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Analytical Solution of Black-Scholes Model for Pricing Barrier Option using Method of Partial Taylor Series Expansion

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Abstract

In this work, Black-Scholes differential equation for barrier/traditional option is solved using partial Taylor series expansion method. The developed solutions are in very good agreement with the closed-form solutions of the Black Scholes equation for the powered ML-payoff functions. Also, the analytical solutions of the new method in this present study give the same expressions as the solutions of projected differential equations and homotopy perturbation method as presented in the literature. Moreover, the reliability, speed, accuracy, and ease of application of the proposed method show its potential for wide areas of applications in science, financial mathematics, and engineering.

1. Introduction

In financial markets, option is taken as one of the best products among various derivatives. Option is a derivative of financial security that gives its owner the right to buy or sell a specified amount of a particular asset at a fixed price, called the exercise (strike) price, on or before a specified date (maturity date) [1]. An option can be an American option or a European option. An America option is an option which can be exercised at any time until expiration or maturity date while a European option is an option which can be exercised only at a fixed expiration or maturity date [2]. The European option can be of two types, namely, call option and put option. These two options are the basis for a wide range of option strategies that are designed for hedging, income, or speculation. Call options give or allow the buyer or the holder the right, but not the obligation, to buy the underlying asset such as stock, bond, or commodity at a

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stated (strike) price within a specific timeframe in the contract. Put options give or allow the buyer or the holder the right, but not the obligation, to sell the underlying asset at a stated (strike) price within a specific timeframe in the contract.

Option pricing has been efficiently modeled by the well-known Black-Scholes second-order partial differential equation [3]. The Black-Scholes model can be used for European or American option pricing [4-6]. In order to achieve this, there have been several attempts to produce analytical solutions to the second-order partial differential equation [6]. However, the coefficients of the Black-Scholes can depend on the time and the asset price. Consequently, the analytical solution of the generalized Black-Scholes model is not a straight-forward task. Therefore, over the years, various numerical methods have been presented to solve the option pricing problems [7-20]. In the computational adventures for the numerical solutions for the Black–Scholes model, different numerical schemes have been developed [21-24]. The limitations of the numerical schemes have led to the development of various approximate analytical and hybrid methods [25-34] which provide series solutions. However, the series solutions provide a non-smooth analytical solution at a single point, i.e., when the exercise or strike price is equal to the stock price [34]. Consequently, the quest for relatively simple method with high level of accuracy continues.

Taylor series expansion method (TSEM) has been used to expand trigonometric, hyperbolic, rational, fractional, special functions, etc, into series forms. Such series expressions have been helpful in differentiating and integrating difficult functions. Moreover, the method has been used to develop approximate analytical solutions to differential equations [35-50]. However, the classical Taylor series expansion method is not frequently applied especially to partial differential equations. This is because it requires more function evaluations than well-known classical algorithms and the over-elaborate tasks of calculations of the higher-order derivatives involve in finding approximate solutions of differential equations. Such limitations are addressed in the proposed new Taylor series expansion method called partial Taylor series expansion method (PTSEM). Therefore, in this work, a new Taylor series expansion method called partial Taylor series expansion method is introduced and used to solve Black-Scholes differential equation for barrier option. An analytical solution for the model of European options for barrier option is presented using the PSTEM. Also, the partly series solution method is used to develop non-series solutions for the call and put options pricing. The results of the solutions of the analytical method are compared with the results of the exact analytical solutions. Moreover, numerical examples are presented to illustrate the effectiveness, speed, high level of accuracy and reliability and ease of applications when compared to other analytical methods are established.

2. The Black-Scholes Model

The Black-Scholes model is a mathematical model for the dynamics of a financial market, and it is used for the valuation of financial options. The Black-Scholes second-order partial differential equation is given as [3, 4 and 5]

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0, \tag{1}$$

where

V is the value of the option which explicitly depends on the current asset price and time.

S is the current price of the underlying asset

r is the interest rate (risk-free rate)

t is time to maturity or expiration

σ is volatility of the underlying asset.

The required value $V(S,t)$ will provide us with the information on how much should be paid now, at time t, to hold that option if the current asset price is S.

The above Black-Scholes Model is based on the following assumptions [3]:

i. The stock price V follows the Geometric Brownian Motion with constant drift μ and volatility σ.

ii. The short selling of securities with full use of proceeds is permitted

iii. There are no transactions costs or taxes. All securities are perfectly divisible.

iv. There are no dividends during the life of the option.

v. There are no riskless arbitrage opportunities.

vi. Security trading is continuous.

vii. The risk-free rate of interest, r, is constant and the same for all maturities.

2.1. Black-Scholes model for pricing barrier option

The focus of the present study is on the barrier options. Such options are only weekly path-dependent (options which payoffs at exercise or expiry depend, in some non-trivial way, on the past history of the underlying asset price as well as its price at exercise or expiry) and satisfy the Black-Scholes equation [53-61]. The options are commonly applied in risk management by retail investors, banks, and businesses [60].

The Black-Scholes equation and boundary conditions for a barrier or the traditional option with an additional constraint involving B, the option value $M(S,t)$ are, as described in.

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0,\quad S>B \tag{2}$$

$$V=0,\quad S\leq B$$

with terminal condition

$$V(S,T)=max(S(T)-E,0). \tag{3}$$

If S reaches B, the option is invalid, i.e., $V(B,t) = 0$.

Therefore, the additional condition

$$V(B,t)=0, \tag{4}$$

ET and $r$are the exercise (strike) price and the expiry, respectively. $Max(S-E,0)$ indicates the large value between $S-E$ and 0.

Calls or puts barrier options are categorized as: up-and-in, down-and-in, up-and-out, down-and-out depending on the time when $S=B$ is reached in respect to the expiry [57-58]. However, the focus of this study is down-and-out option that is constructed with only one asset.

The above partial differential equation with variable coefficient as presented in Eq. (2) can be transformed to partial differential equation with constant coefficient using the following variable transformations:

$$S=Be^x,\quad t=T-\frac{\tau}{\frac{1}{2}\sigma^2},\quad V=v(x,t). \tag{5}$$

Applying Eq. (5) in Eqs. (2), (3) and (4), we have

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv, \tag{6}$$

where

$$k = \frac{2r}{\sigma^2} \tag{7}$$

with initial condition

$$v(x,0) = max(Be^x - E, 0). \tag{8}$$

and boundary condition

$$v(0,\tau) = 0. \tag{9}$$

3. Analytical Solutions Black-Scholes Model using Partial Taylor Series Method

Indisputably, the Black-Scholes model can easily be solved numerically. However, in the analysis of the transient problems, we made recourse to symbolic solutions of the problems. Such symbolic solution will provide better physical insights into the importance of model parameters than the numerical methods. In the generation of the analytical solutions to differential equations, the partial Taylor series expansion method is used to solve Black-Scholes classical and generalized differential equations. An analytical solution for the model of European option is presented with the aid of the method.

3.1. The basic principle of partial Taylor series expansion methods

The basic principle of the partial Taylor series method for solving partial differential equation is as follows:

Given an ordinary differential equation

$$f\left(x, y, y', y'', \ldots, y^n\right) = 0. \tag{10}$$

From the n^{th}-order Taylor series of a smooth function about the point $x = a$, the series solution of the differential equation is given by

$$y(x) = y(a) + (x-a)y'(a) + \frac{1}{2!}(x-a)^2 y''(a)$$
$$+\frac{1}{3!}(x-a)^3 y'''(a) + \ldots + \frac{1}{n!}(x-a)^n y^n(a) + R(x) \tag{11}$$

$R(x)$ represents the remainder. With the aid of Eq. (11), each term in the differential equation in Eq. (10) can be found to solve the differential equation.

The Taylor series expansion for two independent variables about the point $x = a$ and $y = b$ is given as

$$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b)$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$
$$+\frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right]$$
$$+\frac{1}{4!}\left[(x-a)^4 f_{xxxx}(a,b) + 4(x-a)^3(y-b)f_{xxxy}(a,b) + 6(x-a)^2(y-b)^2 f_{xxyy}(a,b)\& + 4(x-a)(y-b)^3 f_{xyyy}(a,b) + (y-b)^4 f_{yyyy}(a,b)\right]$$
$$+\cdots + R(x,\tau). \tag{12}$$

However, the partial Taylor series expansion for two independent variables about a point where $y = b$ but varying points of $x = x$ is given as

$$f(x,y) = f(x,b) + (y-b)f_y(x,b) + \frac{1}{2!}(y-b)^2 f_{yy}(x,b) + \frac{1}{3!}(y-b)^3 f_{yyy}(x,b)$$
$$+\frac{1}{4!}(y-b)^4 f_{yyyy}(x,b) + \cdots + \frac{1}{5!}(y-b)^4 f_{yyyyy}(x,b)$$
$$+\cdots + \frac{1}{n!}(y-b)^n f_{yyyyy\ldots y}(x,b) + R(x,b). \tag{13}$$

In the partial Taylor series expansion method, the expansion is carried out with respect to a specific variable of the partial differential equation. Such an approach of partial expansion of a several variable function with respect to a specific variable make the application of Taylor series method to be much simpler than the standard Taylor series method for several variables.

3.2. Application of partial Taylor series expansion method to Black-Scholes models

Given that the classical Black-Scholes model as

$$v_\tau(x,\tau) = v_{xx}(x,\tau) + (q-1)v_x(x,\tau) - qv(x,\tau). \tag{14}$$

In solving the above problem by the partial Taylor series expansion method, From the n^{th}-order partial Taylor series expansion of a smooth function about the point $\tau = \tau_0$, the seriessolution of the differential equation is given by

$$v(x,\tau) \simeq v(x,\tau_o) + (\tau - \tau_o)v_\tau(x,\tau_o) + \frac{1}{2!}(\tau - \tau_o)^2 v_{\tau\tau}(x,\tau_o)$$
$$+\frac{1}{3!}(\tau - \tau_o)^3 v_{\tau\tau\tau}(x,\tau_o) + \frac{1}{4!}(\tau - \tau_o)^4 v_{\tau\tau\tau\tau}(x,\tau_o)$$
$$+\frac{1}{5!}(\tau - \tau_o)^5 v_{\tau\tau\tau\tau\tau}(x,\tau_o) + \cdots + \frac{1}{n!}(\tau - \tau_o)^n v_{\tau\tau\tau\tau\tau\ldots\tau}(x,\tau_o). \quad (15)$$

From the n^{th}-order partial Taylor series of a smooth function about the point $\tau = 0$, the series solution of the differential equation is given by

3.2.1. Partial Taylor series expansion method to Black-Scholes model for pricing barrier option

Given that that the

$$v(x,0) = max(Be^x - E, 0).$$

Therefore,

$$v_x(x,0) = max(Be^x, 0), v_{xx}(x,0) = max(Be^x, 0), v_{xxx}(x,0) = max(Be^x, 0),$$
$$v_{xxxx}(x,0) = max(Be^x, 0), v_{xxxxx}(x,0) = max(Be^x, 0), v_{xxxxxx}(0,0) = max(Be^x, 0) \quad (16)$$
$$v_{x\tau}(x,0) = v_{xx\tau}(x,0) = v_{x\tau\tau}(x,0) = v_{xxx\tau}(x,0) = v_{xx\tau\tau}(x,0) = v_{x\tau\tau\tau}(x,0) = 0,$$
$$v_{xxxx\tau}(x,0) = v_{xxx\tau\tau}(x,0) = v_{xx\tau\tau\tau}(x,0) = v_{x\tau\tau\tau\tau}(x,0) = \cdots = 0.$$

From the governing Eq. (6), one has

$$v_\tau(x,0) = v_{xx}(x,0) + (k-1)v_x(x,0) - kv(x,0). \quad (17)$$

On substituting Eq. (16) into Eq. (17), this gives

$$v_\tau(x,0) = max(Be^x, 0) + (k-1)max(Be^x, 0) - kmax(Be^x - E, 0). \quad (18)$$

Therefore

$$v_\tau(x,0) = k[max(Be^x, 0) - max(Be^x - E, 0)]. \quad (19)$$

Now, differentiating Eq. (11) with respect to "τ", we have

$$v_{\tau\tau}(x,\tau) = v_{xx\tau}(x,\tau) + (k-1)v_{x\tau}(x,\tau) - kv_\tau(x,\tau) \quad (20)$$

and

$$v_{\tau\tau}(x,0) = v_{xx\tau}(x,0) + (k-1)v_{x\tau}(x,0) - kv_{\tau}(x,0). \tag{21}$$

Again, substitute Eq. (16) into Eq. (22), this produces

$$v_{\tau\tau}(x,0) = -k^2[max(Be^x,0) - max(Be^x - E,0)]. \tag{22}$$

Now, differentiating Eq. (20) with respect to "τ", we have

$$v_{\tau\tau\tau}(x,\tau) = v_{xx\tau\tau}(x,\tau) + (k-1)v_{x\tau\tau}(x,\tau) - kv_{\tau\tau}(x,\tau) \tag{23}$$

and

$$v_{\tau\tau\tau}(x,0) = v_{xx\tau\tau}(x,0) + (k-1)v_{x\tau\tau}(x,0) - kv_{\tau\tau}(x,0). \tag{24}$$

After the substitution of the corresponding terms in Eq. (16) into Eq. (22), one arrives at

$$v_{\tau\tau\tau}(x,0) = k^3[max(Be^x,0) - max(Be^x - E,0)] \tag{25}$$

similarly

$$v_{\tau\tau\tau\tau}(x,0) = -k^4[max(Be^x,0) - max(Be^x - E,0)] \tag{26}$$

$$v_{\tau\tau\tau\tau\tau}(x,0) = k^5[max(Be^x,0) - max(Be^x - E,0)] \tag{27}$$

$$\vdots$$

$$\vdots$$

$$v_{\tau\tau\tau\tau\tau\ldots\tau}(x,0) = (-1)^{n+1}k^n[max(Be^x,0) - max(Be^x - E,0)]. \tag{28}$$

On substituting Eqs. (8), (19), (22), (25), (26), (27) and (28) into Eq. (13), we have

$$\begin{aligned} v(x,\tau) \simeq\ & max(Be^x - E,0) + k[max(Be^x,0) - max(Be^x - E,0)]\tau \\ & -\frac{k^2}{2!}[max(Be^x,0) - max(Be^x - E,0)]\tau^2 \\ & +\frac{k^3}{3!}[max(Be^x,0) - max(Be^x - E,0)]\tau^3 \\ & -\frac{k^4}{4!}[max(Be^x,0) - max(Be^x - E,0)]\tau^4 \\ & + \cdots + (-1)^{n+1}\frac{k^n}{n!}[max(Be^x,0) - max(Be^x - E,0)]\tau^n \end{aligned} \tag{29}$$

which gives,

$$v(x,\tau) \simeq max(Be^x - E,0) + k\tau max(Be^x,0) - k\tau max(Be^x - E,0)$$
$$-\frac{1}{2!}(k\tau)^2 max(Be^x,0) + \frac{1}{2!}(k\tau)^2(Be^x - E,0)$$
$$+\frac{1}{3!}(k\tau)^3 max(Be^x,0) - \frac{1}{3!}(k\tau)^3(Be^x - E,0) - \frac{1}{4!}(k\tau)^4 max(Be^x,0)$$
$$+\frac{1}{4!}(k\tau)^4(Be^x - E,0)$$
$$+\cdots+(-1)^{n+1}\left[\frac{1}{n!}(k\tau)^n max(Be^x,0) - \frac{1}{n!}(k\tau)^n(Be^x - E,0)\right]. \quad (30)$$

We can write that

$$v(x,\tau) \simeq max(Be^x - E,0)$$
$$+\sum_{n=1}^{N}(-1)^{n+1}\left[\frac{1}{n!}(k\tau)^n max(Be^x,0) - \frac{1}{n!}(kt)^n max(Be^x - E,0)\right] \quad (31)$$

i.e.

$$v(x,\tau) = max(Be^x - E,0)$$
$$+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(k\tau)^n}{n!}[max(Be^x,0) - max(Be^x - E,0)]. \quad (32)$$

Recall that

$$S = Be^x, \quad t = T - \frac{\tau}{\frac{1}{2}\sigma^2}, \quad v(x,\tau) = V(x,t), \quad k = \frac{2r}{\sigma^2}.$$

Therefore,

$$V(S,t) = max(S - E,0)$$
$$+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}[r(T-t)]^n}{n!}[max(S,0) - max(S - E,0)]. \quad (33)$$

Furthermore, we can the collect like terms in Eq. (30) to have

$$v(x,\tau) \simeq k\tau max(Be^x,0) - \frac{1}{2!}(k\tau)^2 max(Be^x,0)\frac{1}{3!}(k\tau)^3 max(Be^x,0)$$

$$-\frac{1}{4!}(k\tau)^4(Be^x,0)+(-1)^{n+1}\left[\frac{1}{n!}(k\tau)^n max(Be^x,0)\right]$$

$$+\cdots+max(Be^x-E,0)-k\tau max(Be^x-E,0)$$

$$+\frac{1}{2!}(k\tau)^2 max(Be^x-E,0)-\frac{1}{3!}(k\tau)^3 max(Be^x-E,0)$$

$$+\frac{1}{4!}(k\tau)^4 max(Be^x-E,0)\cdots-(-1)^{n+1}\frac{1}{n!}(k\tau)^n max(Be^x-E,0) \qquad (34)$$

which can also be expressed as

$$v(x,\tau)\simeq\left\{1-k\tau+\frac{1}{2!}(k\tau)^2-\frac{1}{3!}(k\tau)^3+\frac{1}{4!}(k\tau)^4-\cdots-(-1)^{n+1}\left[\frac{1}{n!}(k\tau)^n\right]\right\}max(Be^x-E,0)$$

$$+\left\{k\tau-\frac{1}{2!}(k\tau)^2+\frac{1}{3!}(k\tau)^3-\frac{1}{4!}(k\tau)^4+\cdots+(-1)^{n+1}\left[\frac{1}{n!}(k\tau)^n\right]\right\}max(Be^x,0) \quad (35)$$

and then

$$v(x,\tau)=\left\{1-k\tau+\frac{1}{2!}(k\tau)^2-\frac{1}{3!}(k\tau)^3+\frac{1}{4!}(k\tau)^4-\cdots\right\}max(Be^x-E,0)$$

$$+\left\{k\tau-\frac{1}{2!}(k\tau)^2+\frac{1}{3!}(k\tau)^3-\frac{1}{4!}(k\tau)^4+\cdots\right\}max(Be^x,0). \qquad (36)$$

Recall that by series expansion,

$$e^{-k\tau}=1-k\tau+\frac{1}{2!}(k\tau)^2-\frac{1}{3!}(k\tau)^3+\frac{1}{4!}(k\tau)^4-\cdots \qquad (37)$$

$$1-e^{-k\tau}=k\tau-\frac{1}{2!}(k\tau)^2+\frac{1}{3!}(k\tau)^3-\frac{1}{4!}(k\tau)^4+\cdots \qquad (38)$$

Therefore, the above Eq. (39) can be expressed as,

$$v(x,\tau)=max(Be^x-K,0)e^{-k\tau}+max(Be^x,0)\left(1-e^{-k\tau}\right). \qquad (39)$$

Recall that

$$S=Ee^x,\ t=T-\frac{\tau}{\frac{1}{2}\sigma^2}, v(x,t)=V(x,t),\ k=\frac{2r}{\sigma^2},\ \ \tau=\frac{1}{2}\sigma^2(T-t)\rightarrow k\tau=r(T-t).$$

Therefore, Eq. (39) becomes

$$V(x,t)=max(S-E,0)e^{-r(T-t)}+max(S,0)\left(1-e^{-r(T-t)}\right). \qquad (40)$$

Again, if we go back to Eq. (36), we have

$$v(x,\tau) = \left\{1 - k\tau + \frac{1}{2!}(k\tau)^2 - \frac{1}{3!}(k\tau)^3 + \frac{1}{4!}(k\tau)^4 - \cdots\right\}(Be^x - E)$$
$$+\left\{k\tau - \frac{1}{2!}(k\tau)^2 + \frac{1}{3!}(k\tau)^3 - \frac{1}{4!}(k\tau)^4 + \cdots\right\}(Be^x) \quad (41)$$

which can be expressed as

$$v(x,\tau) = \left\{1 - k\tau + \frac{1}{2!}(k\tau)^2 - \frac{1}{3!}(k\tau)^3 + \frac{1}{4!}(k\tau)^4 - \cdots\right\}(Be^x)$$
$$-\left\{1 - k\tau + \frac{1}{2!}(k\tau)^2 - \frac{1}{3!}(k\tau)^3 + \frac{1}{4!}(k\tau)^4 - \cdots\right\}E$$
$$+\left\{k\tau - \frac{1}{2!}(k\tau)^2 + \frac{1}{3!}(k\tau)^3 - \frac{1}{4!}(k\tau)^4 + \cdots\right\}(Be^x). \quad (42)$$

Then

$$v(x,\tau) = Be^x - E\left\{1 - k\tau + \frac{1}{2!}(k\tau)^2 - \frac{1}{3!}(k\tau)^3 + \frac{1}{4!}(k\tau)^4 - \cdots\right\}. \quad (43)$$

The above Eq. (43) can be written as

$$v(x,\tau) = S - E\left\{1 - r(T-t) + \frac{1}{2!}r^2(T-t)^2 - \frac{1}{3!}r^3(T-t)^3 + \frac{1}{4!}r^4(T-t)^4 - \cdots\right\}. \quad (44)$$

Alternatively, we have

$$v(x,\tau) = S + E\left\{-1 + r(T-t) - \frac{1}{2!}r^2(T-t)^2 + \frac{1}{3!}r^3(T-t)^3 - \frac{1}{4!}r^4(T-t)^4 + \cdots\right\}. \quad (45)$$

Eq. (45) is the extended form of the solution that was arrived at by Dehghan and Pourghanbar [57] using homotopy perturbation method.

In non-series solution, Eq. (41) can be easily expressed as

$$v(x,\tau) = (Be^x - E)e^{-k\tau} + Be^x\left(1 - e^{-k\tau}\right) \quad (46)$$

which reduces to

$$v(x,\tau) = Be^x - Ee^{-k\tau}. \quad (47)$$

Recall that

$$S = Ee^{x}, \quad v(x,t) = V(x,t), \quad k\tau = r(T-t).$$

Therefore, we arrived at

$$V(S,t) = S - Ee^{-r(T-t)}. \tag{48}$$

The accuracy of the solutions can be tested when the results is verified with the results of the analytical solution which is obtained by Fourier transformation. The analytical solution is given as

$$V(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2), \tag{49}$$

where

$$d_1 = \frac{In\left(\frac{S}{B}\right) + \frac{1}{2}\sigma^2(T-t)\left(\frac{r}{\frac{1}{2}\sigma^2} + 1\right)}{\sigma\sqrt{T-t}}, \quad d_2 = \frac{In\left(\frac{S}{B}\right) + \frac{1}{2}\sigma^2(T-t)\left(\frac{r}{\frac{1}{2}\sigma^2} - 1\right)}{\sigma\sqrt{T-t}}, \tag{50}$$

$$N(\eta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\eta} e^{-\frac{1}{2}S^2}\, dS.$$

$T - t$ is the time remaining till expiration as at time t; $N(\eta)$ is the cumulative normal density function.

Numerical Example and Parametric Studies

In the parametric study, the valuation of barrier option is considered. The stock price is given $S = (11, 13, 15, 17, 19)$ the exercise/strike price of $E = 10$, the risk-neutral interest rate is 0.05 per year, and the volatility is 0.05 per year. Therefore, we have the following: $S = (11, 13, 15, 17, 19)), K = 10, T = 0.25, r = 0.05, \sigma = 0.05$.

Table 1: Comparison of the results when $E = 10$, $B = 9$, $T = 0.25$, $r = 0.05$, $\sigma = 0.05$, $t = 0$.

Stock Price	Exact Solution [57]	VIM [57]	HPM [57]	PTSEM (Present study)
11	1.12422244	1.12422199	1.12421875	1.12422199
13	3.12422199	3.12422199	3.12421875	3.12422199
15	5.12422199	5.12422199	5.12421875	5.12422199
17	7.12422199	7.12422199	7.12421875	7.12422199
19	9.12422199	9.12422199	9.12421875	9.12422199

Table 2: Comparison of the results when $E = 10,\ B = 9,\ T = 0.50,\ r = 0.05,\ \sigma = 0.05,\ t = 0.$

Stock Price	Exact Solution [57]	VIM [57]	HPM [57]	PTSEM (Present study)
11	1.24693225	1.24690087	1.24687500	1.24690087
13	3.24690087	3.24690087	3.24687500	3.24690087
15	5.24690087	5.24690087	5.24687500	5.24690087
17	7.24690087	7.24690087	7.24687500	7.24690087
19	9.24690087	9.24690087	9.24687499	9.24690087

Tables 1 and 2 show the comparison of results of the partial Taylor series expansion method (PTSEM) and that of the existing methods in literature [57]. While the results of the homotopy perturbation method are not in perfect agreements with the results of the exact analytical solution, it could be seen that the results are of the present method are in excellent agreements with the results of the exact analytical solutions and variational iteration method. However, the results of the partial Taylor series expansion method are obtained with less computations and converge faster than that of the exact solutions. This attests to the efficiency and accuracy of the method of partial Taylor series expansion.

4. Conclusion

In this study, a new method called method of partial Taylor series expansion has been successfully applied to solve Black-Scholes equations for pricing of barrier options. It was found that the results are of the method of partial Taylor series expansion are in excellent agreements with the results of the exact analytical solutions and variational iteration method. In fact, the new method displayed high level of simplicity and low cost of computation. Therefore, it could be stated that the PTSEM is very efficient, reliable, and fast and very easy in application. It is hoped that this method will be widely applied in science, financial mathematics, and engineering.

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