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Construction of Lyapunov Functions for the Stability of Sixth Order Ordinary Differential Equation

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Abstract

This study employed Lyapunov function method to investigate the stability of nonlinear ordinary differential equations. Using Lyapunov direct method, we constructed Lyapunov function to investigate the stability of sixth order nonlinear ordinary differential equations. We find $V(x)$, a quadratic form, positive definite and $U(x)$ which is also positive definite was chosen such that the derivative of $V(x)$ with respect to time was equal to the negative value of $U(x)$.

1 Introduction

In real life, most problems that occur are non-linear in nature and may not have analytic solutions except by approximations or stimulations and so trying to find an explicit solution may in general be complicated and sometimes impossible. Lyapunov functions are useful tools in determining stability, asymptotic stability, uniform stability, global stability or out-right instability of differential system and boundedness of solution of a real scalar fourth-order differential equation [1-4]. Asymptotic stability is intimately linked to the existence of a Lyapunov's function, that is, a proper, non-negative function varnishing only on an invariant set and decreasing along those curved paths of the system not evolving in the

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invariant set. Lyapunov theorem allows stability of linear and nonlinear system to be verified without differential equations solution being required. The presence of Lyapunov function implies asymptotic stability for linear time-invariant systems [9].

The concept of stability in problems arising from theory and application of differential equations is very important and an effective approach is the second approach of Lyapunov [7]. The method of Lyapunov functions was introduced by Aleksandra M. Lyapunov, a Russian Mathematician. The fundamental of his proof was centred on the established fact that the sum of energy in a system is decreasing or constant as it approaches state of equilibrium. Lyapunov functions have been constructed for linear equations on the platform that given any that is definite positive, we have another definite positive function U such that $-U = V^*$ and for the nonlinear case, a correlation is taken between the constant coefficient equations of linear and nonlinear equations which leads to the appropriate Lyapunov functions for the nonlinear case [1], [5], [6], [8]. Many authors have obtained useful and valid results using Lyapunov second method (direct method) for stability analysis and construction of appropriate Lyapunov function for some differential equations [3], [5]. This paper is motivated by reviewing [11] where the authors constructed Lyapunov function for fifth order differential equation, this work extended [11] to sixth order differential equation.

2 Statement of Problems, Preliminaries and Definitions

Consider the sixth-order differential equation

$$x^{(6)} + ax^{(5)} + bx^{(4)} + c\dddot{x} + d\ddot{x} + e\dot{x} + fx = 0, \tag{1}$$

where a, b, c, d, e and f are constants with $a > 0, b > 0, c > 0, d > 0, e > 0$ and $f > 0$. The equation (1) is equivalent to the following six system of equations:

$$\dot{x} = y$$
$$\dot{y} = z$$
$$\dot{z} = w$$
$$\dot{z} = r$$
$$\dot{r} = s$$
$$\dot{s} = -as - br - cw - dz - ey - fx. \tag{2}$$

The system (2) has negative real parts if and only if $a > 0, b > 0, c > 0, d > 0, e > 0$ and $f > 0$. There is therefore need to have a positive definite continuous quadratic function V and another positive quadratic form V such that

$$\dot{V} = -U \tag{3}$$

along the solution paths of (1) or (2). Before now, the result in equation (3) has been extended and is established to hold for positive semi definite quadratic $U(x)$ as well. It is our interest therefore to construct a Lyapunov function that would ultimately satisfy equation (3).

Lyapunov's Direct (Second) Method

Given a set of nonlinear first order differential equations

$$\dot{x}_i = f_i(x_1, x_2, ..., x_n) \text{ for } i = 1, 2, ..., n. \tag{4}$$

where $x_i = x_i(t)$ for some t and $\dot{x}_i$ stands for the time derivative of x_i for $i = 1, 2, ..., n$. Whereas f_i are analytic functions such that $f_i(0, ..., 0) = 0$ for $i = 1, 2, ..., n$ so that the origin $x = 0$ is an equilibrium point.

Lyapunov Test Function

For a function, $V(x)$, where $x = (x_1, x_2, ..., x_n)$, if the following conditions are satisfied:

(i) $V(x)$ and $\frac{\partial V}{\partial x_i}$ are continuous, for all $x \in \mathbb{R}^n$ and $i = 1, 2, ..., n$, not necessarily at the origin.
(ii) $V(0) = 0$.
Then we say that $V(x)$ is a possible Lyapunov test function for system (4).

Definition

i. A continuous function $V(x,t) = V(x_1, x_2, ..., x_n, t)$ is *positive definite* if $\lim_{|x| \longrightarrow 0} V(x,t) = 0$ and there exist $\varphi(\|x\|)$ such that

$$V(x,t) > \varphi(\|x\|).$$

ii. A continuous function $V(x,t) = V(x_1, x_2, ..., x_n, t)$ is *positive semi-definite* if $\lim_{|x| \longrightarrow 0} V(x,t) = 0$ and there exist $\varphi(\|x\|)$ such that

$$V(x,t) \leq \varphi(\|x\|).$$

iii. A continuous function $V(x,t) = V(x_1, x_2, ..., x_n, t)$ is *negative definite* if $\lim_{|x| \longrightarrow 0} V(x,t) = 0$ and there exist $\varphi(\|x\|)$ such that

$$V(x,t) < -\varphi(\|x\|).$$

iv. A continuous function $V(x,t) = V(x_1, x_2, ..., x_n, t)$ is *negative semi-definite* if $\lim_{|x| \longrightarrow 0} V(x,t) = 0$ and there exist $\varphi(\|x\|)$ such that

$$V(x,t) \leq -\varphi(\|x\|).$$

v. A continuous function $V(x,t) = V(x_1, x_2, ..., x_n, t)$ is *indefinite* if it assumes both positive and negative values in an arbitrary neighbourhood of the origin in a domain D.
vi. A continuous function $v : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be *radially unbounded* if is positive definite and $v(x) \longrightarrow \infty$ as $\|x\| \longrightarrow \infty$.
vii. A continuous function $V(x,t) = V(x_1, x_2, ..., x_n, t)$ is said to be *decresent* if

a positive definite function v such that

$$|V(t,x)| \geq v(x), \forall t \geq 0 \text{ and } \forall\ x \in B(r),\ r > 0$$

In another Lyapunov theorem, we reaffirm the definitions in this context. Given a differential equation

$$\dot{x} = f(t,x),\ f(t,0) = 0, \tag{5}$$

where f is continuous in (t,x).

Theorem (Sufficient Conditions for Stability)

The equilibrium point $x = 0$ of equation (5) is stable if $\exists$ a C^1 function V which is positive definite and such that its derivative along the solution of (9) is negative semi-definite, or identically zero (i.e. $\dot{V}(t,x) \leq 0$ or $\dot{V}(t,x) \equiv 0$).

Theorem (Sufficient Conditions for Asymptotically Stability)

The trivial solution $x = 0$ of the equation (5) is asymptotically stable, if $\exists$ a C^1 function V which is positive definite and whose derivative along the solution of (5) is negative definite.

Theorem (Lassale's Invariant Principle)

Assume that $V(x)$ is a Lyapunov function of (5) on a subset $G \subset \mathbb{R}^n$, $n \geq 1$. Define $S = x \in \overline{G} : V(x) = 0$, where $\overline{G}$ is the closure of G. Let M be maximal subset S. Then for $t \leq 0$, every bounded trajectory of (5) that remains in G approaches the set M as $t \longrightarrow +\infty$.

3 Methodology and Discussion

The system under investigation is

$$x^{(6)} + ax^{(5)} + bx^{(4)} + c\dddot{x} + d\ddot{x} + e\dot{x} + fx = 0.$$

The above sixth order differential equation can be expressed in a compact form as:

$$\dot{X} = AX = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ -f & -e & -d & -c & -b & -a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \\ r \\ s \end{pmatrix} \tag{6}$$

where $a > 0, b > 0, c > 0, d > 0, e > 0, f > 0$ for the system to have a negative real path. The required quadratic form in this case is given as

$$\begin{aligned} 2\dot{V} = {} & k_1x^2 + k_2y^2 + k_3z^2 + k_4w^2 + k_5r^2 + k_6s^2 + 2k_7xy + 2k_8xz + 2k_9xw \\ & + 2k_{10}xr + 2k_{11}xs + 2k_{12}yz + 2k_{13}yw + 2k_{14}yr + 2k_{15}ys \\ & + 2k_{16}zw + 2k_{17}zr + 2k_{18}zs + 2k_{19}wr + 2k_{20}ws + 2k_{21}rs. \end{aligned} \tag{7}$$

Differentiating (7) with respect to the system (2), we obtain:

$$\begin{aligned} 2\dot{V} = {} & 2k_1x\dot{x} + 2k_2y\dot{y} + 2k_3z\dot{z} + 2k_4w\dot{w} + 2k_5r\dot{r} + 2k_6s\dot{s} + 2k_7(x\dot{y} + y\dot{x}) \\ & + 2k_8(x\dot{z} + z\dot{x}) + 2k_9(x\dot{w} + w\dot{x}) + 2k_{10}(x\dot{r} + r\dot{x}) + 2k_{11}(x\dot{s} + s\dot{x}) \\ & + 2k_{12}(y\dot{z} + z\dot{y}) + 2k_{13}(y\dot{w} + w\dot{y}) + 2k_{14}(y\dot{r} + r\dot{y}) + 2k_{15}(y\dot{s} + s\dot{y}) \\ & + 2k_{16}(z\dot{w} + w\dot{z}) + 2k_{17}(z\dot{r} + r\dot{z}) + 2k_{18}(z\dot{s} + s\dot{z}) \\ & + 2k_{19}(w\dot{r} + r\dot{w}) + 2k_{20}(w\dot{s} + s\dot{w}) + 2k_{21}(r\dot{s} + s\dot{r}). \end{aligned} \tag{8}$$

Dividing equation (8) by 2, we get:

$$\begin{aligned}\dot{V} &= k_1x\dot{x} + k_2y\dot{y} + k_3z\dot{z} + k_4w\dot{w} + k_5r\dot{r} + k_6s\dot{s} + k_7(x\dot{y} + y\dot{x}) \\ &+ k_8(x\dot{z} + z\dot{x}) + k_9(x\dot{w} + w\dot{x}) + k_{10}(x\dot{r} + r\dot{x}) + k_{11}(x\dot{s} + s\dot{x}) \\ &+ k_{12}(y\dot{z} + z\dot{y}) + k_{13}(y\dot{w} + w\dot{y}) + k_{14}(y\dot{r} + r\dot{y}) + k_{15}(y\dot{s} + s\dot{y}) \\ &+ k_{16}(z\dot{w} + w\dot{z}) + k_{17}(z\dot{r} + r\dot{z}) + k_{18}(z\dot{s} + s\dot{z}) \\ &+ k_{19}(w\dot{r} + r\dot{w}) + k_{20}(w\dot{s} + s\dot{w}) + k_{21}(r\dot{s} + s\dot{r}). \end{aligned} \tag{9}$$

Substituting $\dot{x}, \dot{y}, \dot{z}, \dot{w}, \dot{r}$, and $\dot{s}$ from equation (6) into equation (9), we obtain:

$$\begin{aligned}\dot{V} &= k_1xy + k_2yz + k_3wz + k_4rw + k_5rs + k_6s(-as - br - cw - dz - ey - fx) \\ &+ k_7(y^2 + xz) + k_8(xw + yz) + k_9(rx + yw) + k_{10}(sx + ry) \\ &+ k_{11}(x(-as - br - cw - dz - ey - fx) + sy) + k_{12}(wy + z^2) \\ &+ k_{13}(ry + wz) + k_{14}(sy + rz) + k_{15}(y(-as - br - cw - dz - ey - fx) + sz) \\ &+ k_{16}(rz + w^2) + k_{17}(sz + rw) + k_{18}(z(-as - br - cw - dz - ey - fx) + sw) \\ &+ k_{19}(sw + r^2) + k_{20}(w(-as - br - cw - dz - ey - fx) + rs) \\ &+ k_{21}(r(-as - br - cw - dz - ey - fx) + s^2). \end{aligned} \tag{10}$$

Simplifying equation (10), we get:

$$\begin{aligned}\dot{V} &= k_1xy + k_2yz + k_3wz + k_4rw + k_5rs - ak_6s^2 - bk_6rs - ck_6sw - dk_6sz \\ &- ek_6sy - fk_6sx + k_7y^2 + k_7xz + k_8xw + k_8yz + k_9rx + k_9yw + k_{10}sx \\ &+ k_{10}ry - ak_{11}xs - bk_{11}rx - ck_{11}wx - dk_{11}xz - ek_{11}xy - fk_{11}x^2 + k_{11}sy \\ &+ k_{12}wy + k_{12}z^2 + k_{13}ry + wzk_{13} + k_{14}sy + k_{14}rz - ak_{15}sy - bk_{15}ry \\ &- ck_{15}wy - dk_{15}yz - ek_{15}y^2 - fk_{15}xy + K_{15}sz + k_{16}rz + k_{16}w^2 + k_{17}sz + k_{17}rw \\ &- ak_{18}sz - bk_{18}rz - ck_{18}wz - dk_{18}z^2 - ek_{18}yz - fk_{18}xz + k_{18}sw + k_{19}sw + k_{19}r^2 \\ &- ak_{20}sw - bk_{20}rw - ck_{20}w^2 - dk_{20}wz - ek_{20}wy - fk_{20}xw + k_{20}rs - ak_{21}rs \\ &- bk_{21}r^2 - ck_{21}rw - dk_{21}rz - ek_{21}ry - fk_{21}rx + k_{21}s^2. \end{aligned} \tag{11}$$

Collecting the like terms and respective coefficients of equation (11), we get:

Terms	Coefficients
x^2	$-fk_{11}$
y^2	$k_7 - ek_{15}$
z^2	$k_{12} - dk_{18}$
w^2	$k_{16} - ck_{20}$
r^2	$k_{19} - bk_{21}$
s^2	$k_{21} - ak_6$
xy	$k_1 - ek_{11} - fk_{15}$
yz	$k_2 + k_8 - ek_{18} - dk_{15}$
wz	$k_3 + k_{13} - ck_{18} - dk_{20}$
rw	$k_4 + k_{17} - bk_{20} - ck_{21}$
rz	$k_{14} + k_{16} - bk_{18} - dk_{21}$
ry	$k_{10} + k_{13} - bk_{15} - ek_{21}$
rx	$k_9 - bk_{11} - fk_{21}$
xz	$k_7 - dk_{11} - fk_{18}$
xw	$k_8 - ck_{11} - fk_{20}$
wy	$k_9 + k_{12} - ck_{15} - ek_{20}$
sx	$k_{10} - fk_6 - ak_{11}$
sy	$k_{11} + k_{14} - ek_6 - ak_{15}$
sz	$k_{15} + k_{17} - ak_{18} - dk_6$
sw	$k_{18} + k_{19} - ck_6 - ak_{20}$
rs	$k_5 + k_{20} - bk_6 - ak_{21}$

Next is to determine $\dot{V}$ such that one of the following axioms holds:

$$(i) \quad \dot{V} \leq -\beta x^2$$
$$(ii) \quad \dot{V} \leq -\beta y^2$$
$$(iii) \quad \dot{V} \leq -\beta z^2$$
$$(iv) \quad \dot{V} \leq -\beta w^2$$
$$(v) \quad \dot{V} \leq -\beta r^2$$
$$(vi) \quad \dot{V} \leq -\beta s^2$$
$$(vii) \quad \dot{V} \leq -\beta(x^2 + y^2 + z^2 + z^2 + w^2 + r^2 + s^2). \tag{12}$$

For the realization of any of these cases; from Table 1.1, we impose the following conditions:

$$k_{11} = 0 \tag{13}$$

$$k_7 - ek_{15} > 0 \tag{14}$$

$$k_{12} - dk_{18} = 0 \tag{15}$$

$$k_{16} - ck_{20} = 0 \tag{16}$$

$$k_{19} - bk_{21} = 0 \tag{17}$$

$$k_{21} - ak_6 = 0 \tag{18}$$

since $k_{11} = 0$, from the table we get:

$$k_1 = fk_{15} \tag{19}$$

$$k_9 = fk_{21} \tag{20}$$

$$k_7 = fk_{18} \tag{21}$$

$$k_8 = fk_{20} \tag{22}$$

$$k_{10} = fk_6 \tag{23}$$

$$k_{12} = ck_{15} + ek_{20} - k_9 \tag{24}$$

$$k_{14} = ek_6 + ak_{15} \tag{25}$$

$$k_{15} = dk_6 + ak_{18} - k_{17} \tag{26}$$

$$k_{18} = ck_6 + ak_{20} - k_{19} \tag{27}$$

$$k_5 = bk_6 + ak_{21} - k_{20} \tag{28}$$

$$k_{10} = bk_{15} + ek_{21} - k_{13} \tag{29}$$

$$k_{14} = bk_{18} + dk_{21} - k_{16} \tag{30}$$

$$k_4 = bk_{20} + ck_{21} - k_{17} \tag{31}$$

$$k_3 = ck_{18} + dk_{20} - k_{13} \tag{32}$$

$$k_2 = ek_{18} + dk_{15} - k_8. \tag{33}$$

Recall that one of the interest is on equation (14)

$$k_7 - ek_{15} > 0.$$

In equation (21), $k_7 = fk_{18}$, thus

$$\implies fk_{18} - ek_{15} > 0. \tag{34}$$

In equation (27),

$$k_{18} = ck_6 + ak_{20} - k_{19} \quad \text{but } k_{19} = abk_6$$

$$\implies k_{18} = ck_6 + ak_{20} - abk_6. \tag{35}$$

From equation (24),

$$k_9 = ck_{15} + ek_{20} - k_{12} \quad \text{but } k_9 = afk_6 \text{ and } k_{12} = dk_{18}$$

$$\implies afk_6 = ck_{15} + ek_{20} - dk_{18}. \tag{36}$$

Substituting (35) into (36), we have

$$k_{15} = \frac{af}{c}k_6 + dk_6 - \frac{abd}{c}k_6 + \frac{ad}{c}k_{20} - \frac{e}{c}k_{20}$$

$$\implies k_{15} = \frac{(af + cd - abd)}{c}k_6 + \frac{(ad - e)}{c}k_{20}. \tag{37}$$

In equation (30),

$$k_{14} = bk_{18} + dk_{21} - k_{16}, \quad \text{but } k_{16} = ck_{20} \quad \text{and } k_{21} = ak_6$$

$$\implies k_{14} = bk_{18} + adk_6 - ck_{20}.$$

But $k_{14} = ek_6 + ak_{15}$ from (25),

$$\implies ek_6 + ak_{15} = bk_{18} + adk_6 - ck_{20}.$$

Substituting (35) and (37) into the above equation, we get:

$$\implies k_{20} = \frac{ce + a^2f + acd - a^2bd - bc^2 + ab^2c - acd}{abc + a^2d - c^2 - ae} k_6.$$

Let

$$\Psi = \frac{ce + a^2f + acd - a^2bd - bc^2 + ab^2c - acd}{abc + a^2d - c^2 - ae}.$$

Then,

$$\implies k_{20} = \Psi k_6.$$

Therefore, we have the following:

$$k_1 = \left(fd - \frac{abdf}{c} - \frac{ef\Psi}{c} - \frac{adf\Psi}{c} + \frac{af^2}{c} \right) k_6$$

$$k_2 = \left(d^2 - \frac{abd^2}{c} - \frac{de\Psi}{c} - \frac{ad^2\Psi}{c} + \frac{adf}{c} + ec - abe + ae\Psi - f\Psi \right) k_6$$

$$k_3 = \left(c^2 - abc + ac\Psi - d\Psi - ae - f - bd + \frac{ab^2d}{c} + \frac{be\Psi}{c} + \frac{abd\Psi}{c} - \frac{abf}{c} \right) k_6$$

$$
\begin{aligned}
k_4 &= (b\Psi - ac - cf + abf - af\Psi)\, k_6 \\
k_5 &= (b + a^2 - \Psi)k_6 \\
k_7 &= (cf - abf + af\Psi)k_6 \\
k_8 &= f\Psi k_6 \\
k_9 &= afk_6 \\
k_{10} &= fk_6 \\
k_{11} &= 0 \\
k_{12} &= (cd - abd + ad\Psi)k_6 \\
k_{13} &= \left(ae - f - bd + \frac{ab^2d}{c} + \frac{be\Psi}{c} + \frac{abd\Psi}{c} - \frac{abf}{c}\right) k_6 \\
k_{14} &= (e + ad - \frac{a^2bd}{c} - \frac{ae\Psi}{c} - \frac{a^2d\Psi}{c} + \frac{a^2f}{c})k_6 \\
k_{15} &= \frac{1}{c}\,(cd - abd - e\Psi - ad\Psi + af)\, k_6 \\
k_{16} &= c\Psi k_6 \\
k_{17} &= (d + ac - a^2b + a^2\Psi - \frac{1}{c}(cd - abd - e\Psi - ad\Psi + af))k_6 \\
k_{18} &= (c - ab + a\Psi)k_6 \\
k_{19} &= abk_6 \\
k_{20} &= \Psi k_6 \\
k_{21} &= ak_6.
\end{aligned}
\tag{38}
$$

From (34),

$$fk_{18} - ek_{15} > 0$$

$$\Longrightarrow f(c - ab + a\Psi)k_6 - \frac{e}{c}\,(cd - abd - e\Psi - ad\Psi + af)\, k_6$$

$$\Longrightarrow (c^2 - abcf + acf\Psi - ced + abde + e^2\Psi + ade\Psi - aef)k_6 > 0.$$

Ploughing (48) back into equation (7) gives

$$
\begin{aligned}
2V = &\left(fd - \frac{abdf}{c} - \frac{ef\Psi}{c} - \frac{adf\Psi}{c} + \frac{af^2}{c}\right)x^2k_6 \\
&+ \left(d^2 - \frac{abd^2}{c} - \frac{de\Psi}{c} - \frac{ad^2\Psi}{c} + \frac{adf}{c} + ec - abe + ae\Psi - f\Psi\right)y^2k_6 \\
&+ \left(c^2 - abc + ac\Psi - d\Psi - ae - f - bd + \frac{ab^2d}{c} + \frac{be\Psi}{c} + \frac{abd\Psi}{c} - \frac{abf}{c}\right)z^6k_6 \\
&+(b\Psi - ac - cf + abf - af\Psi)\, w^2k_6 + (b+a^2-\Psi)r^2k_6 + s^2k_6 + 2(cf - abf + af\Psi)xyk_6 \\
&+ 2f\Psi xzk_6 + 2afxwk_6 + 2fxrk_6 + 2(cd - abd + ad\Psi)yzk_6 \\
&+ 2\left(ae - f - bd + \frac{ab^2d}{c} + \frac{be\Psi}{c} + \frac{abd\Psi}{c} - \frac{abf}{c}\right)ywk_6 \\
&+ 2(e + ad - \frac{a^2bd}{c} - \frac{ae\Psi}{c} - \frac{a^2d\Psi}{c} + \frac{a^2f}{c})ryk_6 \\
&+ \frac{2}{c}(cd - abd - e\Psi - ad\Psi + af)\, syk_6 + 2c\Psi wzk_6 \\
&+ 2(d + ac - a^2b + a^2\Psi - \frac{1}{c}(cd - abd - e\Psi - ad\Psi + af))rzk_6 \\
&+ 2(cab + a\Psi)szk_6 + 2abrwk_6 + 2\Psi swk_6 + 2arsk_6. \qquad (39)
\end{aligned}
$$

By setting $k_6 = 1$ in (39) and dividing through by 2, we obtain:

$$
\begin{aligned}
V = &\frac{1}{2}\left(fd - \frac{abdf}{c} - \frac{ef\Psi}{c} - \frac{adf\Psi}{c} + \frac{af^2}{c}\right)x^2 \\
&+\frac{1}{2}\left(d^2 - \frac{abd^2}{c} - \frac{de\Psi}{c} - \frac{ad^2\Psi}{c} + \frac{adf}{c} + ec - abe + ae\Psi - f\Psi\right)y^2 \\
&+\frac{1}{2}\left(c^2 - abc + ac\Psi - d\Psi - ae - f - bd + \frac{ab^2d}{c} + \frac{be\Psi}{c} + \frac{abd\Psi}{c} - \frac{abf}{c}\right)z^6 \\
&+\frac{1}{2}(b\Psi - ac - cf + abf - af\Psi)\, w^2 + \frac{1}{2}(b + a^2 - \Psi)r^2 \\
&+\frac{1}{2}s^2 + (cf - abf + af\Psi)xy \\
&+f\Psi xz + afxw + fxr + (cd - abd + ad\Psi)yz \\
&+\left(ae - f - bd + \frac{ab^2d}{c} + \frac{be\Psi}{c} + \frac{abd\Psi}{c} - \frac{abf}{c}\right)yw
\end{aligned}
$$

$$+(e + ad - \frac{a^2bd}{c} - \frac{ae\Psi}{c} - \frac{a^2d\Psi}{c} + \frac{a^2f}{c})ry$$
$$+\frac{1}{c}\,(cd - abd - e\Psi - ad\Psi + af)\,sy + c\Psi wz$$
$$+(d + ac - a^2b + a^2\Psi - \frac{1}{2c}(cd - abd - e\Psi - ad\Psi + af))rz$$
$$+(cab + a\Psi)sz + abrw + \Psi sw + ars.$$

If $ad < e$, $ab < c$ and if we choose $a = b = c = d = e = f > 1$, then these values of the constants guarantee the positive definiteness of V. The corresponding time derivative:

$$\dot{V} = -(c^2 - abcf + acf\Psi - ced + abde + e^2\Psi + ade\Psi - aef)y^2 \qquad (40)$$

since equation (41) satisfies V defined by equation (40) satisfied equation (1) if U is replaced by $\dot{V} = -(c^2 - abcf + acf\Psi - ced + abde + e^2\Psi + ade\Psi - aef)y^2$, then V defined by (40) is a Lyapunov function for the sixth order system (2). The existence of Lyapunov function guarantee the stability of nonlinear ordinary differential equations and by Lasalleâs theorem on stability of a system in (Theorem), the system is locally and globally asymptotically stable.

4 Conclusion

We have seen that the Lyapunov function candidate constructed in this work is a good techniques in the stability analysis of dynamical systems. Without solving the systems of differential equations involved, we were able to obtain the qualitative behaviour of the systems near their equilibrium points. A valid quadratic form and positive definite $V(x)$ and also positive definite $U(x)$ was chosen such that the derivative of $V(x)$ with respect to time along the solution paths of the six scales system is equal to the negative $U(x)$, that is, $\dot{V} = -U$. The existence of Lyapunov function for the sixth order nonlinear system guarantee local and global asymptotic stability of the system as corroborated by Lassale's Invariant theorem.

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