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m-commuting Additive Maps on Upper Triangular Matrices Rings

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Abstract

Let $T_n(R)$ be the upper triangular matrix ring over a unital commutative ring whose characteristic is not a divisor of m. Suppose that $f: T_n(R) \to T_n(R)$ is an additive map such that $X^m f(X) = f(X) X^m$ for all $X \in T_n(R)$, where $m \geq 1$ is an integer. We consider the problem of describing the form of the map $X \to f(X)$.

1. Introduction and Results

For a ring R we say that the map $f: R \to R$ is commuting if $[f(x), x] = 0$ for every $x \in R$, where $[a, b] = ab - ba$ denotes the standard commutator. The study of such maps was inspired by Posner [13] who proved that if a prime ring has a nonzero centralizing derivation, then it must be commutative. This theorem was generalized in many ways (see for instance [1, 10, 11, 12, 15]). The first general result regarding commuting maps comes from Brešar [4] who has shown that additive commuting maps f over a simple unital ring R must be of the form $f(x) = \lambda x + \mu : R \to Z(R)$, where $Z(R)$ denotes the center of R. This form is usually called a standard form for the commuting map.

There are plenty of results on commuting maps (for example [7, 8, 9, 16]) and the

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readers are referred to the survey paper [5] for acquaintance with the development of the theory of commuting maps and the various results that have been established.

In 2000, Beidar et al. [2] proved that a similar result holds true over $T_r(F)$, the ring of $r \times r$ upper triangular matrices over a field F. Their work showed that any linear commuting map $f: T_r(F) \to T_r(F)$ is again of the standard form, so $f(x) = \lambda x + \mu(x)$ for some $\lambda \in F$ and linear map $\mu : T_r(F) \to Z(T_r(F))$. In [6] Cheung extended this result to triangular algebras.

Recently, in [3], Bounds extended some of these results to the case $N_r(F)$ - the ring of strictly upper triangular matrices over a field F of characteristic zero. The author proved that if $f: N_r(F) \to N_r(F)$ is a commuting linear map, then there exists $\lambda \in F$ and an additive map $\mu : N_r(F) \to \Omega$ such that $f(x) = \lambda x + \mu(x)$ for all $x \in N_r(F)$, where $\Omega = \{ae_{1,r-1} + be_{1,r} + ce_{2,r} : a, b, c \in F\}$ and $e_{i,j}$ denotes the standard matrix unit.

For a positive integer m, a map $f: R \to R$ is said to be *m-power commuting* if $[f(x), x^m] = 0$ for all $x \in R$. Clearly, every commuting map is a 1-power commuting map. In [17] Brešar and Hvala studied 2-power commuting additive maps and showed that if R is a prime ring with the extended centroid C, $charR = 2$ and $f: R \to R$ is a 2-power commuting additive map, then there exist $\lambda \in C$ and an additive map $\delta : R \to C$ such that $f(x) = \lambda x + \delta(x)$ for all $x \in R$. Later, Beidar et al. [18] extended this result to m-power commuting additive maps and proved that if R is a prime ring with the extended centroid C, $charR = 0$ or $charR > m$ and $f: R \to R$ is an m-power commuting additive map, then there exist $\lambda \in C$ and an additive map $\delta : R \to C$ such that $f(x) = \lambda x + \delta(x)$ for all $x \in R$. Recently, in [19], Liu and Yang characterize the m-power commuting additive maps on invertible or singular matrices.

In this paper, we examine m-power commuting additive maps over the ring of upper triangular matrices $T_n(R)$ over a commutative ring R whose characteristic is not a divisor of m. Precisely, we will show that the following theorems are true.

Theorem 1. *Let R be a commutative ring. If $f: T_n(R) \to T_n(R)$ is an additive commuting map, then there exist $\lambda \in R$ and an additive map $\delta : T_n(R) \to R$ such that $f(X) = \lambda X + \delta(X) I_n$ for all $X \in T_n(R)$.*

Theorem 2. *Let R be a commutative ring whose characteristic is not a divisor of m. If* $f : T_n(R) \to T_n(R)$ *is an additive m-commuting map, then there exist* $\lambda \in R$ *and an additive map* $\delta : T_\infty(R) \to R$ *such that* $f(X) = \lambda X + \delta(X) I_n$ *for all* $X \in T_n(R)$.

2. Preliminaries and Commuting Additive Maps of $T_n(R)$

We denote by $T_n(R)$ a set of upper triangular matrices over a commutive ring *R*. For $1 \le i < j$ by e_{ij} we mean the matrix unit - the matrix whose only nonzero entry is 1 in the (i, j)th position. It is known that the product of e_{ij} and e_{kl} is equal to $e_{ij} \times e_{kl} = \delta_{jk} e_{il}$, where δ_{ij} is the Kronecker delta.

For any $A = (a_{ij}) \in T_n(R)$, to abbreviate notation and facilitate calculations, we will write $A = \sum_{1 \le i \le j \le n} a_{ij} e_{i,j}$. In particular, we put $I_n = \sum_{i=1}^{n} e_{i,i}$ and we write $Z(T_n(R))$ for the center of $T_n(R)$, $Z(T_n(R)) = RI_n$.

In order to prove our main result, we first need to establish following lemmas.

Lemma 2.1. *Suppose that* $f : T_n(R) \to T_n(R)$ *is an additive map satisfying* $Xf(X) = f(X)X$ *for all* $X \in T_n(R))$. *Then* $f(X)Y + f(Y)X - Xf(Y) - Yf(X) = 0$ *for all* $X, Y \in T_n(R)$.

Lemma 2.2. *Suppose that* $f : T_n(R) \to T_n(R)$ *is an additive map satisfying* $Xf(X) = f(X)X$ *for all* $X \in T_n(R)$ *and* $r \in R$. *Then there exists* $\lambda \in R$ *such that* $f(rI_n) = \lambda r I_n$.

Proof. Let $r \in R$ for every $X \in T_n(R)$, $f(X)(rI_n) + f(rI_n)X - Xf(rI_n) - (rI_n)f(X) = 0$. Then $[f(rI_n), X] = 0$. Thus, $f(rI_n) \in Z(T_n(R))$. Consequently, there exists $\lambda \in R$ such that $f(rI_n) = \lambda r I_n$.

Lemma 2.3. *Suppose that* $f : T_n(R) \to T_n(R)$ *is an additive map satisfying* $Xf(X) = f(X)X$ *for all* $X \in T_n(R)$. *Then* $f(\alpha e_{ii})$ *is a diagonal matrix with* $\alpha \in R$, $f(\alpha e_{ii}) e_{ii} = e_{ii} f(\alpha e_{ii})$ *and* $f(\alpha e_{ij})(\alpha e_{ij}) = (\alpha e_{ij}) f(\alpha e_{ij})$ *for all distinct integers* i, j *with* $1 \le i < j \le n$.

Proof. Write $f(\alpha e_{ii}) = \sum_{1\leq r\leq s\leq n} a_{rs}^{ii}(\alpha)e_{r,s}$, where each $a_{rs}^{ii} : F \to F$ is an additive map for $1 \leq s \leq t \leq n$. We have $(\alpha e_{ii})f(\alpha e_{ii}) = f(\alpha e_{ii})(\alpha e_{ii})$ and $\alpha \neq 0$. This implies that $f(\alpha e_{ii})e_{ii} = e_{ii}f(\alpha e_{ii})$.

Let j be an integer such that $1 \leq j$ with $j \neq i$. Then we have $e_{ii}f(\alpha e_{ii})e_{jj} = e_{jj}f(\alpha e_{ii})e_{ii} = 0$. This implies $a_{ij}^{ii}(\alpha) = a_{ij}^{ii}(\alpha) = 0$.

Let j, k be an integer such that $1 \leq j, k$ with $j, k \neq i$ and $\alpha \neq 0$. For $X = \alpha e_{ii}$ and $Y = \beta e_{jj}$, we have

$$f(\alpha e_{ii})(\alpha e_{jj}) + f(\alpha e_{jj})(\alpha e_{ii}) - (\alpha e_{ii})f(\alpha e_{jj}) - (\alpha e_{jj})f(\alpha e_{ii}) = 0$$

we obtain

$$e_{ii}f(\alpha e_{jj}) + e_{jj}f(\alpha e_{ii}) = f(\alpha e_{jj})e_{ii} + f(\alpha e_{ii})e_{jj}.$$

Multiplying by e_{kk} from the left and by e_{jj} from the right, we get $e_{kk}f(\alpha e_{ii})e_{jj} = 0$. This implies $a_{kj}^{ii}(\alpha) = 0$. Thus $f(\alpha e_{ii})$ is a diagonal matrix, as desired. Write $f(\alpha e_{ii}) = \sum_{1\leq r\leq n} a_{rr}^{ii}(\alpha)e_{r,r}$. From $f(\alpha e_{ij} + I_n)(\alpha e_{ij} + I_n) = (\alpha e_{ij} + I_n)f(\alpha e_{ij} + I_n)$ we have $f(\alpha e_{ij})(\alpha e_{ij}) = (\alpha e_{ij})f(\alpha e_{ij})$ and $\alpha \neq 0$. This implies $f(\alpha e_{ij})e_{ij} = e_{ij}f(\alpha e_{ij})$.

Lemma 2.4. *For all $\alpha \in R$, there exist $\lambda_i \in R$, $\delta_i : R \to R$ additive map such that $f(\alpha e_{ii}) = \lambda_i \alpha e_{ii} + \delta_i(\alpha)I_n$ for all integers i with $i \geq 1$.*

Proof. Write $f(\alpha e_{ii}) = \sum_{1\leq r\leq n} a_{rr}^{ii}(\alpha)e_{r,r}$ for all distinct integers i, j, k with $(k \leq l, \neq i)$, we have

$$f(\alpha e_{ii})\beta e_{kl} - \beta e_{kl}f(\alpha e_{ii}) + f(\beta e_{kl})\alpha e_{ii} - \alpha e_{ii}f(\beta e_{kl}) = 0.$$

Multiplying by e_{kk} from the left and by e_{ll} from the right, we get

$$a_{kk}^{ii}(\alpha)\beta e_{kl} - \beta a_{ll}^{ii}(\alpha)e_{kl} = 0.$$

This implies $a_{kk}^{ii}(\alpha) = a_{ll}^{ii}(\alpha) = \delta_i(\alpha)$, it follows that

$$f(\alpha e_{ii}) = \sum_{1\le r\le n} a_{rr}^{ii}(\alpha)e_{r,r} = (a_{ii}^{ii}(\alpha) - \delta_i(\alpha))e_{ii} + \delta_i(\alpha)I_n.$$

For all distinct integers i, j with $(i < j)$ we have

$$f(\alpha e_{ii})\beta e_{ij} - \beta e_{ij} f(\alpha e_{ii}) + f(\beta e_{ij})\alpha e_{ii} - \alpha e_{ii} f(\beta e_{ij}) = 0.$$

Multiplying by e_{ii} from the left and by e_{jj} from the right, this implies

$$a_{ii}^{ii}(\alpha)\beta e_{ij} - \beta\delta_i(\alpha)e_{ij} - \alpha e_{ii} f(\beta e_{ij})e_{jj} = 0,$$

it follows that $a_{ii}^{ii}(\alpha)\beta - \beta\delta_i(\alpha) - \alpha a_{ij}^{ij}(\beta) = 0.$ Thus for $\beta = 1$ we get

$$a_{ii}^{ii}(\alpha) - \delta_i(\alpha) = \alpha a_{ij}^{ij}(1) = \alpha\lambda_{ij} = \alpha\lambda_i.$$

Hence $f(\alpha e_{ii}) = \alpha\lambda_i e_{ii} + \delta_i(\alpha)I_n.$

Lemma 2.5. *For all $\alpha, \beta \in R$, for all distinct integers i, j with $1 \le i < j \le n$, then there exists $\lambda \in R$, and the additives maps $\delta_i, a_{ii}^{ii} : R \to R$ such that $f(\alpha e_{ii}) = \lambda\alpha e_{ii} + \delta_i(\alpha)I_n$ and $f(\beta e_{ij}) = \lambda\beta e_{ij} + a_{ii}^{ii}(\beta)I_n$.*

Proof. Write $f(\alpha e_{ij}) = \sum_{1\le r\le s\le n} a_{rs}^{ii}(\alpha)e_{r,s}$, for all distinct integers i, j with $1 \le i < j \le n.$ Then by assumption,

$$f(\alpha e_{ii} + \beta e_{ij})(\alpha e_{ii} + \beta e_{ij}) = (\alpha e_{ii} + \beta e_{ij})f(\alpha e_{ii} + \beta e_{ij}).$$

Hence,

$$f(\beta e_{ij})\alpha e_{ii} + f(\alpha e_{ii})\beta e_{ij} = \alpha e_{ii} f(\beta e_{ij}) + \beta e_{ij} f(\alpha e_{ii}).$$

Multiplying by e_{ii} from the left and by e_{jj} from the right, this implies that

$$e_{ii} f(\alpha e_{ii})\beta e_{ij} = \alpha e_{ii} f(\beta e_{ij})e_{jj} + \beta e_{ij} f(\alpha e_{ii})e_{jj},$$

we obtain $\lambda_i\beta = a_{ij}^{ij}(\beta).$

For $k < i$. Multiplying by e_{kk} from the left and by e_{ii} from the right, this implies $a_{ij}^{ki}(\beta) = 0$.

For $i < l$ $(l \neq j)$. Multiplying by e_{ii} from the left and by e_{ll} from the right, this implies $a_{ij}^{il}(\beta) = 0$.

Similarly, using $f(\alpha e_{jj} + \beta e_{ij})(\alpha e_{jj} + \beta e_{ij}) = (\alpha e_{jj} + \beta e_{ij}) f(\alpha e_{jj} + \beta e_{ij})$ we have $\beta\lambda_j = a_{ij}^{ij}(\beta)$. Hence $\beta\lambda_j = a_{ij}^{ij}(\beta) = \beta\lambda_i$.

We notice $\lambda = \lambda_i = \lambda_j$. In particular, $f(\alpha e_{ii}) = \lambda\alpha e_{ii} = \delta_i(\alpha) I_\infty$ for $k < j$, $(k \neq i)$, by assumption, $f(\beta e_{ij})\alpha e_{jj} + f(\alpha e_{jj})\beta e_{ij} = \alpha e_{jj} f(\beta e_{ij}) + \beta e_{ij} f(\alpha e_{jj})$. Multiplying by e_{kk} from the left and by e_{jj} from the right, hence $f(\beta e_{ij})\alpha e_{ii}$ $+ f(\alpha e_{ii})\beta e_{ij} = \alpha e_{ii} f(\beta e_{ij}) + \beta e_{ij} f(\alpha e_{ii})$.

Multiplying by e_{ii} from the left and by e_{jj} from the right, this implies that $a_{ij}^{kj}(\beta) = 0$.

Similarly, using for $j < l$, $(l \neq i)$ multiplying by e_{jj} from the left and by e_{ll} from the right, it follows that $a_{ij}^{jl}(\beta) = 0$.

Let $l < k$ and $\notin \{i, j\}$. From $f(e_{kk} + \alpha e_{ij})(e_{kk} + \alpha e_{ij}) = (e_{kk} + \alpha e_{ij}) f(e_{kk} + \alpha e_{ij})$ and multiplying by e_{ll} from the left and by e_{kk} from the right, we get $a_{ij}^{lk}(\beta) = 0$.

Hence $f(\beta e_{ij}) = a_{ij}^{ij}(\beta) e_{ij} + \sum_{1 \leq r} a_{rr}^{ii}(\beta) e_{r,r}$.

From $f(\beta e_{ij})(\alpha e_{ij}) = (\beta e_{ij}) f(\alpha e_{ij})$, multiplying by e_{ii} from the left and by e_{jj} from the right, we get $a_{ii}^{ij}(\beta) = a_{jj}^{ij}(\beta)$.

From $f(\alpha e_{ij} + \beta e_{lj})(\alpha e_{ij} + \beta e_{lj}) = (\alpha e_{ij} + \beta e_{lj}) f(\alpha e_{ij} + \beta e_{lj})$. Multiplying by e_{ll} from the left and by e_{jj} from the right, we get $a_{ll}^{ij}(\alpha)\beta = \beta a_{jj}^{ij}(\alpha)$. Then $f(\beta e_{ij}) = a_{ij}^{ij}(\beta) e_{ij} + a_{ii}^{ij}(\beta) I_n = f(\beta e_{ij}) = \beta\lambda e_{ij} + a_{ii}^{ij}(\beta) I_n$.

Proof of Theorem 1. Suppose that $f : T_n(R) \to T_n(R)$ is an additive map satisfying $Xf(X) = f(X)X$ for all $X \in T_n(R)$. We put $X = \sum_{1 \le i \le j \le n} a_{ij}e_{i,\,j}$, then

$$\begin{aligned}
f(X) &= f\left(\sum_{1 \le i \le j \le n} x_{ij}e_{i,\,j}\right)\\
&= \sum_{1 \le i \le j \le n} f(x_{ij}e_{i,\,j})\\
&= \sum_{1 \le i \le n} f(x_{ii}e_{i,\,i}) + \sum_{1 \le i < j \le n} f(x_{ij}e_{i,\,j})\\
&= \sum_{1 \le i \le n} (\lambda x_{ii}e_{ii} + \delta_i(x_{ii})I_n) + \sum_{1 \le i < j \le n} x_{ij}\lambda e_{ij} + a_{ii}^{ij}(x_{ij})I_n\\
&= \lambda X + \sum_{1 \le i \le n} \delta_i(x_{ii})I_n + \sum_{1 \le i < j \le n} a_{ii}^{ij}(x_{ij})I_n\\
&= \lambda X + \delta(X)I_n.
\end{aligned}$$

This proves Theorem 1.

3. *m*-commuting Additive Maps of $T_n(R)$

Lemma 3.1. *Let m be a natural number and let R be a commutative ring whose characteristic is not a divisor of m. Suppose that* $f : T_n(R) \to T_n(R)$ *is an additive map satisfying* $X^m f(X) = f(X)X^m$ *for all* $X \in T_n(R)$ *and* $r \in R$, *then* $f(rI_n) \in Z(T_n(R))$.

Proof. Let $\alpha, \beta \in R$. Moreover, let $i < j$. Clearly,

$$(\alpha e_{ij} + \beta I_\infty)^m f(\alpha e_{ij} + \beta I_n) = (\alpha e_{ij} + \beta I_\infty)^m f(\alpha e_{ij} + \beta I_n).$$

This implies

$$(m\alpha\beta^{m-1}e_{ij} + \beta^m I_n) f(\alpha e_{ij} + \beta I_n) = f(\alpha e_{ij} + \beta I_n)(m\alpha\beta^{m-1}e_{ij} + \beta^m I_n).$$

Hence

$$e_{ij} f(\alpha e_{ij} + \beta I_n) = f(\alpha e_{ij} + \beta I_n) e_{ij}.$$

Similarly, using $r, \alpha, \beta \in R$ we obtain

$$e_{ij} f(\alpha e_{ij} + (\beta + r) I_n) = f(\alpha e_{ij} + (\beta + r) I_n) e_{ij}.$$

The difference of above two relations yields $e_{ij} f(rI_n) = f(rI_n) e_{ij}$, then $f(rI_n) \in Z(T_n(R))$.

Lemma 3.2. *Let m be a natural number and let R be a commutative ring whose characteristic is not a divisor of m. Suppose that* $f : T_n(R) \to T_n(R)$ *is an additive map satisfying* $X^m f(X) = f(X) X^m$ *for all* $X \in T_n(R)$, *then* $Xf(X) = f(X)X$ *for all* $X \in T_n(R)$.

Proof. We have $X^m f(X) = f(X) X^m$ for every $X \in T_n(R)$. Clearly,

$$[f(X), (X + pI_n)^m] + [f(pI_n), (X + pI_n)^m] = 0.$$

Recall that $f(rI_n) \in Z(T_n(R))$. Thus,

$$\sum_{1 \le k \le m \le n} p^{m-k} C_m^k [f(X), X^k] = 0.$$

Using matrix notation we can rewrite these systems in the following way: For $p = 1, p = 2, \ldots, p = m - 1$

$$\begin{pmatrix} 1C_m^1 & 1C_m^2 & 1C_m^3 & \cdots & 1C_m^m \\ 2^{m-1}C_m^1 & 2^{m-2}C_m^2 & 2^{m-3}C_m^3 & \cdots & 1C_m^m \\ 3^{m-1}C_m^1 & 3^{m-2}C_m^2 & 3^{m-3}C_m^3 & \cdots & 1C_m^m \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ m^{m-1}C_m^1 & m^{m-2}C_m^2 & m^{m-3}C_m^3 & \cdots & 1C_m^m \end{pmatrix} \begin{pmatrix} [f(X), X] \\ [f(X), X^2] \\ [f(X), X^3] \\ \vdots \\ [f(X), X^m] \end{pmatrix} = 0.$$

Because the determinant of the Vandermonde matrix formed by the coefficients of the system is not zero, we get that $[f(X), X] = 0$. This proves Theorem 2.

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