus\dots\oplus\text{Ч}_k)^{\mathfrak{y}}(\text{Ч}_1\oplus\text{Ч}_2\oplus\dots\oplus\text{Ч}_k)^{\mathfrak{y}}[(\text{Ч}_1\oplus\text{Ч}_2\oplus\dots\oplus\text{Ч}_k)^*)]^{\mathfrak{y}}$$

$$= [\ {\text{Ч}_1}^*)^{\mathfrak{f}}\oplus ({\text{Ч}_2}^*)^{\mathfrak{f}}\oplus\dots\oplus ({\text{Ч}_k}^*)^{\mathfrak{f}}][[({\text{Ч}_1}^*)^{\mathfrak{y}}\oplus ({\text{Ч}_2}^*)^{\mathfrak{y}}\oplus\dots\oplus ({\text{Ч}_k}^*)^{\mathfrak{y}}]({\text{Ч}_1}^{\mathfrak{y}}\oplus{\text{Ч}_2}^{\mathfrak{y}}\oplus\dots\oplus{\text{Ч}_k}^{\mathfrak{y}})$$

$$({\text{Ч}_1}^{\mathfrak{y}}\oplus{\text{Ч}_2}^{\mathfrak{y}}\oplus\dots\oplus{\text{Ч}_k}^{\mathfrak{y}})\ [({\text{Ч}_1}^*)^{\mathfrak{y}}\oplus ({\text{Ч}_2}^*)^{\mathfrak{y}}\oplus\dots\oplus ({\text{Ч}_k}^*)^{\mathfrak{y}}]$$

$$= [({\text{Ч}_1}^*)^{\mathfrak{f}}(\ ({\text{Ч}_1}^*)^{\mathfrak{y}}{\text{Ч}_1}^{\mathfrak{y}}{\text{Ч}_1}^{\mathfrak{y}}\ ({\text{Ч}_1}^*)^{\mathfrak{y}})\ ]\oplus[({\text{Ч}_2}^*)^{\mathfrak{f}}(\ ({\text{Ч}_2}^*)^{\mathfrak{y}}{\text{Ч}_2}^{\mathfrak{y}}{\text{Ч}_2}^{\mathfrak{y}}\ ({\text{Ч}_2}^*)^{\mathfrak{y}})\ ]\oplus\dots\oplus[\ ({\text{Ч}_k}^*)^{\mathfrak{f}}(\ ({\text{Ч}_k}^*)^{\mathfrak{y}}{\text{Ч}_k}^{\mathfrak{y}}{\text{Ч}_k}^{\mathfrak{y}}\ ({\text{Ч}_k}^*)^{\mathfrak{y}})\ ]$$

$$= [({\text{Ч}_1}^*)^{\mathfrak{f}}(({\text{Ч}_1}^*\text{Ч}_1\text{Ч}_1{\text{Ч}_1}^*)^{\mathfrak{y}})\ ]\oplus[({\text{Ч}_2}^*)^{\mathfrak{f}}({\text{Ч}_2}^*\text{Ч}_2\text{Ч}_2{\text{Ч}_2}^*)^{\mathfrak{y}}\ ]\oplus\dots\oplus[({\text{Ч}_k}^*)^{\mathfrak{f}}({\text{Ч}_k}^*\text{Ч}_k\text{Ч}_k{\text{Ч}_k}^*)^{\mathfrak{y}}\ ]$$

$$= [\ \mathfrak{Q}(({\text{Ч}_1}^*\text{Ч}_1\text{Ч}_1{\text{Ч}_1}^*)^{\mathfrak{y}})\ ({\text{Ч}_1}^*)^{\mathfrak{f}}]\oplus[\ \mathfrak{Q}({\text{Ч}_2}^*\text{Ч}_2\text{Ч}_2{\text{Ч}_2}^*)^{\mathfrak{y}}\ ({\text{Ч}_2}^*)^{\mathfrak{f}}]\oplus\dots\oplus[\mathfrak{Q}({\text{Ч}_k}^*\text{Ч}_k\text{Ч}_k{\text{Ч}_k}^*)^{\mathfrak{y}}({\text{Ч}_k}^*)^{\mathfrak{f}}]$$

$$= [\mathfrak{Q}(({\text{Ч}_1}^*)^{\mathfrak{y}}{\text{Ч}_1}^{\mathfrak{y}}{\text{Ч}_1}^{\mathfrak{y}}\ ({\text{Ч}_1}^*)^{\mathfrak{y}})\ ({\text{Ч}_1}^*)^{\mathfrak{f}}]\oplus[\ \mathfrak{Q}(({\text{Ч}_2}^*)^{\mathfrak{y}}{\text{Ч}_2}^{\mathfrak{y}}{\text{Ч}_2}^{\mathfrak{y}}({\text{Ч}_2}^*)^{\mathfrak{y}})\ ({\text{Ч}_2}^*)^{\mathfrak{f}}]\oplus\dots\oplus[\ \mathfrak{Q}(\ ({\text{Ч}_k}^*)^{\mathfrak{y}}{\text{Ч}_k}^{\mathfrak{y}}{\text{Ч}_k}^{\mathfrak{y}}\ ({\text{Ч}_k}^*)^{\mathfrak{y}})\ ({\mathfrak{A}_n}^*)^{\mathfrak{f}}]$$

$$= [\mathfrak{Q}[({\text{Ч}_1}^*)^{\mathfrak{y}}\oplus ({\text{Ч}_2}^*)^{\mathfrak{y}}\oplus\dots\oplus ({\text{Ч}_n}^*)^{\mathfrak{y}}]({\text{Ч}_k}^{\mathfrak{y}}\oplus{\text{Ч}_k}^{\mathfrak{y}}\oplus\dots\oplus{\text{Ч}_k}^{\mathfrak{y}})\ ({\text{Ч}_1}^{\mathfrak{y}}\oplus{\text{Ч}_2}^{\mathfrak{y}}\oplus\dots\oplus{\text{Ч}_k}^{\mathfrak{y}})\ [({\text{Ч}_1}^*)^{\mathfrak{y}}\oplus ({\text{Ч}_2}^*)^{\mathfrak{y}}\oplus\dots\oplus ({\text{Ч}_k}^*)^{\mathfrak{y}}]$$

$$[({\text{Ч}_1}^*)^{\mathfrak{f}}\oplus ({\text{Ч}_2}^*)^{\mathfrak{f}}\oplus\dots\oplus ({\text{Ч}_k}^*)^{\mathfrak{f}}]$$

$$= \mathfrak{Q}((\text{Ч}_1\oplus\text{Ч}_2\oplus\dots\oplus\text{Ч}_k)^*(\text{Ч}_1\oplus\text{Ч}_2\oplus\dots\oplus\text{Ч}_k)(\text{Ч}_1\oplus\text{Ч}_2\oplus\dots\oplus\text{Ч}_k)(\text{Ч}_1\oplus\text{Ч}_2\oplus\dots\oplus\text{Ч}_k)^*)^{\mathfrak{y}}\ ((\text{Ч}_1\oplus\text{Ч}_2\oplus\dots\oplus\text{Ч}_k)^*)^{\mathfrak{f}}$$

Hence, $(\text{Ч}_1\oplus\text{Ч}_2\oplus\dots\oplus\text{Ч}_k)$ is a $(\mathfrak{f}^* - \mathfrak{Q})$ quasi binormal operator of order $\mathfrak{y}$.

Theorem 1.12. *Let* $\text{Ч}_1, \text{Ч}_2, \dots, \text{Ч}_k$ *be* $(\mathfrak{f}^* - \mathfrak{Q})$ *quasi binormal operators of order* $\mathfrak{y}$. *Then the tenser product* $(\text{Ч}_1\otimes\text{Ч}_2\otimes\dots\otimes\text{Ч}_k)$ *is a* $(\mathfrak{f}^* - \mathfrak{Q})$ *quasi binormal operator of order* $\mathfrak{y}$.

Proof. Since every operator of $\text{Ч}_1, \text{Ч}_2, \dots, \text{Ч}_k$ is $(\mathfrak{f}^* - \mathfrak{Q})$ quasi binormal operator of order $\mathfrak{y}$, we have

$$(Ч_j{}^*)^{\mathfrak{f}}(Ч_j{}^*Ч_jЧ_jЧ_j{}^*)^{\mathfrak{y}} = Ꙍ(Ч_j{}^*Ч_jЧ_jЧ_j{}^*)^{\mathfrak{y}}(Ч_j{}^*)^{\mathfrak{f}},\ \text{for all } j = 1,2,\dots,k.$$

Then

$$((Ч_1 \otimes Ч_2 \otimes \dots \otimes Ч_k)^*)^{\mathfrak{f}}((Ч_1 \otimes Ч_2 \otimes \dots \otimes Ч_k)^*(Ч_1 \otimes Ч_2 \otimes \dots \otimes Ч_k)(Ч_1 \otimes Ч_2 \otimes \dots \otimes Ч_k)(Ч_1 \otimes Ч_2 \otimes \dots \otimes Ч_k)^*)^{\mathfrak{y}}$$
$$(x_1 \otimes x_2 \otimes \dots \otimes x_k)$$
$$= [(Ч_1{}^*)^{\mathfrak{f}} \otimes (Ч_2{}^*)^{\mathfrak{f}} \otimes \dots \otimes (Ч_k{}^*)^{\mathfrak{f}}][((Ч_1 \otimes Ч_2 \otimes \dots \otimes Ч_k)^*]^{\mathfrak{y}}(Ч_1 \otimes Ч_2 \otimes \dots \otimes Ч_k)^{\mathfrak{y}}(Ч_1 \otimes Ч_2 \otimes \dots \otimes Ч_k)^{\mathfrak{y}}[(Ч_1 \otimes Ч_2 \otimes \dots \otimes Ч_k)^*)]^{\mathfrak{y}}$$
$$(x_1 \otimes x_2 \otimes \dots \otimes x_k)$$
$$= [(Ч_1{}^*)^{\mathfrak{f}} \otimes (Ч_2{}^*)^{\mathfrak{f}} \otimes \dots \otimes (Ч_k{}^*)^{\mathfrak{f}}][[(Ч_1{}^*)^{\mathfrak{y}} \otimes (Ч_2{}^*)^{\mathfrak{y}} \otimes \dots \otimes (Ч_k{}^*)^{\mathfrak{y}}](Ч_1{}^{\mathfrak{y}} \otimes Ч_2{}^{\mathfrak{y}} \otimes \dots \otimes Ч_k{}^{\mathfrak{y}})\ (Ч_1{}^{\mathfrak{y}} \otimes Ч_2{}^{\mathfrak{y}} \otimes \dots \otimes Ч_k{}^{\mathfrak{y}})$$
$$[(Ч_1{}^*)^{\mathfrak{y}} \otimes (Ч_2{}^*)^{\mathfrak{y}} \otimes \dots \otimes (Ч_k{}^*)^{\mathfrak{y}}](x_1 \otimes x_2 \otimes \dots \otimes x_k)$$
$$= [(Ч_1{}^*)^{\mathfrak{f}}((Ч_1{}^*)^{\mathfrak{y}}Ч_1{}^{\mathfrak{y}}Ч_1{}^{\mathfrak{y}}\ (Ч_1{}^*)^{\mathfrak{y}})\ ](x_1) \otimes [\ (Ч_2{}^*)^{\mathfrak{f}}((Ч_2{}^*)^{\mathfrak{y}}Ч_2{}^{\mathfrak{y}}Ч_2{}^{\mathfrak{y}}\ (Ч_2{}^*)^{\mathfrak{y}})\ ](x_2) \otimes \dots \otimes$$
$$[(Ч_k{}^*)^{\mathfrak{f}}((Ч_k{}^*)^{\mathfrak{y}}Ч_k{}^{\mathfrak{y}}Ч_k{}^{\mathfrak{y}}\ (Ч_k{}^*)^{\mathfrak{y}})\ (x_n)]$$
$$= [(Ч_1{}^*)^{\mathfrak{f}}((Ч_1{}^*Ч_1Ч_1Ч_1{}^*)^{\mathfrak{y}})\ ](x_1) \otimes [\ (Ч_2{}^*)^{\mathfrak{f}}(Ч_2{}^*Ч_2Ч_2Ч_2{}^*)^{\mathfrak{y}}\ ](x_2) \otimes \dots \otimes [(Ч_k{}^*)^{\mathfrak{f}}(Ч_k{}^*Ч_kЧ_kЧ_k{}^*)^{\mathfrak{y}}\ (x_k)]$$
$$= [\ Ꙍ((Ч_1{}^*Ч_1Ч_1Ч_1{}^*)^{\mathfrak{y}})\ (Ч_1{}^*)^{\mathfrak{f}}](x_1) \otimes [\ Ꙍ(Ч_2{}^*Ч_2Ч_2Ч_2{}^*)^{\mathfrak{y}}\ (Ч_2{}^*)^{\mathfrak{f}}](x_2) \otimes \dots \otimes [Ꙍ(Ч_k{}^*Ч_kЧ_kЧ_k{}^*)^{\mathfrak{y}}(\mathfrak{A}_n{}^*)^{\mathfrak{f}}(x_k)]$$
$$= [Ꙍ((Ч_1{}^*)^{\mathfrak{y}}Ч_1{}^{\mathfrak{y}}Ч_1{}^{\mathfrak{y}}\ (Ч_1{}^*)^{\mathfrak{y}})\ (Ч_1{}^*)^{\mathfrak{f}}](x_1) \otimes [\ Ꙍ(\ (Ч_2{}^*)^{\mathfrak{y}}Ч_2{}^{\mathfrak{y}}Ч_2{}^{\mathfrak{y}}\ (Ч_2{}^*)^{\mathfrak{y}})\ (Ч_2{}^*)^{\mathfrak{f}}](x_2) \otimes \dots \otimes$$
$$[Ꙍ((Ч_k{}^*)^{\mathfrak{y}}Ч_k{}^{\mathfrak{y}}Ч_k{}^{\mathfrak{y}}\ (Ч_k{}^*)^{\mathfrak{y}})\ (Ч_k{}^*)^{\mathfrak{f}}](x_k)$$
$$= [Ꙍ[(Ч_1{}^*)^{\mathfrak{y}} \otimes (Ч_2{}^*)^{\mathfrak{y}} \otimes \dots \otimes (Ч_k{}^*)^{\mathfrak{y}}](Ч_1{}^{\mathfrak{y}} \otimes Ч_2{}^{\mathfrak{y}} \otimes \dots \otimes Ч_k{}^{\mathfrak{y}})\ (Ч_1{}^{\mathfrak{y}} \otimes Ч_2{}^{\mathfrak{y}} \otimes \dots \otimes Ч_n{}^{\mathfrak{y}})\ [(Ч_1{}^*)^{\mathfrak{y}} \otimes (Ч_2{}^*)^{\mathfrak{y}} \otimes \dots \otimes (Ч_n{}^*)^{\mathfrak{y}}]$$
$$[(Ч_1{}^*)^{\mathfrak{f}} \otimes (Ч_2{}^*)^{\mathfrak{f}} \otimes \dots \otimes (Ч_k{}^*)^{\mathfrak{f}}](x_1 \otimes x_2 \otimes \dots \otimes x_k)$$
$$= Ꙍ((Ч_1 \otimes Ч_2 \otimes \dots \otimes Ч_k)^*(Ч_1 \otimes Ч_2 \otimes \dots \otimes Ч_k)(Ч_1 \otimes Ч_2 \otimes \dots \otimes Ч_k)(Ч_1 \otimes Ч_2 \otimes \dots \otimes Ч_k)^*)^{\mathfrak{y}}\ ((Ч_1 \otimes Ч_2 \otimes \dots \otimes Ч_k)^*)^{\mathfrak{f}}$$
$$(x_1 \otimes x_2 \otimes \dots \otimes x_k).$$

Hence, $(Ч_1 \otimes Ч_2 \otimes \dots \otimes Ч_k)$ is a $(\mathfrak{f}^* - Ꙍ)$ quasi binormal operator of order $\mathfrak{y}$.

References

[1] A. Brown, On a class of operators, *Proc. Amer. Math. Soc.* 4 (1953), 723-728. https://doi.org/10.1090/S0002-9939-1953-0059483-2

[2] S. K. Berberian, *Introduction to Hilbert Space*, 2nd ed., Chelsea Publishing Co., New York, 1976.

[3] Stephen L. Campbell, Linear operators for which T^*T and TT^* commute, *Proc. Amer. Math. Soc.* 34 (1972), 177-180. https://doi.org/10.2307/2037922

[4] J. B. Conway, A course in functional analysis, Graduate Texts in Mathematics, 96, *Springer-Verlag, New York*, 1985. https://doi.org/10.1007/978-1-4757-3828-5

[5] E. A. Kuffi and H. A. Satar, A new linear normal operators and its relations, *Journal of College of Education, Al-Mustansiriya University, Baghdad, Iraq* 5 (2014).