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The Kumaraswamy Unit-Gompertz Distribution and its Application to Lifetime Datasets

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Abstract

This paper presents a new generalized bounded distribution called the Kumaraswamy unit-Gompertz (KUG) distribution. Some of the Mathematical properties which include; the density function, cumulative distribution function, survival and hazard rate functions, quantile, mode, median, moment, moment generating function, Renyi entropy and distribution of order statistics are derived. We employ the maximum likelihood estimation method to estimate the unknown parameters of the proposed KUG distribution. A Monte Carlo simulation study is carried out to investigate the performance of the maximum likelihood estimates of the unknown parameters of the proposed distribution. Two real datasets are used to illustrate the applicability of the proposed KUG distribution in lifetime data analysis.

1. Introduction

Kumaraswamy [10] introduced the Kumaraswamy distribution defined on a unit interval $[0,1]$ with the cumulative distribution function (cdf) given by

$$G(x) = 1 - (1 - x^{\alpha})^{\beta}, \quad 0 < x < 1,\ \alpha, \beta > 0, \tag{1}$$

and probability density function defined as

$$g(x) = \alpha\beta x^{\alpha-1}(1 - x^{\alpha})^{\beta}, \quad 0 < x < 1,\ \alpha, \beta > 0. \tag{2}$$

This distribution has found its application in many natural phenomena whose outcomes have lower and upper bounds such as hydrological data (daily rainfall, daily stream flow).

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Jones [9] gave a comprehensive background of the Kumaraswamy distribution, and more importantly, pointed out some advantages of the Kumaraswamy distribution over the beta distribution. Although, the two distributions are defined on a unit interval $[0,1]$, the Kumaraswamy distribution has an explicit expression for the cumulative distribution function and the quantile function does not involve special functions.

Suppose $G(x)$ denotes the base line cumulative distribution function of a random variable X. Cordeiro and de Castro [3] introduced a generalized form of the distribution called the Kumaraswamy-*G* distribution with cumulative distribution function defined by

$$F(x) = 1 - (1 - [G(x)]^{\alpha})^{\beta}, \quad 0 < x < 1,\ \alpha, \beta > 0, \tag{3}$$

and probability density function given by

$$f(x) = \alpha\beta g(x)(1 - [G(x)]^{\alpha})^{\beta-1}[G(x)]^{\alpha-1},\ \ 0 < x < 1,\ \alpha, \beta > 0. \tag{4}$$

Several generalizations of the Kumaraswamy distribution can be found in the works of Cordeiro *et al.* [4], who introduced the Kumaraswamy Weibull distribution, Paranaba *et al.* [15] introduced the Kumaraswamy Burr distribution, Bourguignon *et al.* [1] studied the Kumaraswamy Pareto distribution, Oluyede *et al.* [12] proposed the Kumaraswamy Power Lindley distribution, Salem and Hagag [18] developed the Kumaraswamy Lindley distribution. Recently, Mazucheli *et al.* [11] proposed the unit-Gompertz distribution with bounded support using a similar approach considered in Grassia [8] for the unit-Gamma distribution.

The probability density function of the unit-Gompertz distribution is defined by

$$g(x) = \lambda\theta x^{-(\theta+1)} e^{-\lambda(x^{-\theta}-1)}, \quad 0 < x < 1,\ \lambda, \theta > 0, \tag{5}$$

and the cumulative distribution function is given by

$$G(x) = e^{-\lambda(x^{-\theta}-1)}, \quad 0 < x < 1,\ \lambda, \theta > 0. \tag{6}$$

In this paper, motivated by the flexibility of the generalized distribution in terms of exhibiting an increasing, decreasing and bathtub shapes hazard rate property, we introduced a new generalized bounded distribution called the Kumaraswamy unit-Gompertz (KUG) distribution. The remaining sections of this paper are organized as follows: Section 2 presents the Mathematical properties of the Kumaraswamy unit-Gompertz distribution. In Section 3, the parameter estimation of the KUG distribution using the maximum likelihood method and a Monte Carlo simulation study to investigate the performance of the maximum likelihood estimators of the KUG distribution are derived. Section 4 presents an application of the KUG distribution to two real datasets and the concluding remark is presented in Section 5.

2. Mathematical Properties of the Kumaraswamy Unit-Gompertz Distribution

2.1. Density, cumulative distribution, survival and hazard functions of the KUG distribution

The probability density function and the cumulative distribution function of the proposed KUG distribution are obtained by substituting the density and the cumulative distribution functions of the unit-Gompertz distribution defined in (5) and (6) into the method of generalization defined in (3) and (4). Thus, we have

$$f(x) = \theta\beta\gamma x^{-(\beta+1)} e^{-\gamma(x^{-\beta}-1)} \left\{1 - e^{-\gamma(x^{-\beta}-1)}\right\}^{\theta-1}, \quad 0 < x < 1, \theta, \beta, \gamma > 0, \quad (7)$$

and

$$F(x) = 1 - \left[1 - e^{-\gamma(x^{-\beta}-1)}\right]^{\theta}, \quad 0 < x < 1,\ \theta, \beta, \gamma > 0,\ \gamma = \alpha\lambda. \quad (8)$$

The density function of the KUG distribution defined in (7) can be represented in series using the binomial expansion given by

$$(1-x)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k x^k, \quad (9)$$

$$\left\{1 - e^{-\gamma(x^{-\beta}-1)}\right\}^{\theta-1} = \sum_{i=0}^{\infty} \binom{\theta-1}{i} (-1)^i e^{-\gamma(x^{-\beta}-1)i},$$

$$e^{-\gamma(x^{-\beta}-1)(i+1)} \approx e^{-\gamma x^{-\beta}(i+1)+\gamma(i+1)},$$

using the exponential series expansion, we have that

$$e^{-\gamma x^{-\beta}(i+1)} = \sum_{j=0}^{\infty} \frac{(-1)^j [\gamma(i+1)]^j x^{-\beta j}}{j!},$$

thus, the series representation of the density function of the KUG distribution is given by

$$f(x) = \theta\beta\gamma \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{\theta-1}{i} (-1)^{i+j} \frac{[\gamma(i+1)]^j}{j!} e^{\gamma(i+1)} x^{-\beta(j+1)-1}. \quad (10)$$

The mode of the KUG distribution is obtained by taking the natural logarithm of the density function and minimizing the function with respect to the random variable x as,

$$ln(f(x)) = ln(\theta\beta\gamma) - (\beta+1)ln(x) - \gamma\left(x^{-\beta} - 1\right) + (\theta - 1)ln\left\{1 - e^{-\gamma(x^{-\beta}-1)}\right\},$$

$$\frac{d}{dx}\{ln(f(x))\} = \beta\gamma x^{-(\beta+1)} - \frac{1+\beta}{x} - \frac{\beta\gamma(\theta-1)x^{-(1+\beta)}e^{\gamma(1-x^{-\beta})}}{1-e^{\gamma(1-x^{-\beta})}}.$$

The mode $x = x_0$ is the root of the equation $\frac{d}{dx}\{ln(f(x))\} = 0$ which implies that x_0 is the unique critical point at which the density function is maximized.

The graphical representation of the KUG distribution for varying value of the parameters are shown in Figure 1.

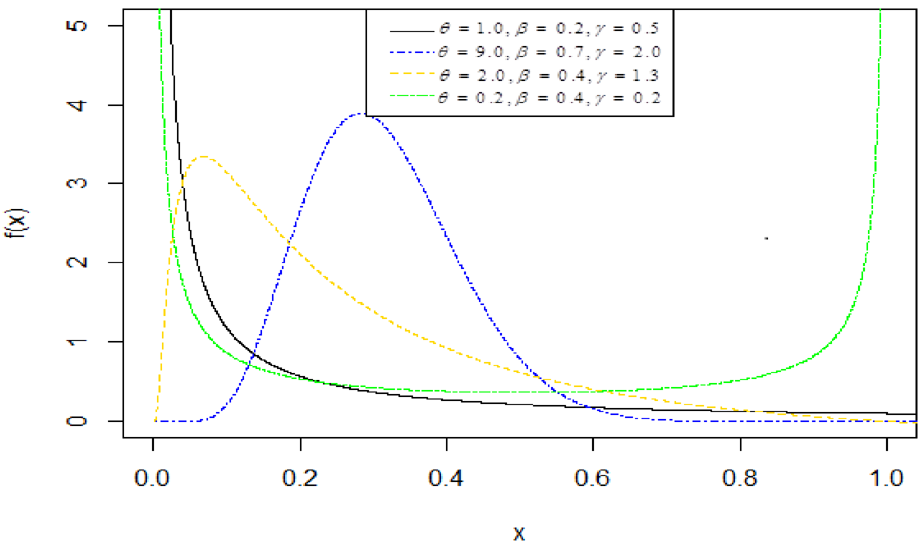

Figure 1: Probability density function of the KUG distribution.

Clearly, Figure 1 indicates that the density function of the KUG distribution accommodates a decreasing (reversed-J), increasing, right-skewed unimodal, symmetric and bathtub shapes.

The mathematical expressions for the survival and hazard rate functions of the KUG distribution are respectively given by

$$s(x) = 1 - F(x) = \left[1 - e^{-\gamma(x^{-\beta}-1)}\right]^{\theta}, \tag{11}$$

and

$$h(x) = \frac{f(x)}{s(x)} = \frac{\theta\beta\gamma x^{-(\beta+1)} e^{-\gamma(x^{-\beta}-1)} \left\{1 - e^{-\gamma(x^{-\beta}-1)}\right\}^{\theta-1}}{\left[1 - e^{-\gamma(x^{-\beta}-1)}\right]^{\theta}},$$

$$= \frac{\theta\beta\gamma x^{-(\beta+1)} e^{-\gamma(x^{-\beta}-1)}}{\left\{1 - e^{-\gamma(x^{-\beta}-1)}\right\}}. \tag{12}$$

The graphical plots of the hazard rate function of the KUG distribution is shown in Figure 2.

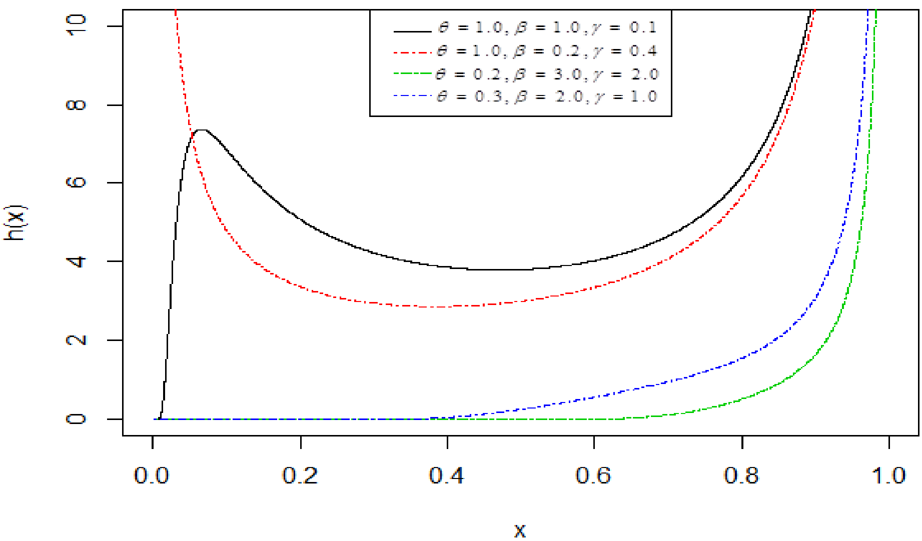

Figure 2: Hazard rate function of the KUG distribution.

Figure 2 shows that the KUG distribution exhibits an increasing, bathtub and inverted bathtub shapes hazard rate properties.

2.2. The quantile function of the KUG distribution

Given the cumulative distribution function $F(x)$ defined in (8), the quantile function of the KUG distribution can be obtained as $Q_X(u) = F^{-1}(u)$.

The u^{th} quantile function is obtained by solving $F(x) = u$, i.e.,

$$\left(1 - e^{-\gamma(x^{-\beta}-1)}\right)^{\theta} = 1 - u,$$

$$e^{-\gamma(x^{-\beta}-1)} = 1 - (1-u)^{\frac{1}{\theta}},$$

$$x^{-\beta} - 1 = \frac{-\log\left[1-(1-u)^{\frac{1}{\theta}}\right]}{\gamma},$$

$$x = \left[1 - \frac{\log\left[1-(1-u)^{\frac{1}{\theta}}\right]}{\gamma}\right]^{-\frac{1}{\beta}}, \quad 0 < x < 1. \tag{13}$$

The median of the KUG distribution is obtained by substituting $u = 0.5$ into (13) which yields,

$$Median = \left[1 - \frac{\log\left(1-\theta\sqrt{0.5}\right)}{\gamma}\right]^{-\frac{1}{\beta}}. \tag{14}$$

Some numerical computation of quantiles from the KUG distribution for different values of the parameters are given in Table 1.

Table 1: Some quantiles from the KUG distribution ($\beta = 2$).

$\boldsymbol{P}$	$\boldsymbol{\theta = 2, \gamma = 2}$	$\boldsymbol{\theta = 3, \gamma = 2}$	$\boldsymbol{\theta = 5, \gamma = 4}$	$\boldsymbol{\theta = 1, \gamma = 1}$
0.1	0.7966	0.8501	0.9470	0.5503
0.2	0.8444	0.8880	0.9620	0.6191
0.3	0.8767	0.9126	0.9712	0.6736
0.4	0.9020	0.9314	0.9779	0.7224
0.5	0.9232	0.9468	0.9831	0.7685
0.6	0.9417	0.9600	0.9875	0.8136
0.7	0.9582	0.9715	0.9912	0.8585
0.8	0.9732	0.9820	0.9945	0.9042
0.9	0.9871	0.9913	0.9974	0.9511

Table 1 reveals that for varying values of the parameters of the KUG distribution, the random samples fall within the unit interval which conforms with the support of the random variable X following the KUG distribution.

2.3. The r^{th} moments and moment generating function of the KUG distribution

Let X be a continuous random variable with probability density function $f(x)$, then the r^{th} moment about the origin of X is defined by,

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx. \tag{15}$$

Substituting the series representation of the density function of the KUG distribution into (15), the r^{th} moment of the KUG distribution is obtained as

$$\mu_r' = \theta\beta\gamma \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{\theta-1}{i} (-1)^{i+j} \frac{[\gamma(i+1)]^j}{j!} e^{\gamma(i+1)} \int_0^1 x^{r-\beta(j+1)-1}\, dx \tag{16}$$

evaluating the integral part of (16) yields,

$$\mu_r' = \theta\beta\gamma \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{\theta-1}{i} (-1)^{i+j} \frac{[\gamma(i+1)]^j e^{\gamma(i+1)}}{j!\,[r-\beta(j+1)]}, \quad r = 1,2,3,4. \tag{17}$$

The first four r^{th} moment of the KUG distribution in terms of infinite series are obtained from (17) as;

$$\mu_1' = \theta\beta\gamma \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{\theta-1}{i} (-1)^{i+j} \frac{[\gamma(i+1)]^j e^{\gamma(i+1)}}{j!\,[1-\beta(j+1)]},$$

$$\mu_2' = \theta\beta\gamma \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{\theta-1}{i} (-1)^{i+j} \frac{[\gamma(i+1)]^j e^{\gamma(i+1)}}{j!\,[2-\beta(j+1)]},$$

$$\mu_3' = \theta\beta\gamma \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{\theta-1}{i} (-1)^{i+j} \frac{[\gamma(i+1)]^j e^{\gamma(i+1)}}{j!\,[3-\beta(j+1)]},$$

$$\mu_4' = \theta\beta\gamma \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{\theta-1}{i} (-1)^{i+j} \frac{[\gamma(i+1)]^j e^{\gamma(i+1)}}{j!\,[4-\beta(j+1)]}.$$

The variance, coefficients of skewness and kurtosis of the KUG distribution can be derived by substituting the values of the r^{th} moments into the expressions below;

$$Variance(\sigma^2) = \left(\mu_2^{'} - \mu^2\right),$$

$$Skewness(S_k) = \frac{\mu_3^{'} - 3\mu_2^{'}\mu + 2\mu^3}{\left(\mu_2^{'} - \mu^2\right)^{3/2}},$$

$$Kurtosis(K_s) = \frac{\mu_4^{'} - 4\mu_3^{'}\mu + 6\mu_2^{'}\mu^2 - 3\mu^4}{\left(\mu_2^{'} - \mu^2\right)^2}.$$

The moment generating function of a continuous random variable X with density function $f(x)$, is defined by

$$M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)dx, \tag{18}$$

thus, the moment generating function of the KUG distribution is defined by

$$M_X(t) = \theta\beta\gamma \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{\theta-1}{i} (-1)^{i+j} \frac{[\gamma(i+1)]^j t^k e^{\gamma(i+1)}}{j!\,k!\,[t-\beta(j+1)]}.$$

Numerical computation of the theoretical moments of the KUG distribution for selected values of the parameters are shown in Table 2.

Table 2: Theoretical moments of the KUG distribution for selected value of the parameters.

θ	β	γ	$\mu_1^{'}$	$\mu_2^{'}$	$\mu_3^{'}$	$\mu_4^{'}$	σ^2	S_k	K_s
1	1	1	0.5963	0.4037	0.2982	0.2339	0.0481	0.0074	1.9908
		2	0.7227	0.5547	0.4453	0.3698	0.0324	-0.4149	2.3363
		4	0.8254	0.6985	0.6031	0.5293	0.0172	-0.8246	3.119
	2	1	0.7579	0.5963	0.4843	0.4037	0.0219	-0.2508	1.6263
		2	0.8427	0.7227	0.6290	0.5547	0.0126	-0.8360	5.5408
		4	0.9054	0.8254	0.7572	0.6985	0.0057	-0.8126	-0.4874
2	1	1	0.4700	0.2526	0.1510	0.0981	0.0317	0.4394	2.6128
		2	0.6199	0.4109	0.2876	0.2103	0.0266	-0.0289	2.2005
		4	0.7525	0.5828	0.4625	0.3747	0.0165	-0.4507	2.6499

	2	1	0.6730	0.4700	0.3395	0.2526	0.0171	0.0950	1.6817
		2	0.7801	0.6199	0.5009	0.4109	0.0113	-0.3187	2.6127
		4	0.8641	0.7525	0.6601	0.5828	0.0058	-0.4816	-3.0588

From Table 2, we observed that the KUG distribution exhibits a right-skewed ($S_k > 0$), left-skewed ($S_k < 0$) and approximately symmetric ($S_k \approx 0$) shapes. Also, considering the peak of the distribution, the KUG distribution can be leptokurtic ($K_s > 3$), platykurtic ($K_s < 3$) and mesokurtic ($K_s \approx 3$). The negative kurtosis belongs to the platykurtic class of distribution with a broad shoulder. This means that the peak of the density function of the KUG distribution is comparatively lower than that of the Normal distribution. See Peter (2014) for more detail on negative kurtosis.

2.4. The Renyi entropy of the KUG distribution

An entropy of a random variable X is a measure of variation of uncertainty associated with the random variable X. Renyi [17] defined the Renyi entropy of X with density function $f(x)$, as

$$\tau_R(\xi) = \frac{1}{1-\xi} ln\left(\int f^{\xi}(x)dx\right), \quad \xi > 0,\ \xi \neq 1. \tag{19}$$

By substituting the density function of the KUG distribution defined in (7) into (19), we obtain

$$\tau_R(\xi) = \frac{1}{1-\xi} ln \int \left[\theta\beta\gamma x^{-(\beta+1)} e^{-\gamma(x^{-\beta}-1)} \left\{1 - e^{-\gamma(x^{-\beta}-1)}\right\}^{\theta-1}\right]^{\xi} dx, \tag{20}$$

using the binomial expansion defined in (9), we obtain

$$\left\{1 - e^{-\gamma(x^{-\beta}-1)}\right\}^{\xi\theta-\xi} = \sum_{i=0}^{\infty} \binom{\xi\theta-\xi}{i} (-1)^{i\xi} e^{-\gamma(x^{-\beta}-1)i\xi},$$

$$e^{-\gamma\xi(i+1)(x^{-\beta}-1)} \approx e^{-\gamma\xi(i+1)x^{-\beta}+\gamma\xi(i+1)},$$

using the exponential series representation, we have

$$e^{-\gamma\xi(i+1)x^{-\beta}} = \sum_{k=0}^{\infty} (-1)^k \frac{[\gamma\xi(i+1)]^k x^{-\beta k}}{k!},$$

So that (20) now becomes

$$\tau_R(\xi) = \frac{1}{1-\xi} ln\left[(\theta\beta\gamma)^{\xi} \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \binom{\xi\theta-\xi}{i} \frac{(-1)^{\xi i+k}[\gamma\xi(i+1)]^k}{k!} e^{\gamma\xi(i+1)} \int_0^1 x^{-\beta(\xi+k)-\xi} dx\right], \tag{21}$$

evaluating the integral part of (21) gives

$$\tau_R(\xi) = \frac{1}{1-\xi} ln\left[(\theta\beta\gamma)^{\xi}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\binom{\xi\theta-\xi}{i}\frac{(-1)^{\xi i+k}[\gamma\xi(i+1)]^k e^{\gamma\xi(i+1)}}{k!\,[1-\beta(\xi+k)-\xi]}\right]. \quad (22)$$

Some significant properties of the measure given in (19) is reported in Opone and Iwerumor [13] as;

(i) The Renyi entropy can be negative;

(ii) For any $\xi_1 < \xi_2, R_{\xi_2} \leq R_{\xi_1}$ and equality holds if and only if X is a uniform random variable.

Numerical computation of the Renyi entropy of the KUG Distribution for varying values of parameter ξ is shown in Table 3.

Table 3: Numerical computation of the Renyi entropy of the KUGD ($\gamma = 1$).

i	ξ_i	$\theta = 2, \beta = 0.2$	$\theta = 1, \beta = 0.2$	$\theta = 2, \beta = 3$	$\theta = 5, \beta = 4$
1	0.01	-0.0113	-0.0022	-0.3851	-0.5451
2	0.03	-0.0339	-0.0065	-0.5399	-0.7255
3	0.5	-0.4994	-0.1176	-1.0527	-1.3556
4	0.8	-0.7384	-0.1964	-1.1358	-1.4644
5	2	-1.3424	-0.5348	-1.2797	-1.6479
6	4	-1.7433	-0.9432	-1.3675	-1.7541
7	6	-1.9120	-1.1476	-1.4094	-1.8029
8	8	-2.0046	-1.2614	-1.4349	-1.8320

Table 3 clearly conform with the conditions that for any two successive values of parameters ξ_i, Say (ξ_1 and ξ_2), the Renyi entropy R_{ξ_i} Say (R_{ξ_1} and R_{ξ_2}), must satisfies $\xi_1 < \xi_2, R_{\xi_2} \leq R_{\xi_1}$ as stated in Opone and Iwerumor [13].

2.5. The distribution of order statistics of the KUG distribution

Suppose that $Y_{1:n} < Y_{2:n} < \cdots < Y_{n:n}$ is the order statistics of a random sample generated from KUG distribution, then the probability density function of the k^{th} order statistics, say $X = Y_{n:n}$ is given by

$$h_k(x) = \frac{n!}{(n-k)!\,(k-1)!}[F(x)]^{k-1}[1-F(x)]^{n-k} f(x), \tag{23}$$

substituting the cumulative distribution function and the density function of KUG distribution defined in (8) and (7) into (23), we have

$$h_k(x) = \frac{n!}{(n-k)!(k-1)!}\left[1-\left[1-e^{-\gamma\left(x^{-\beta}-1\right)}\right]^{\theta}\right]^{k-1}\left[1-\left(1-\left[1-e^{-\gamma\left(x^{-\beta}-1\right)}\right]^{\theta}\right)\right]^{n-k}$$

$$\times\, \theta\beta\gamma x^{-(\beta+1)} e^{-\gamma\left(x^{-\beta}-1\right)}\left\{1-e^{-\gamma\left(x^{-\beta}-1\right)}\right\}^{\theta-1}. \tag{24}$$

Using the binomial series expansion defined in (9), we have

$$\left[1-\left(1-\left[1-e^{-\gamma\left(x^{-\beta}-1\right)}\right]^{\theta}\right)\right]^{n-k} = \left\{1-e^{-\gamma\left(x^{-\beta}-1\right)}\right\}^{\theta(n-k)}$$

$$= \sum_{i=0}^{\infty}\binom{\theta(n-k)}{i}(-1)^i e^{-\gamma\left(x^{-\beta}-1\right)i}$$

$$\left\{1-\left[1-e^{-\gamma\left(x^{-\beta}-1\right)}\right]^{\theta}\right\}^{k-1} = \sum_{j=0}^{\infty}\binom{k-1}{j}(-1)^j\left\{1-e^{-\gamma\left(x^{-\beta}-1\right)}\right\}^{\theta j}$$

$$\left[1-e^{-\gamma\left(x^{-\beta}-1\right)}\right]^{\theta(j+1)-1} = \sum_{m=0}^{\infty}\binom{\theta(j+1)-1}{m}(-1)^m e^{-\gamma\left(x^{-\beta}-1\right)m}$$

Using the exponential series expansion,

$$e^{-x} = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!},$$

$$e^{-\gamma\left(x^{-\beta}-1\right)[m+i+1]} \approx e^{-\gamma[m+i+1]x^{-\beta}+\gamma[m+i+1]},$$

so that

$$e^{-\gamma[m+i+1]x^{-\beta}} = \sum_{p=0}^{\infty}\frac{[-\gamma(m+i+1)]^p x^{-\beta p}}{p!},$$

hence, (24) now becomes

$$h_k(x) = \frac{n!\,\theta\beta\gamma}{(n-k)!\,(k-1)!}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\binom{\theta(n-k)}{i}\binom{k-1}{j}\binom{\theta(j+1)-1}{m}$$

$$\times \frac{(-1)^{i+j+m}[-\gamma(m+i+1)]^p e^{\gamma[m+i+1]}}{p!} x^{-\beta(p+1)-1}. \quad (25)$$

The z^{th} moment of the k^{th} order statistics from the KUG distribution is defined by

$$E(X_k^z) = \int_0^1 x^z h_k(x)dx \quad (26)$$

$$E(X_k^z) = \frac{n!\,\theta\beta\gamma}{(n-k)!\,(k-1)!} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty} \binom{\theta(n-k)}{i}\binom{k-1}{j}\binom{\theta(j+1)-1}{m}$$

$$\times (-1)^{i+j+m}[-\gamma(m+i+1)]^p \frac{e^{\gamma[m+i+1]}}{p!} \int_0^1 x^{z-\beta(p+1)-1}dx. \quad (27)$$

Differentiating the integral part of (27) yields

$$E(X_k^z) = \frac{n!\,\theta\beta\gamma}{(n-k)!\,(k-1)!} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty} \binom{\theta(n-k)}{i}\binom{k-1}{j}\binom{\theta(j+1)-1}{m}$$

$$\times (-1)^{i+j+m}[-\gamma(m+i+1)]^p \frac{e^{\gamma[m+i+1]}}{p![z-\beta(p+1)]}. \quad (28)$$

3. Parameter Estimation

3.1. Maximum likelihood estimation

In this subsection, we present the maximum likelihood estimates (MLEs) of the parameters of KUG distribution (θ, β, γ). Let $x_1, x_2, x_3 \ldots, x_n$ be random samples from the KUG distribution with density function defined in (7), then the log-likelihood function is given by

$$\ell(x,\varphi) = \sum_{i=1}^{n} ln[f(x)], \quad (29)$$

$$= \sum_{i=1}^{n} ln\left[\theta\beta\gamma x^{-(\beta+1)} e^{-\gamma(x^{-\beta}-1)}\left\{1-e^{-\gamma(x^{-\beta}-1)}\right\}^{\theta-1}\right], \quad \varphi = (\gamma,\theta,\beta),$$

$$= ln(\theta\beta\gamma) - (\beta+1)\sum_{i=1}^{n} lnx_i - \gamma\sum_{i=1}^{n}\left(x_i^{-\beta}-1\right) + (\theta-1)\sum_{i=1}^{n} ln\left(1-e^{-\gamma(x_i^{-\beta}-1)}\right). \quad (30)$$

The estimates of the unknown parameters of the KUG distribution are obtained by differentiating the log-likelihood function with respect to the parameters of the distribution and then equate the corresponding equation to zero. Thus, we have

$$\frac{\partial(\ell,\varphi)}{\partial\theta}=\frac{n}{\theta}+\sum_{i=0}^{n}ln\left[1-e^{-\gamma(x_i^{-\beta}-1)}\right],$$

$$\frac{\partial(\ell,\varphi)}{\partial\beta}=\frac{n}{\beta}-\sum_{i=1}^{n}ln\,(x_i)+\gamma\sum_{i=1}^{n}ln\,(x_i)x_i^{-\beta}-(\theta-1)\sum_{i=1}^{n}\frac{e^{-\gamma(x_i^{-\beta}-1)}x_i^{-\beta}\gamma ln(x_i)}{1-e^{-\gamma(x_i^{-\beta}-1)}},$$

$$\frac{\partial(\ell,\varphi)}{\partial\gamma}=\frac{n}{\gamma}-\sum_{i=1}^{n}\left(x_i^{-\beta}-1\right)+(\theta-1)\sum_{i=1}^{n}\frac{-e^{-\gamma(x_i^{-\beta}-1)}+x_i^{-\beta}e^{-\gamma(x_i^{-\beta}-1)}}{1-e^{-\gamma(x_i^{-\beta}-1)}}.$$

The maximum likelihood estimates $\hat{\varphi}$ of the parameters φ are obtained by solving the system of non-linear equation $\frac{\partial\ell(x,\varphi)}{\partial\varphi}=0$. This equation can be solved using a numerical method known as New Raphson iterative scheme given by

$$\hat{\varphi}=\varphi_k-H^{-1}(\varphi_k)U(\varphi_k),\quad \hat{\varphi}=\left(\hat{\theta},\hat{\beta},\hat{\gamma}\right)^T.$$

Where $U(\varphi_k)$ is the score function and $H(\varphi_k)$ is the Hessian matrix which is the second partial derivative of the log-likelihood function. The "bbmle" package in R statistical software program is used to evaluate the maximum likelihood estimates of the parameters of the KUG distribution.

3.2. Interval estimate

The asymptotic confidence intervals (CIs) for the parameters of KUG(θ,β,γ) distribution are obtained according to the asymptotic distribution of the maximum likelihood estimates of the parameters. Suppose $\hat{\varphi}=\left(\hat{\theta},\hat{\beta},\hat{\gamma}\right)$ be MLE of φ, then the estimators are approximately bi-variate normal with mean(θ,β,γ) and the Fisher information matrix is given by

$$I(\varphi_k)=-E\left(H(\varphi_k)\right). \tag{31}$$

The approximate (1-δ)100 CIs for the parameters θ,β and γ are respectively given by

$$\hat{\theta}\pm Z_{\frac{\delta}{2}}\sqrt{var\left(\hat{\theta}\right)},\quad \hat{\beta}\pm Z_{\frac{\delta}{2}}\sqrt{var\left(\hat{\beta}\right)}\quad \text{and}\quad \hat{\gamma}\pm Z_{\frac{\delta}{2}}\sqrt{var(\hat{\gamma})}$$

where $var\left(\hat{\theta}\right)$, $var\left(\hat{\beta}\right)$ and $var(\hat{\gamma})$ are the variance of $\theta,\ \beta$ and γ which are given by the diagonal elements of the variance-covariance matrix $I^{-1}(\varphi_k)$ and $Z_{\delta/2}$ is the upper $({}^{\delta}/_{2})$ percentile of the standard normal distribution.

3.3. Simulation Study

In this subsection, we investigate the asymptotic behaviour of the maximum likelihood estimate of the parameters of the Kumaraswamy unit-Gompertz distribution (KUGD) through a simulation study. A Monte Carlo simulation study is repeated 10,000 times for different sample sizes n = 30, 50, 100, 200 and parameter values $(\theta = 0.5, \beta = 0.1, \gamma = 0.3)$, $(\theta = 0.5, \beta = 0.1, \gamma = 0.5)$, $(\theta = 0.5, \beta = 0.5, \gamma = 0.3)$, $(\theta = 0.5, \beta = 0.5, \gamma = 0.5)$, $(\theta = 0.8, \beta = 0.1, \gamma = 0.3)$, $(\theta = 0.8, \beta = 0.1, \gamma = 0.5)$, $(\theta = 0.8, \beta = 0.5, \gamma = 0.3)$, $(\theta = 0.8, \beta = 0.5, \gamma = 0.5)$. Three quantities which include the Bias, Root Mean Square Error (*RMSE*) and the Coverage Probability (*CP*) of the 95% Confidence Intervals (CIs) for the parameter estimates will be considered as Statistical measures for investigating the asymptotic behaviour of the maximum likelihood estimate of the parameters of the Kumaraswamy unit-Gompertz distribution.

Table 4: Monte Carlo simulation results for Bias, RMSE and CP of parameter estimates of KUGD.

θ	β	γ	n	$bias(\theta)$	$bias(\beta)$	$bias(\gamma)$	$RMSE(\theta)$	$RMSE(\beta)$	$RMSE(\gamma)$	$CP(\theta)$	$CP(\beta)$	$CP(\gamma)$
0.5	0.1	0.3	30	0.0384	0.0148	0.2307	0.2074	0.0557	0.7701	0.9571	0.9857	0.8486
			50	0.0142	0.0093	0.1170	0.1244	0.0415	0.5051	0.9543	0.9629	0.8543
			100	0.0069	0.0045	0.0605	0.0837	0.0270	0.2757	0.9571	0.9543	0.8886
			200	0.0060	0.0019	0.0225	0.0574	0.0184	0.1489	0.9543	0.9486	0.9029
		0.5	30	0.0095	0.0272	0.2283	0.1620	0.0744	0.9408	0.9400	0.9886	0.7886
			50	0.0166	0.0153	0.1602	0.1148	0.0502	0.7405	0.9571	0.9800	0.8057
			100	0.0082	0.0057	0.0957	0.0823	0.0320	0.4707	0.9629	0.9457	0.8629
			200	0.0039	0.0028	0.0435	0.0552	0.0214	0.2740	0.9571	0.9629	0.9143
	0.5	0.3	30	0.0428	0.0947	0.2210	0.1866	0.3132	0.8304	0.9714	0.9743	0.8314
			50	0.0209	0.0480	0.1508	0.1219	0.2155	0.7397	0.9657	0.9571	0.8343
			100	0.0148	0.0236	0.0580	0.0881	0.1369	0.2910	0.9486	0.9543	0.8657
			200	0.0049	0.0157	0.0162	0.0604	0.0920	0.1475	0.9400	0.9543	0.9057
		0.5	30	0.0276	0.1541	0.1932	0.1678	0.3521	1.1068	0.9600	0.9914	0.7914
			50	0.0171	0.0805	0.1997	0.1183	0.2568	0.9671	0.9686	0.9629	0.8114
			100	0.0021	0.0459	0.0537	0.0729	0.1627	0.4300	0.9600	0.9600	0.8600
			200	0.0009	0.0234	0.0215	0.0542	0.1085	0.2659	0.9657	0.9543	0.9086

0.8	0.1	0.3	30	0.1135	0.0125	0.2526	0.4125	0.0575	0.8160	0.9343	0.9914	0.8257
			50	0.0581	0.0080	0.1361	0.2749	0.0398	0.5081	0.9657	0.9743	0.8743
			100	0.0299	0.0035	0.0748	0.1841	0.0273	0.3042	0.9371	0.9286	0.8743
			200	0.0140	0.0019	0.0250	0.1160	0.0172	0.1552	0.9400	0.9600	0.9114
		0.5	30	0.0559	0.0169	0.2529	0.3226	0.0590	0.9389	0.9314	0.9971	0.8286
			50	0.0220	0.0094	0.1752	0.2219	0.0421	0.7406	0.9571	0.9743	0.8771
			100	0.0207	0.0056	0.0719	0.1650	0.0290	0.3865	0.9543	0.9657	0.8857
			200	0.0068	0.0030	0.0267	0.0987	0.0202	0.2214	0.9514	0.9514	0.9114
	0.5	0.3	30	0.1211	0.0513	0.2640	0.4609	0.2593	0.7932	0.9457	0.9857	0.8429
			50	0.0627	0.0332	0.1821	0.3020	0.2073	0.6058	0.9200	0.9486	0.8343
			100	0.0235	0.0161	0.0679	0.1825	0.1288	0.3300	0.9457	0.9657	0.8943
			200	0.0191	0.0041	0.0373	0.1256	0.0877	0.1775	0.9286	0.9514	0.9257
		0.5	30	0.0489	0.0900	0.2706	0.3360	0.3072	0.9779	0.9400	0.9943	0.8286
			50	0.0453	0.0496	0.2189	0.2425	0.2227	0.8356	0.9600	0.9771	0.8514
			100	0.0332	0.0280	0.0964	0.1752	0.1485	0.4840	0.9543	0.9543	0.8943
			200	0.0022	0.0195	0.0244	0.1023	0.1012	0.2290	0.9486	0.9657	0.8914

Table 4 shows the Monte Carlo simulation results for Bias, RMSE and CP of parameter estimates of Kumaraswamy unit-Gompertz distribution. From the Table, we observe that the bias and the root mean square error of the parameter estimates decreases as the sample size n increases, which validates the consistency property of an estimator. Finally, the coverage probability of the 95% confidence interval of the parameter estimates are very close to the nominal level of 95%.

4. Application of the KUG Distribution to Lifetime Datasets

In this section, we apply the KUG distribution together with some existing lifetime distributions with bounded support to two real datasets. The parameter estimates of the distributions, Log-likelihood, Akaike Information Criterion (AIC), Crammer-von Mises test statistic (W^*) and the Anderson Darling test statistic (A^*) with their respective p-values will be employed as statistical tools for suitable model selection. These lifetime distributions with their density function include;

1. Marshall-Olkin Extended Kumaraswamy Distribution (MOEKD) due to George and Thobias [6];

$$f(x) = \frac{\alpha a b x^{a-1}(1-x^a)^{b-1}}{[1-\alpha(1-x^a)^b]^2},$$

2. Kumaraswamy Distribution due to Kumaraswamy [10];

$$f(x) = abx^{a-1}(1-x^a)^{b-1},$$

3. Unit-Gompert Distribution due to Mazucheli et al. [11];

$$f(x) = \lambda\theta x^{-(\theta+1)}e^{-\lambda(x^{-\theta}-1)}.$$

Dataset 1: The dataset consists of 48 rock samples from a petroleum reservoir reported in Cordeiro and Brito [2]. The dataset is defined on a unit interval which is positively (right) skewed with skewness value (S_k = 1.1330) and leptokurtic with kurtosis value ($K_s = 3.9404$). The data set is shown in Table 5.

Table 5: Rock samples from a petroleum reservoir.

0.0903296	0.2036540	0.2043140	0.2808870	0.1976530	0.3286410
0.1486220	0.1623940	0.2627270	0.1794550	0.3266350	0.2300810
0.1833120	0.1509440	0.2000710	0.1918020	0.1541920	0.4641250
0.1170630	0.1481410	0.1448100	0.1330830	0.2760160	0.4204770
0.1224170	0.2285950	0.1138520	0.2252140	0.1769690	0.2007440
0.1670450	0.2316230	0.2910290	0.3412730	0.4387120	0.2626510
0.1896510	0.1725670	0.2400770	0.3116460	0.1635860	0.1824530
0.1641270	0.1534810	0.1618650	0.2760160	0.2538320	0.2004470

Dataset 2: The second dataset represents 20 observations of the maximum flood level (in millions of cubic feet per second) for Susquehanna River at Harrisburg, Pennsylvania.

The data set include; 0.26, 0.27, 0.30, 0.32, 0.32, 0.34, 0.38, 0.38, 0.39, 0.40, 0.41, 0.42, 0.42, 0.42, 0.45, 0.48, 0.49, 0.61, 0.65, 0.74.

The data set was first reported in Dumonceaux and Antle [5], and was recently used in Opone and Osemwenkhae [14] to illustrate the potentials of the transmuted Marshall-Olkin extended Topp-Leone distribution in real life data fitting. The dataset is right skewed with skewness value ($S_k = 0.9939$) and leptokurtic with kurtosis value ($K_s = 3.3053$).

Table 6: Summary statistics for rock sample data set.

Distributions	*Parameter Estimate*	*Log-Lik*	*AIC*	W^* ***(p-value)***	A^* ***(p-value)***
KUGD	$\theta = 3.1154$ $\beta = 1.6812$ $\gamma = 0.1162$	58.3549	-110.7097	0.0294 (**0.9793**)	0.1917 (**0.9926**)
MOEKD	$\alpha = 0.0214$ $a = 4.8120$ $b = 48.3554$	57.7042	-109.4084	0.0462 (**0.9013**)	0.3262 (**0.9169**)
Unit Gompertz	$\lambda = 0.0053$ $\theta = 2.9893$	56.6437	-109.2874	0.0433 (**0.9176**)	0.3572 (**0.8893**)
Kumaraswamy	$a = 2.7187$ $b = 44.6604$	52.4915	-100.9831	0.2060 (**0.2566**)	1.2892 (**0.2358**)

Table 7: Summary statistics for flood level data set.

Distributions	*Parameter Estimate*	*Log-Lik*	*AIC*	W^* ***(p-value)***	A^* ***(p-value)***
KUGD	$\theta = 1.5622$ $\beta = 3.1566$ $\gamma = 0.0587$	16.5086	-27.0172	0.0463 (**0.9031**)	0.2701 (**0.9585**)
MOEKD	$\alpha = 0.0153$ $a = 6.4543$ $b = 5.4128$	15.9235	-25.8471	0.0410 (**0.9320**)	0.3133 (**0.9271**)
Unit Gompertz	$\alpha = 0.0151$ $\beta = 4.1149$	16.3664	-28.7329	0.0532 (**0.8621**)	0.2934 (**0.9425**)
Kumaraswamy	$\alpha = 3.3777$ $b = 12.0057$	12.9733	-21.9465	0.1653 (**0.3482**)	0.9366 (**0.3911**)

Tables 6 and 7 respectively reveal the summary statistics for the rock samples from a petroleum reservoir and the maximum flood level datasets. The parameter estimates, log-likelihood, Akaike Information Criterion (AIC), Crammer-von Mises test statistic (W^*)

and Anderson Darling test statistic (A^*) with their respective *p*-values of the distributions were computed for each data sets. The Tables indicate that the proposed Kumaraswamy unit-Gompertz distribution, having the maximized log-likelihood value and the least value in terms of AIC, W^* and A^* test statistics with the highest corresponding *p-values*, outperforms the Marshall-Olkin Kumaraswamy distribution, Kumaraswamy distribution and the unit-Gompertz distribution in analyzing the two real datasets under study. Further illustration of the flexibility of the proposed KUG distribution was investigated by considering the density fit and the Quantile-Quantile (Q-Q) plots of the distributions for the two datasets as shown in Figures 3 and 4 respectively.

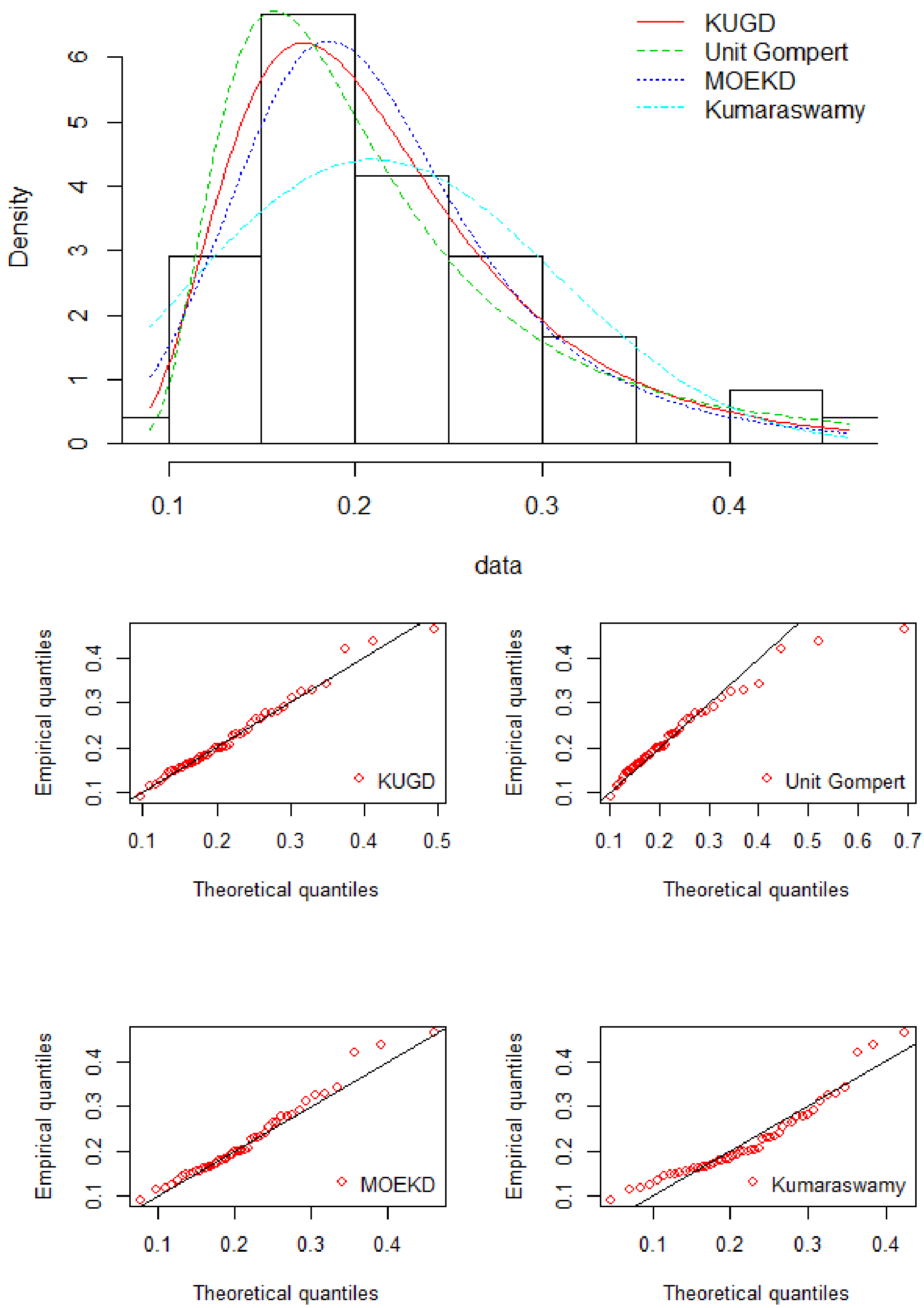

Figure 3: Density fit and Quantile-Quantile (Q-Q) plots of the rock samples dataset.

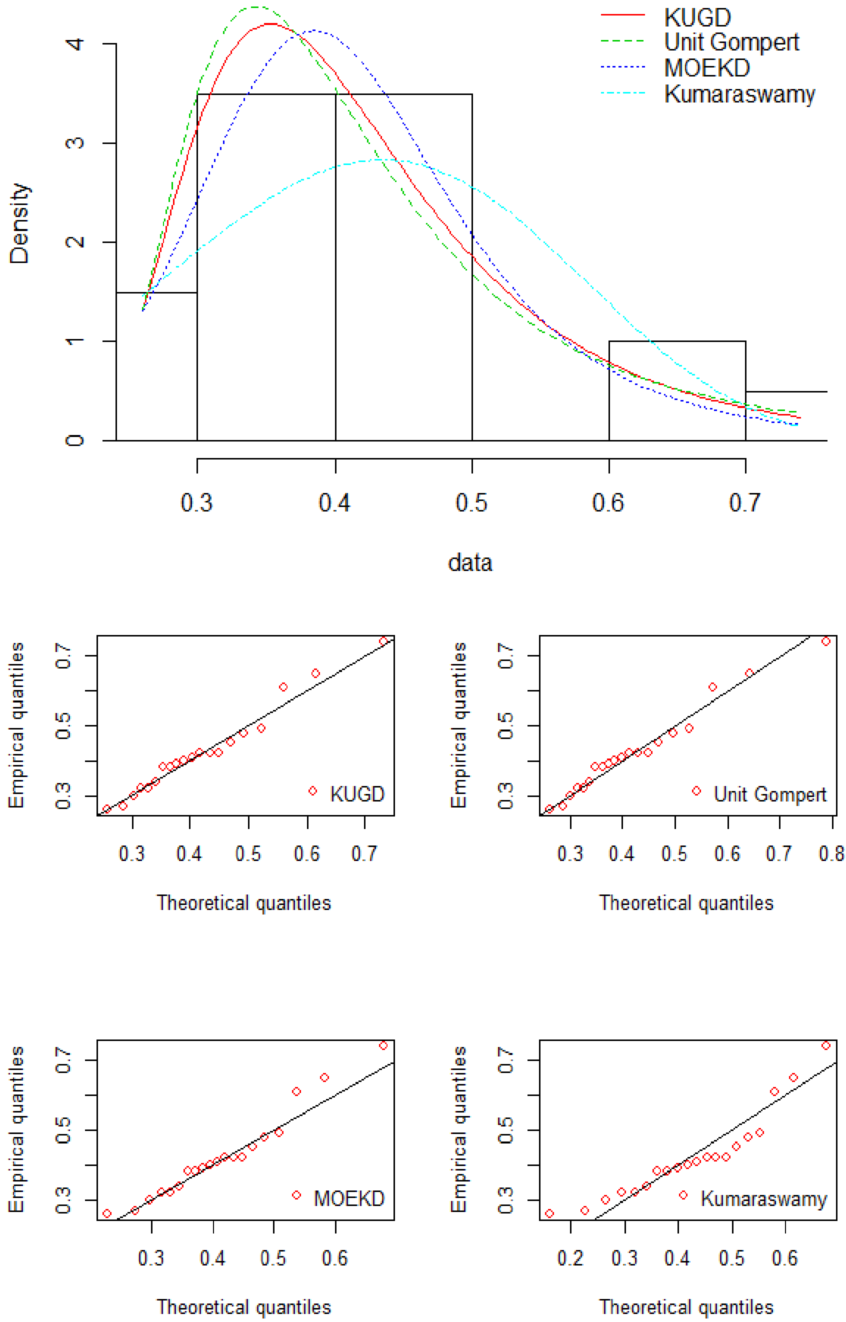

Figure 4: Density fit and Quantile-Quantile (Q-Q) plots of the flood level dataset.

5. Concluding Remark

In this paper, we introduced a new generalized bounded distribution called the Kumaraswamy unit-Gompertz distribution. The mathematical properties of the proposed KUG distribution which include; the density function, cumulative distribution function, survival function, hazard rate function, moments, mode, median, quantile function, Renyi entropy and the distribution of order statistics were obtained. Numerical computations of the quantiles, moments as well as the Renyi entropy of the KUG distribution were established and the method of maximum likelihood estimation was used in estimating the unknown parameters of the proposed KUG distribution. A Monte Carlo simulation study was carried out to investigate the asymptotic behaviour of the parameter estimates of the KUG distribution. Finally, two real bounded datasets were used to illustrate the applicability of the proposed KUG distribution in lifetime data analysis.

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