New Differential Operator for Holomorphic Functions

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Abstract

In this paper, by making use of binomial series, we define a new differential operator of holomorphic functions in the open unit disk. Also, we introduce and investigate two new classes containing this new operator associated with differential subordinations and superordinations. Furthermore, we determine important properties for functions belonging to these classes.

1. Introduction

Indicate by $\mathcal{H}(\mathbb{D})$ the collection of holomorphic functions in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For $n \in \mathbb{N}$ and $a \in \mathbb{C}$, let $\mathcal{H}[a, n] = \{f \in \mathcal{H}(\mathbb{D}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}$.

Let we name by the notation $\mathcal{A}$ the collection of all holomorphic functions in $\mathbb{D}$ and having the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \mathbb{D}. \tag{1.1}$$

"With a view to recalling the principal of subordination between holomorphic

functions, let the functions f and g be holomorphic in $\mathbb{D}$, we say that the function f is subordinate to g, if there exists a Schwarz function w holomorphic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{D})$ such that $f(z) = g(w(z))$. This subordination is indicated by $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbb{D})$. Furthermore, if the function g is univalent in $\mathbb{D}$, then we have the following equivalent (see [4]), $f(z) \prec g(z) \Leftrightarrow f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$."

Definition 1.1 [6]. Let $p, h \in \mathcal{H}(\mathbb{D})$ and $\psi(r, s; z) : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$. If p and $\psi(p(z), zp'(z); z)$ are univalent functions in $\mathbb{D}$ and if p satisfies the first-order differential superordination

$$h(z) \prec \psi(p(z), zp'(z); z), \tag{1.2}$$

then p is called a solution of the differential superordination (1.2). (If f is subordinate to g, then g is superordinate to f). An analytic function q is called a subordinate of (1.2), if $q \prec p$ for all p satisfying (1.2). A univalent subordinate $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all the subordinants q of (1.2) is called the best subordinant.

Now, by making use of the binomial series

$$(1 - \alpha)^k = \sum_{m=0}^{k} \binom{k}{m}(-1)^m \alpha^m, \quad (k \in \mathbb{N} = \{1, 2, ...\}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$$

and for $f \in \mathcal{A}$, $\alpha \in \mathbb{R}$, $\beta \geq 0$ with $\alpha + \beta > 0$ and $\delta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we define the differential operator $W_{\alpha,\beta}^{k,\delta} : \mathcal{A} \to \mathcal{A}$ as follows:

$$W_{\alpha,\beta}^{k,0} f(z) = f(z),$$

$$W_{\alpha,\beta}^{k,1} f(z) = \frac{(1 - (1-\alpha)^k) f(z) + (1 - (1-\beta)^k) zf'(z)}{2 - (1-\alpha)^k - (1-\beta)^k} = W_{\alpha,\beta}^k f(z)$$

$$\vdots$$

$$W_{\alpha,\beta}^{k,\delta} f(z) = W_{\alpha,\beta}^k (W_{\alpha,\beta}^{k,\delta-1} f(z)). \tag{1.3}$$

If f is given by (1.1), then from (1.3), we can easily see that

$$W_{\alpha,\beta}^{k,\delta} f(z) = z + \sum_{n=2}^{\infty} \left[\sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m} \right) \right]^{\delta} a_n z^n. \tag{1.4}$$

Remark 1.1. It should be remarked that the operator $W_{\alpha,\beta}^{k,\delta}$ generalizes some known operators considered earlier

(1) For $k = 1,$ the operator $W_{\alpha,\beta}^{1,\delta} = I_{\alpha,\beta}^{\delta}$ was introduced and studied by Swamy [8].

(2) For $\beta = 1$ and $\alpha > -1,$ the operator $W_{\alpha,1}^{1,\delta} = I_{\alpha}^{\delta}$ was investigated by Cho and Kim [2] and Cho and Srivastava [3].

(3) For $k = 1, \alpha = 1 - \beta$ and $\beta \geq 0,$ the operator $W_{1-\beta,\beta}^{1,\delta} = D_{\beta}^{\delta}$ was given by Al-Oboudi [1].

(4) For $k = 1, \alpha = 0$ and $\beta = 1,$ the operator $W_{0,1}^{1,\delta} = S^{\delta}$ was considered by Sălăgean [7].

We will require the following definition and lemmas in proving our present investigation.

Definition 1.2 [5]. Let Q be the family of all functions that are holomorphic and injective on $\overline{\mathbb{D}} \setminus \mathcal{E}(q),$ where

$$\mathcal{E}(q) = \{ r \in \partial\mathbb{D} : \lim_{z \to r} q(z) = \infty \},$$

and $q'(r) \neq 0$ for $r \in \partial\mathbb{D} \setminus \mathcal{E}(q).$ The subfamily of Q with $q(0) = a$ is denoted by $Q(a).$

Lemma 1.1 [5]. *Suppose that h is holomorphic and convex univalent function in* $\mathbb{D},$ $h(0) = a, \lambda \neq 0$ *and* $\operatorname{Re}(\lambda) \geq 0.$ *If* $p \in \mathcal{H}[a, n]$ *and*

$$p(z) + \frac{1}{\lambda} z p'(z) \prec h(z), \tag{1.5}$$

then

$$p(z) \prec q(z) \prec h(z),$$

where $q(z) = \lambda z^{-\lambda} \int_0^z h(t) t^{\lambda-1} dt$ *is convex and it is the best dominant of* (1.5).

Lemma 1.2 [6]*. Suppose that the function h is convex in* $\mathbb{D}$, $h(0) = a$, $\lambda \neq 0$ *and* $\text{Re}(\lambda) \geq 0$. *If* $p \in \mathcal{H}[a, n] \cap Q$, $p(z) + \dfrac{1}{\lambda} zp'(z)$ *is univalent in* $\mathbb{D}$ *and*

$$h(z) \prec p(z) + \frac{1}{\lambda} zp'(z), \tag{1.6}$$

then

$$q(z) \prec p(z),$$

where $q(z) = \lambda z^{-\lambda} \int_0^z h(t) t^{\lambda-1} dt$ *is convex and it is the best subordinant of* (1.6).

2. Key Lemma and Main Results

Before stating and proving our main results, we first establish the following identities involving the operator defined by (1.4):

Lemma 2.1. *Assume that* $f \in \mathcal{A}$, $\alpha \in \mathbb{R}$, $\beta > 0$, $\alpha + \beta > 0$, $\delta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *and the operator* $W_{\alpha, \beta}^{k, \delta}$ *is defined by* (1.4). *Then*

$$z(W_{\alpha, \beta}^{k, \delta} f(z))' \cong \Upsilon(\alpha, \beta, m, k) W_{\alpha, \beta}^{k, \delta+1} f(z) - \Phi(\alpha, \beta, m, k) W_{\alpha, \beta}^{k, \delta} f(z), \tag{2.1}$$

where $\Upsilon(\alpha, \beta, m, k)$ *and* $\Phi(\alpha, \beta, m, k)$ *have the following binomial series representations:*

$$\Upsilon(\alpha, \beta, m, k) = \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta} \right)^m + 1 \right) \tag{2.2}$$

and

$$\Phi(\alpha, \beta, m, k) = \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta} \right)^m. \tag{2.3}$$

Proof. In the light of (1.4), we find that

$$\Upsilon(\alpha, \beta, m, k) W_{\alpha, \beta}^{k, \delta+1} f(z) - \Phi(\alpha, \beta, m, k) W_{\alpha, \beta}^{k, \delta} f(z)$$

$$= \sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)W_{\alpha,\beta}^{k,\delta+1}f(z) - \sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}W_{\alpha,\beta}^{k,\delta}f(z)$$

$$= z + \sum_{n=2}^{\infty}\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha^{m}+n\beta^{m}}{\alpha^{m}+\beta^{m}}\right)\right]^{\delta+1}a_{n}z^{n}$$

$$- \sum_{n=2}^{\infty}\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha^{m}+n\beta^{m}}{\alpha^{m}+\beta^{m}}\right)\right]^{\delta}a_{n}z^{n}$$

$$= z + \sum_{n=2}^{\infty}\left[\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\right)\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\frac{\alpha^{m}+n\beta^{m}}{\alpha^{m}+\beta^{m}}\right)\right.$$

$$- \sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\left]\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\frac{\alpha^{m}+n\beta^{m}}{\alpha^{m}+\beta^{m}}\right]^{\delta}a_{n}z^{n}\right.$$

$$= z + \sum_{n=2}^{\infty}n\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\frac{\alpha^{m}+n\beta^{m}}{\alpha^{m}+\beta^{m}}\right]^{\delta}a_{n}z^{n-1}$$

$$= z(W_{\alpha,\beta}^{k,\delta}f(z))',$$

which establishes the identity (2.1).

Definition 2.1. Assume that ψ is an holomorphic and convex univalent function in $\mathbb{D}$ with $\psi(0) = 1$ and $\alpha \in \mathbb{R}$, $\eta, \beta > 0$, $\alpha + \beta > 0$, $\delta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A function $f \in \mathcal{A}$ is called in the family $E(\eta, \alpha, \beta, \delta, k, m; \psi)$ if the following differential subordination is satisfied:

$$\frac{1}{z}[(1 - \eta\Upsilon(\alpha, \beta, m, k))W_{\alpha,\beta}^{k,\delta}f(z) + \eta\Upsilon(\alpha, \beta, m, k)W_{\alpha,\beta}^{k,\delta+1}f(z)] \prec \psi(z).$$

A function $f \in \mathcal{A}$ is called in the family $F(\eta, \alpha, \beta, \delta, k, m; \psi)$ if the following differential superordination is satisfied:

$$\psi(z) \prec \frac{1}{z}[(1 - \eta\Upsilon(\alpha, \beta, m, k))W_{\alpha,\beta}^{k,\delta}f(z) + \eta\Upsilon(\alpha, \beta, m, k)W_{\alpha,\beta}^{k,\delta+1}f(z)],$$

where $\Upsilon(\alpha, \beta, m, k)$ is given by (2.2).

Theorem 2.1. *Suppose that* ψ *is a convex function in* $\mathbb{D}$ *with* $\psi(0) = 1$ *and* $\eta > 0$.
If $f \in E(\eta, \alpha, \beta, \delta, k, m; \psi)$, *then there is a convex function q such that* $q(z) \prec \psi(z)$
and $f \in E(0, \alpha, \beta, \delta, k, m; q)$.

Proof. Assume that

$$p(z) = \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} = 1 + \sum_{n=2}^{\infty}\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m}\right)\right]^{\delta} a_n z^{n-1}. \qquad (2.4)$$

It is obvious that $p \in \mathcal{H}[1, 1]$.

Since $f \in E(\eta, \alpha, \beta, \delta, k, m; \psi)$, we find that

$$\frac{1}{z}[(1 - \eta\Upsilon(\alpha, \beta, m, k))W_{\alpha,\beta}^{k,\delta} f(z) + \eta\Upsilon(\alpha, \beta, m, k)W_{\alpha,\beta}^{k,\delta+1} f(z)] \prec \psi(z). \qquad (2.5)$$

Now from (2.1), (2.4) and (2.5), it is evident that

$$\frac{1}{z}[(1 - \eta\Upsilon(\alpha, \beta, m, k))W_{\alpha,\beta}^{k,\delta} f(z) + \eta\Upsilon(\alpha, \beta, m, k)W_{\alpha,\beta}^{k,\delta+1} f(z)]$$

$$= p(z) + \eta z p'(z) \prec \psi(z).$$

By making use of Lemma 1.1 with $\lambda = \dfrac{1}{\eta}$, we deduce that

$$p(z) \prec q(z) \prec \psi(z).$$

It follows from (2.4) that

$$\frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \prec q(z) \prec \psi(z),$$

where

$$q(z) = \frac{1}{\eta} z^{-\frac{1}{\eta}} \int_0^z \psi(t) t^{\frac{1}{\eta}-1} \, dt.$$

This shows that $f \in E(0, \alpha, \beta, \delta, k, m; q)$ and the proof is completed.

Theorem 2.2. *Suppose that* ψ *is a convex function in* $\mathbb{D}$, *with* $\psi(0) = 1$ *and* $\eta > 0$.

If $f \in F(\eta, \alpha, \beta, \delta, k, m; \psi)$, $\dfrac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \in \mathcal{H}[1, 1] \cap Q$ *and*

$$\frac{1}{z}[(1 - \eta \Upsilon(\alpha, \beta, m, k))W_{\alpha,\beta}^{k,\delta} f(z) + \eta \Upsilon(\alpha, \beta, m, k)W_{\alpha,\beta}^{k,\delta+1} f(z)]$$

is univalent in $\mathbb{D}$, *then there is a convex function* q *such that* $f \in F(0, \alpha, \beta, \delta, k, m; q)$.

Proof. Assume that the function p is defined by (2.4). It is evident that $p \in \mathcal{H}[1, 1] \cap Q$. By routine computations and considering $f \in F(\eta, \alpha, \beta, \delta, k, m; \psi)$, we conclude that

$$\psi(z) \prec p(z) + \eta z p'_z(z).$$

By making use of Lemma 1.2 with $\lambda = \dfrac{1}{\eta}$, we obtain

$$q(z) \prec p(z).$$

In view of (2.4), yields

$$q(z) \prec \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z},$$

where

$$q(z) = \frac{1}{\eta} z^{-\frac{1}{\eta}} \int_0^z \psi(t) t^{\frac{1}{\eta}-1} \, dt.$$

If we combine the results of Theorem 2.1 and Theorem 2.2, we obtain "sandwich theorem".

Theorem 2.3. *Assume that* ψ_1 *and* ψ_2 *are two convex functions in* $\mathbb{D}$ *with* $\psi_1(0) = \psi_2(0) = 1$ *and* $\eta > 0$. *If* $f \in E(\eta, \alpha, \beta, \delta; k, m; \psi_1) \cap F(\eta, \alpha, \beta, \delta, k, m; \psi_2)$, $\dfrac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \in \mathcal{H}[1, 1] \cap Q$ *and*

$$\frac{1}{z}[(1 - \eta \Upsilon(\alpha, \beta, m, k))W_{\alpha,\beta}^{k,\delta} f(z) + \eta \Upsilon(\alpha, \beta, m, k)W_{\alpha,\beta}^{k,\delta+1} f(z)]$$

is univalent in $\mathbb{D}$, *then*

$$f \in E(0, \alpha, \beta, \delta, k, m; q_1) \bigcap F(0, \alpha, \beta, \delta, k, m; q_2),$$

where

$$q_1(z) = \frac{1}{\eta} z^{-\frac{1}{\eta}} \int_0^z \psi_1(t) t^{\frac{1}{\eta}-1} dt$$

and

$$q_2(z) = \frac{1}{\eta} z^{-\frac{1}{\eta}} \int_0^z \psi_2(t) t^{\frac{1}{\eta}-1} dt.$$

The functions q_1 *and* q_2 *are convex.*

Theorem 2.4. *Suppose that* ψ *is a convex function in* $\mathbb{D}$ *with* $\psi(0) = 1$ *and*

$$T(z) = \frac{\tau+2}{z^{\tau+1}} \int_0^z t^\tau f(t) dt, \quad (z \in \mathbb{D}, \, \mathrm{Re}(\tau) > -2). \tag{2.6}$$

If $f \in E(1, \alpha, \beta, \delta, k, m; \psi)$, *then there is a convex function* q *such that* $q(z) \prec \psi(z)$ *and* $T \in E(1, \alpha, \beta, \delta, k, m; q)$.

Proof. Let we choose p by

$$p(z) = \left(W_{\alpha,\beta}^{k,\delta}T(z)\right)', \quad (z \in \mathbb{D}). \tag{2.7}$$

Equation (2.7) shows that $p \in \mathcal{H}[1, 1]$. Consider (2.6), we deduce that

$$z^{\tau+1}T(z) = (\tau+2)\int_0^z t^\tau f(t) dt. \tag{2.8}$$

By routine calculations of (2.8), we find that

$$(\tau+2)f(z) = (\tau+1)T(z) + zT'(z)$$

and

$$(\tau+2)W_{\alpha,\beta}^{k,\delta}f(z) = (\tau+1)W_{\alpha,\beta}^{k,\delta}T(z) + z\left(W_{\alpha,\beta}^{k,\delta}T(z)\right)'.$$

Differentiating the last relation with respect to z, we have

$$(W_{\alpha,\beta}^{k,\delta} f(z))' = (W_{\alpha,\beta}^{k,\delta} T(z))' + \frac{z}{\tau+2}(W_{\alpha,\beta}^{k,\delta} T(z))''. \qquad (2.9)$$

According to $f \in E(1, \alpha, \beta, \delta, k, m; \psi)$, we can write

$$\frac{1}{z}[\Upsilon(\alpha, \beta, m, k)W_{\alpha,\beta}^{k,\delta+1} f(z) - \Phi(\alpha, \beta, m, k)W_{\alpha,\beta}^{k,\delta} f(z)] \prec \psi(z), \qquad (2.10)$$

where $\Upsilon(\alpha, \beta, m, k)$ and $\Phi(\alpha, \beta, m, k)$ have the forms (2.2) and (2.3), respectively.

Implementing (2.1) in (2.10), gives us

$$(W_{\alpha,\beta}^{k,\delta} f(z))' \prec \psi(z). \qquad (2.11)$$

In the light of (2.9) and (2.11), we deduce that

$$(W_{\alpha,\beta}^{k,\delta} T(z))' + \frac{z}{\tau+2}(W_{\alpha,\beta}^{k,\delta} T(z))'' \prec \psi(z). \qquad (2.12)$$

Replacing (2.7) in (2.12), we can easily obtain that

$$p(z) + \frac{1}{\tau+2} z p'(z) \prec \psi(z).$$

Applying Lemma 1.1 with $\lambda = \tau + 2$, leads us to

$$p(z) \prec q(z) \prec \psi(z).$$

By virtue of (2.7), yields

$$(W_{\alpha,\beta}^{k,\delta} T(z))' \prec q(z) \prec \psi(z),$$

where

$$q(z) = (\tau + 2) z^{-(\tau+2)} \int_0^z \psi(t) t^{\tau+1} dt.$$

Theorem 2.5. *Suppose that ψ is a convex function in $\mathbb{D}$ with $\psi(0) = 1$ and $T(z)$ is defined by* (2.6). *If $f \in F(1, \alpha, \beta, \delta, k, m; \psi)$, $(W_{\alpha,\beta}^{k,\delta} T(z))' \in \mathcal{H}[1, 1] \cap Q$ and $\frac{1}{z}[\Upsilon(\alpha, \beta, m, k)W_{\alpha,\beta}^{k,\delta+1} f(z) - \Phi(\alpha, \beta, m, k)W_{\alpha,\beta}^{k,\delta} f(z)]$ is univalent in $\mathbb{D}$, then there exists a convex function q such that $T \in F(1, \alpha, \beta, \delta, k, m; q)$.*

Proof. Let the function p be defined by (2.7). It is evident that $p \in \mathcal{H}[1,1] \cap Q$. By routine computations and considering $f \in F(1, \alpha, \beta, \delta, k, m; \psi)$, we conclude that

$$\psi(z) \prec p(z) + \frac{1}{\tau + 2} z p'(z).$$

Applying Lemma 1.2 with $\lambda = \tau + 2$, yields $q(z) \prec p(z)$.

From (2.7), we can write

$$q(z) \prec (W_{\alpha, \beta}^{k, \delta} T(z))',$$

where

$$q(z) = (\tau + 2) z^{-(\tau+2)} \int_0^z \psi(t) t^{\tau+1} dt.$$

Combining results of Theorem 2.4 and Theorem 2.5, we state "sandwich theorem".

Theorem 2.6. *Assume that* ψ_1 *and* ψ_2 *are two convex functions in* $\mathbb{D}$ *with* $\psi_1(0) = \psi_2(0) = 1$ *and* $T(z)$ *is defined by* (2.6). *If* $f \in E(1, \alpha, \beta, \delta, k, m; \psi_1) \cap$ $F(1, \alpha, \beta, \delta, k, m; \psi_2)$, $(W_{\alpha, \beta}^{k, \delta} T(z))' \in \mathcal{H}[1,1] \cap Q$ *and* $\frac{1}{z}[\Upsilon(\alpha, \beta, m, k) W_{\alpha, \beta}^{k, \delta+1} f(z)$ $- \Phi(\alpha, \beta, m, k) W_{\alpha, \beta}^{k, \delta} f(z)]$ *is univalent in* $\mathbb{D}$, *then*

$$T \in E(1, \alpha, \beta, \delta, k, m; q_1) \cap F(1, \alpha, \beta, \delta, k, m; q_2),$$

where

$$q_1(z) = (\tau + 2) z^{-(\tau+2)} \int_0^z \psi_1(t) t^{\tau+1} dt \quad and \quad q_2(z) = (\tau + 2) z^{-(\tau+2)} \int_0^z \psi_2(t) t^{\tau+1} dt.$$

The functions q_1 *and* q_2 *are convex.*

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