Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 10, Number 2, 2022, Pages 365-384
https://doi.org/10.34198/ejms.10222.365384

Certain Subclasses of Univalent Functions Linked with q-Chebyshev Polynomial

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Abstract

The solutions provided in this work address the classic but still relevant topic of establishing new classes of univalent functions linked to q-Chebyshev polynomials and examining coefficient estimates features. Aspects of quantum calculus are also considered in this research to make it more unique and produce more pleasing outcomes. We introduce new classes of univalent functions connected to q-Chebyshev polynomials, which generalize certain previously investigated classes. The link among the previously published findings and the current ones are noted. For each of the new classes, estimates for the Taylor-Maclaurin coefficients $|r_2|$ and $|r_3|$ are derived and the much-studied Fekete-Szegö functional.

1 Introduction and Definitions

The normal calculus is replaced by quantum calculus, which does not have the concept of limits. In mathematics and physics, it has a wide range of applications. Jackson [1, 2] introduces both q-derivative and q-integral as fundamental tools in

Received: June 30, 2022; Accepted: July 30, 2022
2020 Mathematics Subject Classification: Primary 05A30, 30C45; Secondary 11B65, 47B38.
Keywords and phrases: coefficient bounds, q-differential operator, q-Chebyshev polynomials, subordination, q-derivative, univalent functions.
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a methodical manner. In reality, the q-calculus theory can be used to describe univalent function theory. Furthermore, in recent years, fractional q-derivative operators and q-integral have been employed to create numerous subclasses of holomorphic functions using q-calculus operators (see for more details [3–11]). Purohit and Raina [9] looked at the usage of q-calculus fractional operators to define several holomorphic functions in $\mathbb{U}$.

Chebyshev polynomials are used to investigate a subclass of univalent functions in [12]. The authors computed the Taylor-Maclaurin coefficients a_2 and a_3 for functions in the class $\mathcal{L}(\alpha, t)$ in [12]. Chebyshev polynomials were utilized by Altinkaya et al. [13] to discover coefficient expansions for a vast subclass of univalent functions denoted by $\mathcal{K}(\lambda, t)$. Recently, the Komatu integral operator [14] was utilized to investigate a novel subclass of univalent functions as well as Chebyshev polynomials. Researchers on the Univalent function using the Chebyshev polynomial have recently contributed [15–19]. The q-analogs of second-order bivariate Chebyshev polynomials were established by Al Salem and Ismail [20]. Johann Cigler [21] developed the q-analogues of bivariate Chebyshev polynomials in 2012, which allows for straightforward generalizations of many features of classical univariate polynomials, as shown in (2.3), for more details see [22, 23]. The Univalent function, on the other hand, has not been investigated using q-Chebyshev polynomials, which have numerous applications in mathematics.

The function class is symbolized by the letter $\mathcal{A}$, which has the following representation:

$$f(z) = z + \sum_{\nu=2}^{\infty} r_\nu z^\nu, \quad (z \in \mathbb{U}), \tag{1.1}$$

that are holomorphic in the region $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$ and satisfy the following normalization conditions:

$$f(0) = 0 = f'(0) - 1.$$

In addition, we will refer to $\mathcal{S}$ as the class of all functions in $\mathcal{A}$ that are univalent in $\mathbb{U}$.

The following important univalence criterion was derived using inclusion relations for the subclass S_n^σ by the author in [24]:

Theorem 1.1. *Let $f \in \mathcal{A}$ satisfy*

$$\Re\left(\frac{2zf'(z)+z^2f''(z)}{f(z)+zf'(z)}\right) > 0. \tag{1.2}$$

Then $f(z)$ is starlike univalent in U.

Consider univalent normalized functions of the kind (1.1), the Fekete-Szegö functional $|r_3 - \phi r_2^2|$ has a long history in geometric function theory. The authors in [25] disproved Paley's conjecture and Littlewood that the coefficients of odd univalent functions are confined by unity in 1933. Since then, the functional has gotten a lot of attention, especially in subclasses of the univalent function family. This problem appears to have piqued the interest of scholars in recent years (see, for example, [26–31]).

Definition 1.1. For $q \in (0,1)$, the q-differentiation of function f can be defined as

$$\mathfrak{D}_q f(0) - f'(0) = 0,\ \mathfrak{D}_q f = (z(1-q))^{-1}[f(z) - f(zq)] \text{ and } \mathfrak{D}_q^2 f = \mathfrak{D}_q(\mathfrak{D}_q f)\ \ (z \neq 0). \tag{1.3}$$

Clearly, if we use (1.3) in place of (1.1), we get

$$\mathfrak{D}_q f(z) = z^{-1}\left[z + \sum_{\nu=2}^{\infty}[\nu]_q r_\nu z^\nu\right] \text{ and } \mathfrak{D}_q^2 f(z) = z^{-1}\left[\sum_{\nu=2}^{\infty}[\nu+1]_q[\nu]_q r_\nu z^{\nu-1}\right] \tag{1.4}$$

where $[\nu]_q = (-q+1)^{-1}(-q^\nu+1)$ and $\lim_{q\longrightarrow 1}[\nu]_q = \nu$.

For example, if $f(z) = z^\nu$, so applying (1.3),

$$\mathfrak{D}_q f(z) = \mathfrak{D}_q(z^\nu) = (1-q)^{-1}(1-q^\nu) \times z^{\nu-1} = [\nu]_q z^{\nu-1}$$

and

$$\lim_{q\longrightarrow 1}\mathfrak{D}_q f(z) = \lim_{q\longrightarrow 1}([\nu]_q z^{\nu-1}) = \nu z^{\nu-1} = f'(z).$$

To the best of our knowledge, there are no studies of q-Chebyshev polynomials connected to univalent functions in the literature. The primary purpose of this study was to investigate the properties of univalent functions associated with q-Chebyshev polynomials. The initial coefficient estimates for the Fekete-Szegö issue for univalent function subclasses $\mathcal{HB}(q,n,a)$, $\mathcal{HN}(q,n,a)$, and $\mathcal{HL}(q,n,a)$ are derived using the q-Chebyshev polynomial expansion in this study. Using particular q-Chebyshev polynomials, the authors concentrated on the bound of coefficient functionals for novel subclasses of univalent functions.

2 Bounds of the Coefficients and Fekete-Szegö Inequalities

If the following subordination holds, a function $f \in \mathcal{A}$ is considered to be in the class $\mathcal{HB}(q,n,a)$, $\frac{1}{2} < a < 1, 0 < q < 1, z \in \mathbb{C}$, $-1 \leq n \leq 2$, $z \in \mathbb{U}$,

$$\frac{2z\mathfrak{D}_q f(z) + z^2\mathfrak{D}_q^2 f(z)}{f(z) + z\mathfrak{D}_q f(z)} \prec \mho(q,n,a) \tag{2.1}$$

where $\mathfrak{D}_q$ is the q-differential operator and $\prec$ is the symbol for subordination [32]. Thus, we can write

$$\mho(q,n,a) = \sum_{\rho=0}^{\infty} H_\rho(q,n,a) z^\rho, \quad \left(\frac{1}{2} < a < 1, 0 < q < 1, -1 \leq n \leq 2\right), \tag{2.2}$$

where

$$\begin{aligned} H_\rho(q,n,a) &= P_{\rho+1}(q,-1,n,a)(-q;q)_\rho \\ &= \sum_{\nu=0}^{\frac{k}{2}} q^{\nu^2} \begin{bmatrix} \rho-\nu \\ \nu \end{bmatrix} (1+q^{\nu+1})\cdots(1+q^{\rho-\nu}) x^\nu t^{\rho-2\nu} \end{aligned} \tag{2.3}$$

are called q-Chebyshev polynomial of the second kind. We have

$$H_\rho(q,n,a) = (1+q^\rho) a H_{\rho-1}(q,n,a) + q^{\rho-1} n H_{\rho-2}(q,n,a)$$

and

$$\begin{aligned}
H_1(q,n,a) &= a + aq \\
H_2(q,n,a) &= nq + a^2 + a^2q^2 + qa^2 + a^2q^3 \\
H_3(q,n,a) &= nqa + nq^3a + nq^2a + nq^4a + a^3[q^6 + q^5 + q^4 + 2q^3 + q^2 + q + 1].
\end{aligned} \tag{2.4}$$

Remark 2.1. We can see that

$$H_\rho(1,-1,a) = H_\rho(a),$$

where $H_\rho(a)$ is the classical Chebyshev polynomial of the second kind.

The estimates on the Taylor-Maclaurin coefficients $|r_2|$ and $|r_3|$ for functions in the class $\mathcal{HB}(q,n,a)$ are determined.

Theorem 2.1. *Let $f(z) \in \mathcal{HB}(q,n,a)$. Then*

$$|r_2| \leq \frac{2a(1+q)}{1+2q}$$

and

$$|r_3| \leq \frac{2a^2\left(2q^3+2q^2+5q+3\right)}{(q+1)^2(1+2q)} + \frac{2a+2aq+nq}{q^3+3q^2+3q+1}$$

and for $\phi \in \mathbb{R}$

$$\left|r_3 - \phi r_2^2\right| \leq \begin{cases} \frac{2a}{(q+1)^2} \\ \text{for all } \phi \in [\phi_1,\phi_2], \\ \frac{2a}{(q+1)^2}\left|\frac{a^2(q+1)\left(q^2+1\right)+qn}{(1+q)a} + \frac{(2+q)(1+q)a}{2q+1} - \frac{2\phi a\left(q^3+3q^2+3q+1\right)(q+1)}{(2q+1)^2}\right| \\ \text{for all } \phi \notin [\phi_1,\phi_2], \end{cases}$$

where

$$\phi_1 = \frac{\begin{array}{r}3a^2 + 4a^2q^3 + 7a^2q^2 + 8a^2q + 2a^2q^4 - a - 5aq + nq \\ - 8aq^2 + 4nq^2 - 4aq^3 + 4nq^3\end{array}}{2a^2q^5 + 10a^2q^4 + 20a^2q^3 + 20a^2q^2 + 10a^2q + 2a^2},$$

$$\phi_2 = \frac{\begin{array}{r}3a^2 + 4a^2q^3 + 7a^2q^2 + 8a^2q + 2a^2q^4 + a + 5aq + nq \\ + 8aq^2 + 4nq^2 + 4aq^3 + 4nq^3\end{array}}{2a^2q^5 + 10a^2q^4 + 20a^2q^3 + 20a^2q^2 + 10a^2q + 2a^2}.$$

Proof. Let $f(z) \in \mathcal{HB}(q, n, a)$. From (2.1), we get

$$\frac{2z\mathfrak{D}_q f(z) + z^2\mathfrak{D}_q^2 f(z)}{f(z) + z\mathfrak{D}_q f(z)} = 1 + H_1(q, n, a)\varpi(z) + H_2(q, n, a)\varpi^2(z) + \cdots, \quad (2.5)$$

for some homorphic functions ϖ such that $\varpi(0) = 0$ and $|\varpi| < 1$, for all $z \in \mathbb{U}$. Taking it from (2.5), we get

$$\frac{2z\mathfrak{D}_q f(z) + z^2\mathfrak{D}_q^2 f(z)}{f(z) + z\mathfrak{D}_q f(z)} = 1 + H_1(q, n, a)s_1 z + [H_1(q, n, a)s_2 + H_2(q, n, a)s_1^2]z^2 + \cdots. \quad (2.6)$$

If $|\varpi(z)| = |s_1 z + s_2 z^2 + s_3 z^3 + \cdots| < 1$ and $z \in \mathbb{U}$, it is a well-known fact that

$$|s_\varsigma| \leq 1, \text{ for all } \varsigma \in \mathbb{N}, \quad (2.7)$$

and

$$|s_2 - \xi s_1^2| \leq \max\{1 + |\xi|\}, \text{ for all } \xi \in \mathbb{R}. \quad (2.8)$$

As a result of (2.6), it follows that

$$\frac{1 + 2q}{2} r_2 = H_1(q, n, a)s_1, \quad (2.9)$$

$$\frac{q^3 + 3q^2 + 3q + 1}{2} r_3 - \frac{2q^2 + 5q + 2}{4} r_2^2 = H_1(q, n, a)s_2 + H_2(q, n, a)s_1^2. \quad (2.10)$$

From (2.4) and (2.9), we get

$$|r_2| \leq \frac{2a(1 + q)}{1 + 2q}.$$

Then, usimg (2.9) in (2.10), we can find the bound on $|r_3|$.

$$r_3 = \frac{2H_1(q,n,a)s_2}{q^3+3q^2+3q+1} + \frac{2H_2(q,n,a)s_1^2}{q^3+3q^2+3q+1} + \frac{(4+10q+4q^2)H_1^2(q,n,a)s_1^2}{4q^5+16q^4+25q^3+19q^2+7q+1}. \tag{2.11}$$

In light of (2.4) and (2.7), we have (2.11)

$$|r_3| \leq \frac{2a^2\left(2q^3+2q^2+5q+3\right)}{(q+1)^2(1+2q)} + \frac{2a+2aq+nq}{q^3+3q^2+3q+1}.$$

From (2.9) and (2.11), for $\phi \in \mathbb{R}$, we get

$$|r_3 - \phi r_2^2| = \frac{2H_1(q,n,a)}{q^3+3q^2+3q+1}\left| s_2 + \left\{ \frac{H_2(q,n,a)}{H_1(q,n,a)} + \frac{2+q}{1+2q}H_1(q,n,a) - 2\phi\frac{q^3+3q^2+3q+1}{(1+2q)^2}H_1(q,n,a) \right\} s_1^2 \right|.$$

In light of (2.8), we arrive to the following conclusion:

$$|r_3 - \phi r_2^2| \leq \frac{2H_1(q,n,a)}{q^3+3q^2+3q+1}\max\left\{1, \left| \frac{H_2(q,n,a)}{H_1(q,n,a)} + \frac{2+q}{1+2q}H_1(q,n,a) - 2\phi\frac{q^3+3q^2+3q+1}{(1+2q)^2}H_1(q,n,a) \right| \right\}. \tag{2.12}$$

Finally, by applying (2.4) in (2.12), we have

$$|r_3 - \phi r_2^2| \leq \frac{2a}{(q+1)^2}\max\left\{1, \left| \frac{a^2(1+q)\left(1+q^2\right)+qn}{(1+q)a} + \frac{(2+q)(1+q)a}{1+2q} - \frac{2\phi a\left(q^3+3q^2+3q+1\right)(1+q)}{(1+2q)^2} \right| \right\}. \tag{2.13}$$

Because a is greater than zero, we have

$$\left| \frac{a^2(1+q)\left(1+q^2\right)+qn}{(1+q)a} + \frac{(2+q)(1+q)a}{1+2q} - \frac{2\phi a\left(q^3+3q^2+3q+1\right)(1+q)}{(1+2q)^2} \right| \leq 1$$

$$\iff \phi_1 \leq \phi \leq \phi_2.$$

□

Theorem 2.1 leads us to Corollary 2.1 in the particular case where $q = 1$ and $n = -1$.

Corollary 2.1. *Let $f(z) \in \mathcal{HB}(a)$. Then*

$$|r_2| \leq \frac{4a}{3}$$

and

$$|r_3| \leq \frac{16a^2 + 4a - 1}{8},$$

and for $\phi \in \mathbb{R}$

$$\left|r_3 - \phi r_2^2\right| \leq \begin{cases} \frac{a}{2} & \text{for all } \phi \in [\phi_1, \phi_2], \\ \frac{a}{2}\left|\frac{4a^2-1}{2a} + 2a - \phi\frac{32}{9}\right| & \text{for all } \phi \notin [\phi_1, \phi_2], \end{cases}$$

where

$$\phi_1 = \frac{24a^2 - 18a - 9}{64a^2}, \quad \phi_2 = \frac{24a^2 + 18a - 9}{64a^2}.$$

Consequently, assuming the following subordination holds true, a function $f \in \mathcal{A}$ is considered to be in the class $\mathcal{HN}(q, n, a)$, $\frac{1}{2} < a < 1, 0 < q < 1, z \in \mathbb{C}$, $-1 \leq n \leq 2$, $z \in \mathbb{U}$,

$$\left(\frac{z\mathfrak{D}_q f(z)}{f(z)}\right)^{\beta} \left(\frac{\mathfrak{D}_q f(z) + z\mathfrak{D}_q^2 f(z)}{\mathfrak{D}_q f(z)}\right)^{1-\beta} \prec \mho(q, n, a). \tag{2.14}$$

Where $\mathfrak{D}_q$ is the q-differential operator and $\mho(q, n, a)$ is given by (2.2).

Theorem 2.2. *Let $f(z) \in \mathcal{HN}(q, n, a, \beta)$. Then*

$$|r_2| \leq \frac{aq + a}{1 - \beta + q}$$

and

$$|r_3| \leq \frac{2a^2\left(q^2+1\right)(1-\beta+q)^2+\left(2q^2-2q\beta+4q-\beta^2-\beta-2q^2\beta+2\right)^2(1+q)\,a^2}{2\,(1-\beta+q)^2\,(q^2+q-\beta-q^2\beta+1)} + \frac{a\,(1+q)+qn}{(1+q)\,(q^2+q-\beta-q^2\beta+1)},$$

and for $\phi \in \mathbb{R}$

$$\left|r_3-\phi r_2^2\right| \leq \begin{cases} \frac{a}{q^2+q-\beta-q^2\beta+1} \\ \text{for all } \phi \in [\phi_1,\phi_2], \\ \frac{a}{q^2+q-\beta-q^2\beta+1}\left|\frac{a^2(1+q)\left(1+q^2\right)+qn}{(1+q)a}+\frac{\left(2q^2-2q\beta+4q-\beta-\beta^2-2q^2\beta+2\right)(1+q)a}{2(1+q-\beta)^2} \right. \\ \left. -\frac{a\phi(q+1)^2\left(q^2+q-\beta-q^2\beta+1\right)}{(q+1-\beta)^2}\right| \\ \text{for all } \phi \notin [\phi_1,\phi_2], \end{cases}$$

where

$$\phi_1 = \frac{(q+1)a^2[2(q^2+1)(q+1-\beta)^2+(2q^2-2q\beta-\beta^2-2q^2\beta+4q-\beta+2)(q+1)a]-2(q+1-\beta)^2[qa+a-qn]}{2(q+1)^3a^2[(1+q+q^2)(1-\beta)+q\beta]},$$

$$\phi_2 = \frac{(1+q)a^2[2(q^2+1)(1+q-\beta)^2+(2q^2-2q\beta-\beta^2-2q^2\beta+4q-\beta+2)(q+1)a]-2(q+1-\beta)^2[qa+a+qn]}{2(q+1)^3a^2[(1+q+q^2)(1-\beta)+q\beta]}.$$

Proof. Let $f(z) \in \mathcal{HN}(q,n,a,\beta)$. From (2.14), we get

$$\left(\frac{z\mathfrak{D}_q f(z)}{f(z)}\right)^{\beta}\left(\frac{\mathfrak{D}_q f(z)+z\mathfrak{D}_q^2 f(z)}{\mathfrak{D}_q f(z)}\right)^{1-\beta} = 1+H_1(q,n,a)\varpi(z)+H_2(q,n,a)\varpi^2(z)+\cdots, \quad (2.15)$$

for some homorphic functions ϖ such that $\varpi(0) = 0$ and $|\varpi| < 1$, for all $z \in \mathbb{U}$. Taking it from (2.15), we get

$$\left(\frac{z\mathfrak{D}_q f(z)}{f(z)}\right)^{\beta}\left(\frac{\mathfrak{D}_q f(z) + z\mathfrak{D}_q^2 f(z)}{\mathfrak{D}_q f(z)}\right)^{1-\beta} = 1 + H_1(q,n,a)m_1 z + [H_1(q,n,a)m_2 + H_2(a,n,q)m_1^2]z^2 + \cdots . \tag{2.16}$$

If $|\varpi(z)| = |m_1 z + m_2 z^2 + m_3 z^3 + \cdots| < 1$ and $z \in \mathbb{U}$, it is a well-known fact that

$$|m_\varsigma| \leq 1, \quad \text{for all } \varsigma \in \mathbb{N}, \tag{2.17}$$

and

$$|m_2 - \zeta m_1^2| \leq \max\{1 + |\zeta|\}, \quad \text{for all } \zeta \in \mathbb{R}. \tag{2.18}$$

As a result of (2.16), it follows that

$$(1 + q - \beta)r_2 = H_1(q,n,a)m_1, \tag{2.19}$$

$$(q^3+2q^2+2q-\beta-q^2\beta-q\beta-q^3\beta+1)r_3 - \frac{2q^2-2q\beta+4q-\beta-\beta^2-2q^2\beta+2}{2}r_2^2 = H_1(q,n,a)m_2 + H_2(q,n,a)m_1^2. \tag{2.20}$$

From (2.4) and (2.19), we get

$$|r_2| \leq \frac{aq+a}{1-\beta+q}.$$

Then, usimg (2.19) in (2.20), we can find the bound on $|r_3|$.

$$r_3 = \frac{H_1(q,n,a)m_2}{(1+q)(q^2+q-\beta-q^2\beta+1)} + \frac{H_2(q,n,a)m_1^2}{(1+q)(q^2+q-\beta-q^2\beta+1)} + \frac{(2q^2-2q\beta+4q-\beta-\beta^2-2q^2\beta+2)H_1^2(q,n,a)m_1^2}{2(1+q-\beta)^2(q^2+q-\beta-q^2\beta+1)(1+q)}. \tag{2.21}$$

In light of (2.4) and (2.17), we have (2.21)

$$|r_3| \leq \frac{2a^2\left(q^2+1\right)(1+q-\beta)^2 + \left(2q^2-2q\beta+4q-\beta^2-\beta-2q^2\beta+2\right)^2(1+q)a^2}{2(1-\beta+q)^2(q^2+q-\beta-q^2\beta+1)} + \frac{a(1+q)+qn}{(1+q)(q^2+q-\beta-q^2\beta+1)}.$$

From (2.19) and (2.21), for $\phi \in \mathbb{R}$, we get

$$|r_3 - \phi r_2^2| = \frac{H_1(q,n,a)}{(1+q)\left(q^2+q-\beta-q^2\beta+1\right)} \Bigg| m_2 + \Bigg\{ \frac{H_2(q,n,a)}{H_1(q,n,a)} + \frac{2q^2-2q\beta+4q-\beta-\beta^2-2q^2\beta+2}{2\left(1+q-\beta\right)^2} H_1(q,n,a) - (1+q)\phi \frac{q^2+q-\beta-q^2\beta+1}{\left(1+q-\beta\right)^2} H_1(q,n,a) \Bigg\} m_1^2 \Bigg|.$$

In light of (2.18), we arrive to the following conclusion:

$$|r_3 - \phi r_2^2| \leq \frac{H_1(q,n,a)}{(1+q)\left(q^2+q-\beta-q^2\beta+1\right)} \max \Bigg\{ 1, \Bigg| \frac{H_2(q,n,a)}{H_1(q,n,a)} + \frac{2q^2-2q\beta+4q-\beta-\beta^2-2q^2\beta+2}{2\left(1+q-\beta\right)^2} H_1(q,n,a) - (1+q)\phi \frac{q^2+q-\beta-q^2\beta+1}{\left(1+q-\beta\right)^2} H_1(q,n,a) \Bigg| \Bigg\}. \tag{2.22}$$

Finally, by applying (2.4) in (2.22), we have

$$|r_3 - \phi r_2^2| \leq \frac{a}{q^2+q-\beta-q^2\beta+1} \max \Bigg\{ 1, \Bigg| \frac{a^2(1+q)\left(1+q^2\right)+qn}{(1+q)\,a} + \frac{\left(2q^2-2q\beta+4q-\beta-\beta^2-2q^2\beta+2\right)(1+q)\,a}{2\left(1+q-\beta\right)^2} - \frac{a\phi\,(q+1)^2\left(q^2+q-\beta-q^2\beta+1\right)}{\left(q+1-\beta\right)^2} \Bigg| \Bigg\}. \tag{2.23}$$

Because a is greater than zero, we have

$$\left| \frac{a^2(1+q)\left(1+q^2\right)+qn}{(1+q)\,a} + \frac{\left(2q^2-2q\beta+4q-\beta-\beta^2-2q^2\beta+2\right)(1+q)\,a}{2\left(1+q-\beta\right)^2} - \frac{a\phi\,(q+1)^2\left(q^2+q-\beta-q^2\beta+1\right)}{\left(q+1-\beta\right)^2} \right| \leq 1$$

$$\iff \phi_1 \leq \phi \leq \phi_2.$$

□

Theorem 2.2 leads us to Corollary 2.2 in the particular case where $\beta = 0$.

Corollary 2.2. *Let $f(z) \in \mathcal{HN}(q,n,a)$. Then*

$$|r_2| \leq a$$

and

$$|r_3| \leq \frac{a^2q^3 + 2a^2q^2 + 3a^2q + 2a^2 + aq + a + nq}{(q+1)\left(q\left(q+1\right)+1\right)},$$

and for $\phi \in \mathbb{R}$

$$\left|r_3 - \phi r_2^2\right| \leq \begin{cases} \frac{a}{1+q+q^2} \\ \\ \textit{for all } \ \phi \in [\phi_1, \phi_2], \\ \\ \frac{a}{1+q+q^2}\left|\frac{2a^2+2a^2q^2+3a^2q+a^2q^3+nq-a^2\phi-a^2q^3\phi-2a^2q\phi-2a^2q^2\phi}{(1+q)a}\right| \\ \\ \textit{for all } \ \phi \notin [\phi_1, \phi_2]. \end{cases}$$

Remark 2.2. Putting $q = 1$ and $n = -1$ in Corollary 2.2, we have Corollary 4 in [12].

Consequently, assuming the following subordination holds true, a function $f \in \mathcal{A}$ is considered to be in the class $\mathcal{HL}(q,n,a)$, $\frac{1}{2} < a < 1, 0 < q < 1, z \in \mathbb{C}$, $-1 \leq n \leq 2$, $z \in \mathbb{U}$,

$$\left(\frac{z\mathfrak{D}_q f(z)}{f(z)}\right)(1-\mu) + \mu\left(\frac{\mathfrak{D}_q f(z) + z\mathfrak{D}_q^2 f(z)}{\mathfrak{D}_q f(z)}\right) \prec \mho(q,n,a). \tag{2.24}$$

Where $\mathfrak{D}_q$ is the q-differential operator and $\mho(q,n,a)$ is given by (2.2).

Theorem 2.3. *Let $f(z) \in \mathcal{HL}(q,n,a,\mu)$. Then*

$$|r_2| \leq \frac{a(1+q)}{q+\mu}$$

and

$$|r_3| \leq \frac{q^2+q^4+2q\mu+2q^3\mu+\mu^2+\mu^2q^2+\mu a^2+qa^2+q^2a^2 + a^2q^2\mu+2q\mu a^2+q^3\mu a^2+q^2\mu a^2}{(1+q)^2\left(q+\mu+q^2\mu\right)} + \frac{a\left(1+q\right)+qn}{\left(1+q\right)\left(q+\mu+q^2\mu\right)}.$$

and for $\phi \in \mathbb{R}$

$$\left|r_3-\phi r_2^2\right| \leq \begin{cases} \frac{a}{q+\mu+q^2\mu} \\ \textit{for all } \ \phi \in [\phi_1,\phi_2], \\ \frac{a}{q+\mu+q^2\mu}\left|\frac{a\mu+aq+aq^2+2aq^2\mu+2aq\mu+aq^3\mu-aq^3\phi-2aq^2\phi - aq\phi-a\mu\phi-2aq^2\mu\phi-aq^4\mu\phi-2aq\mu\phi-2aq^3\mu\phi}{(q+\mu)^2} + \frac{a^2(1+q)\left(1+q^2\right)+qn}{(1+q)a}\right| \\ \textit{for all } \ \phi \notin [\phi_1,\phi_2], \end{cases}$$

where

$$\phi_1 = \frac{a^2q^4+a^2\mu^2+a^2\mu+qa^2+q^5a^2+q^2a^2+2a^2q^2+a^2\mu^2q^2+2q^3a^2+q^3a^2\mu^2+qa^2\mu^2 +5qa^2\mu+3a^2q^3\mu+2a^2q^2\mu+4q^2a^2\mu+3q^4a^2\mu+2q^3a^2\mu-q^2a-q^3a+nq^3-2qa\mu -2q^2a\mu+2nq^2\mu-a\mu^2-qa\mu^2+nq\mu^2}{(1+q)^3a^2\left(q+\mu+q^2\mu\right)},$$

$$\phi_2 = \frac{a^2q^4+a^2\mu^2+a^2\mu+qa^2+q^5a^2+q^2a^2+2a^2q^2+a^2\mu^2q^2+2q^3a^2+q^3a^2\mu^2+qa^2\mu^2 +5qa^2\mu+3a^2q^3\mu+2a^2q^2\mu+4q^2a^2\mu+3q^4a^2\mu+2q^3a^2\mu+q^2a+q^3a+nq^3 +2qa\mu+2q^2a\mu+2nq^2\mu+a\mu^2+qa\mu^2+nq\mu^2}{(1+q)^3a^2\left(q+\mu+q^2\mu\right)}.$$

Proof. Let $f(z) \in \mathcal{HL}(q, n, a, \mu)$. From (2.24), we get

$$\left(\frac{z\mathfrak{D}_q f(z)}{f(z)}\right)(1-\mu) + \mu\left(\frac{\mathfrak{D}_q f(z) + z\mathfrak{D}_q^2 f(z)}{\mathfrak{D}_q f(z)}\right) = 1 + H_1(q,n,a)\varpi(z) + H_2(q,n,a)\varpi^2(z) + \cdots, \tag{2.25}$$

for some homorphic functions ϖ such that $\varpi(0) = 0$ and $|\varpi| < 1$, for all $z \in \mathbb{U}$. Taking it from (2.25), we get

$$\left(\frac{z\mathfrak{D}_q f(z)}{f(z)}\right)(1-\mu) + \mu\left(\frac{\mathfrak{D}_q f(z) + z\mathfrak{D}_q^2 f(z)}{\mathfrak{D}_q f(z)}\right) = 1 + H_1(q,n,a)b_1 z + [H_1(q,n,a)b_2 + H_2(q,n,a)b_1^2]z^2 + \cdots. \tag{2.26}$$

If $|\varpi(z)| = |b_1 z + b_2 z^2 + b_3 z^3 + \cdots| < 1$ and $z \in \mathbb{U}$, it is a well-known fact that

$$|b_\iota| \leq 1, \quad \text{for all } \iota \in \mathbb{N}, \tag{2.27}$$

and

$$|b_2 - \kappa b_1^2| \leq \max\{1 + |\kappa|\}, \quad \text{for all } \kappa \in \mathbb{R}. \tag{2.28}$$

As a result of (2.26), it follows that

$$(q+\mu)r_2 = H_1(q,n,a)b_1, \tag{2.29}$$

$$(1+q)\left(q+\mu+q^2\mu\right)r_3 + (-qu - q - u - q^2 u)r_2^2 = H_1(q,n,a)b_2 + H_2(q,n,a)b_1^2. \tag{2.30}$$

From (2.4) and (2.29), we get

$$|r_2| \leq \frac{a(1+q)}{q+\mu}.$$

Then, usimg (2.29) in (2.30), we can find the bound on $|r_3|$.

$$r_3 = \frac{H_1(q,n,a)b_2}{(1+q)\left(q+\mu+q^2\mu\right)} + \frac{H_2(q,n,a)b_1^2}{(1+q)\left(q+\mu+q^2\mu\right)} + \frac{(\mu+q^2\mu+q\mu+q)H_1^2(q,n,a)b_1^2}{(q+\mu)^2(1+q)\left(q+\mu+q^2\mu\right)}. \tag{2.31}$$

In light of (2.4) and (2.27), we have (2.31)

$$|r_3| \leq \frac{\begin{array}{r} q^2+q^4+2q\mu+2q^3\mu+\mu^2+\mu^2q^2+\mu a^2+qa^2+q^2a^2 \\ +a^2q^2\mu+2q\mu a^2+q^3\mu a^2+q^2\mu a^2 \end{array}}{(1+q)^2\left(q+\mu+q^2\mu\right)} + \frac{a(1+q)+qn}{(1+q)\left(q+\mu+q^2\mu\right)}.$$

From (2.29) and (2.31), for $\phi \in \mathbb{R}$, we get

$$|r_3-\phi r_2^2| = \frac{H_1(q,n,a)}{(1+q)\left(q+\mu+q^2\mu\right)}\left|b_2+\left\{\frac{H_2(q,n,a)}{H_1(q,n,a)}+ \frac{\mu+q^2\mu+q\mu+q}{(q+\mu)^2}H_1(q,n,a)-\frac{(1+q)\,\phi\left(q+\mu+q^2\mu\right)}{(q+\mu)^2}H_1(q,n,a)\right\}b_1^2\right|.$$

In light of (2.28), we arrive to the following conclusion:

$$|r_3-\phi r_2^2| \leq \frac{H_1(q,n,a)}{(1+q)\left(q+\mu+q^2\mu\right)}\max\left\{1,\left|\frac{H_2(q,n,a)}{H_1(q,n,a)}+ \frac{\mu+q^2\mu+q\mu+q}{(q+\mu)^2}H_1(q,n,a)-\frac{(1+q)\,\phi\left(q+\mu+q^2\mu\right)}{(q+\mu)^2}H_1(q,n,a)\right|\right\}. \tag{2.32}$$

Finally, by applying (2.4) in (2.32), we have

$$|r_3-\phi r_2^2| \leq \frac{a}{q+\mu+q^2\mu}\max\left\{1,\left|\frac{a^2+a^2q^2+a^2q+a^2q^3+nq}{a+aq}+ \frac{\begin{array}{r} a\mu+aq+aq^2+2aq^2\mu+2aq\mu+aq^3\mu-aq^3\phi \\ -2aq^2\phi-aq\phi-a\mu\phi-2aq^2\mu\phi-aq^4\mu\phi-2aq\mu\phi-2aq^3\mu\phi \end{array}}{(q+\mu)^2}\right|\right\}.$$

Because a is greater than zero, we have

$$\left|\frac{a^2+a^2q^2+a^2q+a^2q^3+nq}{a+aq}+ \frac{\begin{array}{r} a\mu+aq+aq^2+2aq^2\mu+2aq\mu+aq^3\mu-aq^3\phi \\ -2aq^2\phi-aq\phi-a\mu\phi-2aq^2\mu\phi-aq^4\mu\phi-2aq\mu\phi-2aq^3\mu\phi \end{array}}{(q+\mu)^2}\right| \leq 1$$

$$\iff \phi_1 \leq \phi \leq \phi_2.$$

□

Theorem 2.3 leads us to Corollary 2.3 in the particular case where $\mu = 1$.

Corollary 2.3. *Let $f(z) \in \mathcal{HN}(q, n, a)$. Then*

$$|r_2| \leq a$$

and

$$|r_3| \leq \frac{a^2\left(q^2+q+2\right)}{1+q+q^2} + \frac{a\,(1+q)+qn}{(1+q)\,(1+q+q^2)}$$

and for $\phi \in \mathbb{R}$

$$\left|r_3 - \phi r_2^2\right| \leq \begin{cases} \frac{a}{1+q+q^2} \\ \text{for all } \phi \in [\phi_1, \phi_2], \\ \frac{a}{1+q+q^2}\left|\frac{2a^2+2a^2q^2+3a^2q+a^2q^3+nq-a^2\phi-a^2q^3\phi-2a^2q\phi-2a^2q^2\phi}{(1+q)a}\right| \\ \text{for all } \phi \notin [\phi_1, \phi_2]. \end{cases}$$

Remark 2.3. Putting $q = 1$ and $n = -1$ in Corollary 2.3, we have Corollary 3.2 in [13].

3 Conclusion

The q-calculus covers a wide range of topics including differential equations, quantum group theory, analytic number theory, special polynomials, combinatorics, special functions, quantum theory, and other related theories. We constructed three new classes of univalent functions linked to the q-Chebyshev polynomial, $\mathcal{HB}(q,n,a)$, $\mathcal{HN}(q,n,a)$, and $\mathcal{HL}(q,n,a)$, building on the work of Altnkaya and Yalcin [13]. For the three new classes, we found the Fekete-Szegö problem and coefficient estimates $|r_2|$ and $|r_3|$. More research is needed to improve the sharpness of the boundaries of the coefficient estimates produced here. The findings are intriguing since they incorporate quantum calculus into the study, which is a common strategy in recently published and acknowledged articles.

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