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Best Bounds for Fekete-Szegö Functional for the k^{th} Root Transform of Certain Subclasses of Sakaguchi Type functions

P. Lokesh[1,*] and Chinthamani S[2]

[1] Department of Mathematics, Adhiparasakthi College of Engineering, Kalavai, India
e-mail: lokeshpandurangan@gmail.com

[2] Department of Mathematics, Stella Maris College (Autonomous), Chennai, India
e-mail: chinthamani@stellamariscollege.edu.in

Abstract

In recent research, working on coefficient bounds is very popular and useful to deal with geometric properties of the underlying functions. In this work, two new subclasses of Sakaguchi type functions with respect to symmetric points through subordination are considered. Moreover, the initial coefficients and the sharp upper bounds for the functional $|\rho_{2k+1} - \mu\rho_{k+1}^2|$ corresponding to k^{th} root transformation belong to the above classes are obtained and thoroughly investigated.

1 Introduction

To define our desired classes and to investigate its Fekete-Szegö functionals which has greater importance in recent research due to its geometrical properties, we define, first of all, the complex function $f(z)$ in usual way as follows:

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

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[*]Corresponding author

These complex functions are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. The class of such functions is denoted by $\mathbb{A}$. When we impose the univalence condition into this class $\mathbb{A}$, we get another subclass dented by S, i.e., S is a subclass of $\mathbb{A}$ which are univalent in $\mathbb{U}$.

Subordination: It is a powerful tool in Geometric Function Theory (GFT) which is defined as follows: "An analytic function f is subordinate to an analytic function g, written as $f(z) \prec g(z)$, if there exists an analytic self map w of $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))(z \in \mathbb{U})$. It follows from the Schwarz lemma that $f(z) \prec g(z)(z \in \mathbb{U}) \Rightarrow f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. If the function g is univalent in $\mathbb{U}$, then

$$f(z) \prec g(z)(z \in \mathbb{U}) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U})\text{''}$$

(See [10]). Based on Bieberbach conjecture [4], one may get idea on coefficients of the function given in (1.1) such that $|a_n| < n$. These coefficient bounds give information about their geometric properties of the functions.

Hankel Determinants: For given parameters $q, n \in \mathbb{N} = \{1, 2, ...\}$, the Hankel determinant $H_{q,n}(f)$ was considered by Noonan and Thomas [11] for a function $f \in \mathbb{S}$ of the form (1.1) given by:

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}$$

where $a_1 = 1$. The growth of $H_{q,n}(f)$ has been investigated for different subfamilies of univalent functions. One can easily observe that the Fekete-Szegö functional is $H_{2,1}(f)$. Fekete-Szegö then further generalized as the estimate $|a_3 - \mu a_2^2|$ where μ is any complex number. The functional $|a_3 - \mu a_2^2|$ for normalized univalent function $f(z)$ of the form (1.1) is notorious in the past and current history of GFT. For detailed study on Fekete-Szegö coefficient functional for univalent and multivalent functions, see [1, 3, 5, 7, 9, 13].

In this paper, the sharp upper bounds of this functional for new subclasses are derived and further investigated. Starlike and Convex functions with respect to symmetric points are defined below:

Definition 1.1. [6] *A function $f \in \mathbb{A}$ is said to be in the class $S_s^\star(\lambda, \delta, m)$, if it satisfies the condition*

$$\mathbb{R}\left[\frac{(1-m)(\lambda\delta z^3 f'''(z) + (2\lambda\delta+\lambda-\delta)z^2 f''(z) + zf'(z))}{\lambda\delta z^2(f''(z) - m^2 f''(mz)) + (\lambda-\delta)z(f'(z) - mf'(mz)) + (1-\lambda+\delta)(f(z) - f(mz))}\right] > 0, \quad (z \in \mathbb{U}) \quad (1.2)$$

where $0 \leq \lambda < 1, 0 \leq \delta < 1, |m| \leq 1$ *but* $m \neq 1$.

The class $S_s^\star(\lambda, \delta, m)$ was introduced and studied by Sakaguchi [16].

Definition 1.2. [6] *A function $f \in \mathbb{A}$ is said to be in the class $C_s(\lambda, \delta, m)$, if it satisfies the condition*

$$\mathbb{R}\left[\frac{(1-m)(\lambda\delta z^3 f'''(z) + (2\lambda\delta+\lambda-\delta)z^2 f''(z) + zf'(z))'}{\lambda\delta z^2(f''(z) - m^2 f''(mz)) + (\lambda-\delta)z(f'(z) - mf'(mz)) + (1-\lambda+\delta)(f(z) - f(mz))'}\right] > 0, \quad (z \in \mathbb{U}) \quad (1.3)$$

where $0 \leq \lambda < 1, 0 \leq \delta < 1, |m| \leq 1$ *but* $m \neq 1$. *It may be noted that* $f \in C_s \Leftrightarrow zf' \in S_s^\star$.

Motivated by the works of Sakaguchi and Panigrahi et al. [14], the subclasses $S_s^\star(\lambda, \delta, m, \phi)$ and $C_s(\lambda, \delta, m, \phi)$ are generalized through subordination and defined as follows:

Definition 1.3. [16] *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + ...$ be a univalent starlike function with respect to 1 which maps the unit disk $\mathbb{U}$ onto a region in the right half plane which is symmetric with respect to the real axis and let $B_1 > 0$. The function*

$f \in \mathbb{A}$ is in the classes $S_s^\star(\lambda,\delta,m,\phi)$ if

$$\frac{(1-m)(\lambda\delta z^3 f'''(z)+(2\lambda\delta+\lambda-\delta)z^2f''(z)+zf'(z))}{\lambda\delta z^2(f''(z)-m^2f''(mz))+(\lambda-\delta)z(f'(z)-mf'(mz))+(1-\lambda+\delta)(f(z)-f(mz))} \prec \phi(z), \tag{1.4}$$

and the class $C_s(\lambda,\delta,m,\phi)$ if

$$\frac{(1-m)(\lambda\delta z^3 f'''(z)+(2\lambda\delta+\lambda-\delta)z^2f''(z)+zf'(z))'}{\lambda\delta z^2(f''(z)-m^2f''(mz))+(\lambda-\delta)z(f'(z)-mf'(mz))+(1-\lambda+\delta)(f(z)-f(mz))'} \prec \phi(z). \tag{1.5}$$

Remark 1.4. Putting $\phi(z)=\frac{1+z}{1-z}$ in (1.4) and (1.5), the classes $S_s^\star$ and C_s studied by Sakaguchi [16] and Das and Singh [6] respectively.

Remark 1.5. Putting $\lambda=0,\delta=0$ and $m=-1$ in (1.4) and (1.5), the classes $S_s^\star$ and C_s studied by Panigrahi et al. [11,12,13].

Let k be a positive integer. For an univalent function f of the form (1.1), the k^{th} root transform is defined by

$$F(z)=\left[f(z^k)\right]^{\frac{1}{k}}=z+\sum_{n=1}^{\infty}\rho_{nk+1}z^{nk+1} \tag{1.6}$$

where the initial coefficients are given by

$$\begin{aligned}\rho_{k+1}&=\frac{a_2}{k}\\ \rho_{2k+1}&=\frac{a_3}{k}+\frac{1-k}{2k^2}a_2^2,\\ \rho_{3k+1}&=\frac{a_4}{k}+\frac{1-k}{k^2}a_2a_3+\frac{(1-k)(1-2k)}{3!k^3}a_2^3,\end{aligned}$$

and so on. Since f is univalent, so $\frac{f(z^k)}{z^k}$ is non-vanishing in $\mathbb{U}$ implies that k^{th} root of f is analytic in $\mathbb{U}$. The Fekete-Szegö coefficient functional of the associated function $F(z))$ is given by $|\rho_{2k+1}-\mu\rho_{k+1}^2|$. This quantity is known as Fekete-Szegö problem of the k^{th} root transform of f. The k^{th} root transform has been widely used in a variety of ways in Complex function theory. In 2009, Ali et al. [2] have investigated the Fekete-Szegö coefficient functional for the k^{th} root transform of functions belonging to several classes of analytic functions defined by means of subordination (also refer [12]).

In this paper, upper bounds for the Fekete-Szegö Coefficient functional $|\rho_{2k+1}-\mu\rho_{k+1}^2|$ associated with the k^{th} root transform of the function f belonging to the classes $S_s^\star(\lambda,\delta,m,\phi)$ and $C_s(\lambda,\delta,m,\phi)$ are obtained.

2 Coefficient bounds for $f \in S_s^\star(\lambda,\delta,m,\phi)$ and $f \in C_s(\lambda,\delta,m,\phi)$

We use the following lemmas to prove our main results:

Lemma 2.1. [3] *Let Δ be the class of the analytic functions w, normalized by $w(0) = 0$ satisfying the condition $|w(z)| < 1$. If $w \in \Delta$ and $w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \ldots$ $(z \in \mathbb{U})$, then for any real numbers t, it is clear that*

$$|w_2 - tw_1^2| \leq \begin{cases} -t, & t \leq -1 \\ 1, & -1 \leq t \leq 1 \\ t, & t \geq 1 \end{cases}$$

For $t < -1$ or $t > 1$, equality holds if and only if $w(z) = z$ or one of its rotations. For $-1 < t < 1$, equality holds if and only if $w(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if

$$w(z) = z\left(\frac{\epsilon + z}{1+\epsilon z}\right)(0 \leq \epsilon \leq 1)$$

or one of its rotations, while for $t = 1$, equality holds if and only if

$$w(z) = -z\left(\frac{\epsilon + z}{1+\epsilon z}\right)(0 \leq \epsilon \leq 1)$$

or one of its rotations.

Lemma 2.2. [8] *If $w \in \Delta$, then*

$$|w_2 - tw_1^2| \leq max\{1, |t|\},$$

for any complex number t. The result is sharp for the function $w(z) = z^2$ or $w(z) = z$.

Next, the bounds for the functional $|\rho_{2k+1} - \mu\rho_{k+1}^2|$ corresponding to the k^{th} root transform of the function f in the classes $S_s^\star(\lambda, \delta, m, \phi)$ and $C_s(\lambda, \delta, m, \phi)$ are investigated.

Theorem 2.3. *For $0 \leq \lambda < 1, 0 \leq \delta < 1, |m| \leq 1$ but $m \neq 1$, let $\phi(z) = 1 + B_1 z + B_2 z^2 + ...$ with $B_1 > 0, B_2 \geq 0$ and $B_n's$ real. If $f \in S_s^\star(\lambda, \delta, m, \phi)$ and F is the k^{th} root transformation of f given by (1.6), then for any complex number μ, then*

$$|\rho_{2k+1} - \mu\rho_{k+1}^2| \leq \frac{B_1}{(2-m-m^2)k} max\Big\{1, \Big|\frac{(k-1)(2-m-m^2)B_1}{2(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2 k} - \frac{(1-m)B_2 + (1+m)B_1^2}{(1-m)(1+2(\lambda-\delta+3\lambda\delta))B_1} + \frac{\mu(2-m-m^2)}{(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2 k}\Big|\Big\} \quad (2.1)$$

The result is sharp.

Proof. Let $f \in S_s^\star(\lambda, \delta, m, \phi)$. Then by (1.4) of definition 1.3, there exists a Schwarz's function $w(z) \in \Delta$ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$\frac{(1-m)(\lambda\delta z^3 f'''(z) + (2\lambda\delta + \lambda - \delta)z^2 f''(z) + zf'(z))}{\lambda\delta z^2(f''(z) - m^2 f''(mz)) + (\lambda-\delta)z(f'(z) - mf'(mz)) + (1-\lambda+\delta)(f(z) - f(mz))} = \phi(w(z)) \quad (2.2)$$

By simple computation from (1.1) it follows that

$$\frac{(1-m)(\lambda\delta z^3 f'''(z) + (2\lambda\delta + \lambda - \delta)z^2 f''(z) + zf'(z))}{\lambda\delta z^2(f''(z) - m^2 f''(mz)) + (\lambda-\delta)z(f'(z) - mf'(mz)) + (1-\lambda+\delta)(f(z) - f(mz))}$$
$$= 1 + (1-m)(1+\lambda-\delta+2\lambda\delta)a_2 z + \big[(1+2(\lambda-\delta+3\lambda\delta))(2-m-m^2)a_3 - (1-m^2)(1+\lambda-\delta+2\lambda\delta)^2 a_2^2\big] z^2 + ... \quad (2.3)$$

Also

$$\phi(w(z)) = 1 + B_1 w_1 z + (B_1 w_2 + B_2 w_1^2) z^2 + ... \tag{2.4}$$

By using (2.3) and (2.4) in (2.2), it is obtained that

$$\begin{aligned} &(1-m)(1+\lambda-\delta+2\lambda\delta)a_2 = B_1 w_1 \\ &\qquad\qquad\Rightarrow a_2 = \frac{B_1 w_1}{(1-m)(1+\lambda-\delta+2\lambda\delta)} \end{aligned} \tag{2.5}$$

$$(2-m-m^2)(1+2(\lambda-\delta+3\lambda\delta))a_3 - (1-b^2)(1+\lambda-\delta+2\lambda\delta)^2 a_2^2 = B_1 w_1 + B_2 w_1^2$$

$$\Rightarrow a_3 = \frac{(1-m)B_1 w_2 + ((1-m)B_2 + (1+m)B_1^2)w_1^2}{(1-m)(2-m-m^2)(1+2(\lambda-\delta+3\lambda\delta))} \tag{2.6}$$

If $F(z)$ is the k^{th} root transform of the function $f(z)$, then from (1.6), (2.5) and (2.6), it is clear that

$$\rho_{k+1} = \frac{a_2}{k} = \frac{B_1 w_1}{(1-m)(1+\lambda-\delta+2\lambda\delta)k} \tag{2.7}$$

and

$$\begin{aligned} \rho_{2k+1} &= \frac{a_3}{k} + \frac{1-k}{2k^2} a_2^2 \\ &= \frac{(1-m)B_1 w_2 + ((1-m)B_2 + (1+m)B_1^2)w_1^2}{(1-m)(2-m-m^2)(1+2(\lambda-\delta+3\lambda\delta))} + \frac{(1-k)}{2k^2} \frac{B_1^2 w_1^2}{(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2} \end{aligned} \tag{2.8}$$

Therefore, for any complex number μ, it is found that

$$|\rho_{2k+1} - \mu\rho_{k+1}^2| = \frac{B_1}{(2-m-m^2)k} |W_2 - t w_1^2|, \tag{2.9}$$

where

$$\begin{aligned} t = \frac{\mu(2-m-m^2)}{(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2 k} B_1 - \frac{(1-m)B_2 + (1+m)B_1^2}{(1-m)(1+2(\lambda-\delta+3\lambda\delta)B_1 k} \\ - \frac{(1-k)(2-m-m^2)B_1}{2(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2 k}, \end{aligned}$$

and

$$W_2 = \frac{w_2}{1+2(\lambda-\delta+3\lambda\delta)}$$

An application of Lemma 2.2 to the right hand side of (2.9) gives the desire result as mentioned in (2.1). The result is sharp and followed by

$$|\rho_{2k+1}-\mu\rho_{k+1}^2| = \begin{cases} \frac{B_1}{(2-m-m^2)k}, & w(z)=z^2 \\ \frac{B_1}{(2-m-m^2)k}\Big|\frac{\mu(2-m-m^2)B_1}{(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2k} - \frac{(1-m)B_2+(1+m)B_1^2}{(1-m)(1+2(\lambda-\delta+3\lambda\delta)B_1k} & \\ -\frac{(1-k)(2-m-m^2)B_1}{2(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2k}\Big|, & w(z)=z \end{cases}$$

This completes the proof of Theorem 2.3. □

It is noted that the result is sharp for the extremal functions given by $\frac{z}{1+z^2}$.

Taking $\phi(z)=\frac{1+z}{1-z}=1+2z+2z^2+...$ and $k=1$, the Fekete-Szegö coefficient functional for the class $S_s^\star(\lambda,\delta,m,\phi)$ is investigated as follows:

Corollary 2.4. *Let the function $f\in\mathbb{A}$ given by (1.1) be in the class $S_s^\star(\lambda,\delta,m,\phi)$. Then for any complex number μ, it is clear that*

$$|a_3-\mu a_2^2| \le max\left\{1,\left|\frac{\mu(2-m-m^2)}{(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2}-\frac{2}{(1-m)(1+2(\lambda-\delta+3\lambda\delta))}\right|\right\}$$

The result is sharp.

Remark 2.5. Putting $\lambda=0,\delta=0$ and $m=-1$ in Corollary 2.4, then

$$|a_3-\mu a_2^2| \le max\left\{1,|\mu-1|\right\}$$

which was studied by Panigrahi et al. [13].

Remark 2.6. Taking $\mu=1$ in Remark 2.5, the result $|a_3-a_2^2|\le 1$ for the class $S_s^\star$ was introduced by Shanmugam et al. [18].

Theorem 2.7. *Let $\phi(z)=1+B_1z+B_2z^2+...$ with $B_1>0, B_2\ge 0$ and $B_n^\prime s$ real. If $f\in C_s(\lambda,\delta,m,\phi)$ and F is the k^{th} root transformation of f given by (1.6), then for any complex number μ, it is clear that*

$$|\rho_{2k+1}-\mu\rho_{k+1}^2| \le \frac{B_1}{3(2-m-m^2)k} max\Big\{1,\Big|\frac{3(k-1)(2-m-m^2)B_1}{8(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2k} - \frac{(1-m)B_2+(1+m)B_1^2}{(1-m)(1+2(\lambda-\delta+3\lambda\delta))B_1}+\frac{3\mu(2-m-m^2)B_1}{4(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2k}\Big|\Big\} \quad (2.10)$$

where $0 \leq \lambda < 1, 0 \leq \delta < 1, |m| \leq 1$ *but* $m \neq 1$.

The result is sharp.

Proof. Let $f \in C_s(\lambda, \delta, m, \phi)$. Then by (1.5) of definition 1.3, it follows that

$$\frac{(1-m)(\lambda\delta z^3 f'''(z) + (2\lambda\delta + \lambda - \delta)z^2 f''(z) + zf'(z))'}{[\lambda\delta z^2(f''(z) - m^2 f''(mz)) + (\lambda - \delta)z(f'(z) - mf'(mz)) + (1 - \lambda + \delta)(f(z) - f(mz))]'} = \phi(w(z)) \quad (2.11)$$

A simple calculation shows that

$$\begin{aligned}&\frac{(1-m)(\lambda\delta z^3 f'''(z) + (2\lambda\delta + \lambda - \delta)z^2 f''(z) + zf'(z))'}{[\lambda\delta z^2(f''(z) - m^2 f''(mz)) + (\lambda - \delta)z(f'(z) - mf'(mz)) + (1 - \lambda + \delta)(f(z) - f(mz))]'} \\ &= 1 + 2(1-m)(1+\lambda-\delta+2\lambda\delta)a_2 z + \big[3(1+2(\lambda-\delta+3\lambda\delta))(2-m-m^2)a_3 \\ &\qquad -4(1-m^2)(1+\lambda-\delta+2\lambda\delta)^2 a_2^2\big] z^2 \end{aligned} \quad (2.12)$$

Proceeding as in Theorem 2.3, it is obtained that

$$a_2 = \frac{B_1 w_1}{2(1-m)(1+\lambda-\delta+2\lambda\delta)}, \quad (2.13)$$

and

$$a_3 = \frac{(1-m)B_1 w_2 + ((1-m)B_2 + (1+m)B_1^2)w_1^2}{3(1-m)(2-m-m^2)(1+2(\lambda-\delta+3\lambda\delta))}, \quad (2.14)$$

If $F(z)$ is the k^{th} root transform of $f(z)$, then

$$\rho_{k+1} = \frac{a_2}{k} = \frac{B_1 w_1}{2(1-m)(1+\lambda-\delta+2\lambda\delta)} \quad (2.15)$$

and

$$\rho_{2k+1} = \frac{a_3}{k} + \frac{1-k}{2k^2}a_2^2$$

$$\begin{aligned}&= \frac{(1-m)B_1 w_2}{3(1-m)(2-m-m^2)(1+2(\lambda-\delta+3\lambda\delta))k} + \frac{((1-m)B_2 + (1+m)B_1^2)w_1^2}{3(1-m)(2-m-m^2)(1+2(\lambda-\delta+3\lambda\delta))k} \\ &\qquad + \frac{(1-k)}{2k^2}\frac{B_1^2 w_1^2}{4(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2}\end{aligned} \quad (2.16)$$

Thus, for any complex number μ, it is found that

$$|\rho_{2k+1} - \mu\rho_{k+1}^2| = \frac{B_1}{3(2-m-m^2)k}|W_2 - tw_1^2|, \quad (2.17)$$

where

$$t = \frac{3(k-1)(2-m-m^2)B_1}{8(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2k} - \frac{(1-m)B_2+(1+m)B_1^2}{(1-m)(1+2(\lambda-\delta+3\lambda\delta)B_1} + \frac{3\mu(2-m-m^2)B_1}{4(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2k},$$

and $W_2 = \frac{w_2}{1+2(\lambda-\delta+3\lambda\delta)}$.

By an application of Lemma 2.2, the required result is derived as in Theorem 2.7. The result is sharp and followed by

$$|\rho_{2k+1}-\mu\rho_{k+1}^2| = \begin{cases} \frac{B_1}{3(2-m-m^2)k}, & w(z)=z^2 \\ \frac{B_1}{3(2-m-m^2)k}\left|\frac{3(k-1)(2-m-m^2)B_1}{8(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2k} - \frac{(1-m)B_2+(1+m)B_1^2}{(1-m)(1+2(\lambda-\delta+3\lambda\delta)B_1} \right. & \\ \left. +\frac{3\mu(2-m-m^2)B_1}{4(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2k}\right|, & w(z)=z \end{cases}$$

This completes the proof of Theorem 2.7. □

The following result is obtained by taking $\phi(z) = \frac{1+z}{1-z}$ and $k=1$.

Corollary 2.8. *Let the function $f \in \mathbb{A}$ given by (1.1) be in the class C_s. For any complex number μ, then*

$$|a_3-\mu a_2^2| \le \frac{1}{3} max\left\{1, \left|\frac{3\mu(2-m-m^2)}{4(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2} - \frac{2}{(1-m)(1+2(\lambda-\delta+3\lambda\delta))}\right|\right\}$$

The result is sharp.

Remark 2.9. Putting $\lambda=0, \delta=0$ and $m=-1$ in Corollary 2.8, it is found that

$$|a_3-\mu a_2^2| \le \frac{1}{3} max\left\{1, \left|\frac{3\mu}{8}-1\right|\right\}$$

The estimate is sharp which was studied by Panigrahi et al. [14].

Remark 2.10. Taking $\mu=1$ in Corollary 2.8, the result $|a_3-a_2^2| \le \frac{1}{3}$ for the class C_s due to Shanmugam et al. [18].

Next, the bounds for the functional $|\rho_{2k+1}-\mu\rho_{k+1}^2|$ for real μ for the classes $S_s^\star(\lambda,\delta,m,\phi)$ and $C_s(\lambda,\delta,m,\phi)$ are determined.

Theorem 2.11. *If $f \in S_s^\star(\lambda, \delta, m, \phi)$ and F is the k^{th} root transform of the function f defined by (1.6), then for any real number μ and $0 \le \lambda < 1, 0 \le \delta < 1, |m| \le 1$ but $m \neq 1$, then*

$$\eta_1 = \frac{2(1-m)(1+\lambda-\delta+2\lambda\delta)^2 k[((1-m)B_2+(1+m)B_1^2)-(1-m)(1+2(\lambda-\delta+3\lambda\delta))B_1]+(1-k)(2-m-m^2)(1+2(\lambda-\delta+3\lambda\delta))B_1^2}{2(2-m-m^2)(1+2(\lambda-\delta+3\lambda\delta))B_1^2} \tag{2.18}$$

$$\eta_2 = \frac{2(1-m)(1+\lambda-\delta+2\lambda\delta)^2 k[((1-m)B_2+(1+m)B_1^2)+(1-m)(1+2(\lambda-\delta+3\lambda\delta))B_1]+(1-k)(2-m-m^2)(1+2(\lambda-\delta+3\lambda\delta))B_1^2}{2(2-m-m^2)(1+2(\lambda-\delta-3\lambda\delta))B_1^2} \tag{2.19}$$

It shows that

$$|\rho_{2k+1}-\mu\rho_{k+1}^2| = \begin{cases} \frac{(2-m-m^2)B_1}{(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2k}\left(\frac{(1-m)B_2+(1+m)B_1^2}{(1-m)(1+2(\lambda-\delta+3\lambda\delta))B_1} - \frac{(k-1)(2-m-m^2)B_1}{2(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2k} - \frac{\mu(2-m-m^2)B_1}{(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2k}\right), & \mu \le \eta_1 \\ \frac{(2-m-m^2)B_1}{(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2k}, & \eta_1 \le \mu \le \eta_2 \\ \frac{(2-m-m^2)B_1}{(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2k}\left(\frac{(k-1)(2-m-m^2)B_1}{2(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2k} - \frac{(1-m)B_2+(1+m)B_1^2}{(1-m)(1+2(\lambda-\delta+3\lambda\delta))B_1} + \frac{\mu(2-m-m^2)B_1}{(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2k}\right), & \mu \ge \eta_2 \end{cases} \tag{2.20}$$

Each of the estimate (2.20) is sharp.

Proof. Let $f \in S_s^\star(\lambda, \delta, m, \phi)$. From (2.9) it is derived that

$$|\rho_{2k+1} - \mu\rho_{k+1}^2| = \frac{B_1}{(2-m-m^2)k}|W_2 - tw_1^2|, \tag{2.21}$$

where

$$t = \frac{\mu(2-m-m^2)B_1}{(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2k} - \frac{(1-m)B_2+(1+m)B_1^2}{(1-m)(1+2(\lambda-\delta+3\lambda\delta)B_1k} - \frac{(1-k)(2-m-m^2)B_1}{2(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2k},$$

and

$$W_2 = \frac{w_2}{1+2(\lambda-\delta+3\lambda\delta)}$$

An application of Lemma 2.1 on right hand side of (2.21) give the following cases:
Case (i): If $\mu \leq \eta_1$, then

$$\mu \leq \frac{2(1-m)(1+\lambda-\delta+3\lambda\delta)^2 k[((1-m)B_2+(1+m)B_1^2)-(1-m)(1+2(\lambda-\delta+3\lambda\delta))B_1]+(1-k)(2-m-m^2)(1+2(\lambda-\delta+3\lambda\delta))B_1^2}{2(2-m-m^2)(1+2(\lambda-\delta+3\lambda\delta))B_1^2} \tag{2.18}$$

which implies, $t \leq -1$ and $|W_2 - tw_1^2| \leq -t$, hence it is obtained that

$$|\rho_{2k+1}-\mu\rho_{k+1}^2| \leq \frac{(2-m-m^2)B_1}{(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2 k}\Bigg|\frac{(1-m)B_2+(1+m)B_1^2}{(1-m)(1+2(\lambda-\delta+3\lambda\delta))B_1} - \frac{(k-1)(2-m-m^2)B_1}{2(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2 k} - \frac{\mu(2-m-m^2)B_1}{(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2 k}\Bigg| \tag{2.22}$$

Case (ii): If $\eta_1 \leq \mu \leq \eta_2$, then

$$\frac{2(1-m)(1+\lambda-\delta+2\lambda\delta)^2 k[((1-m)B_2+(1+m)B_1^2)-(1-m)(1+2(\lambda-\delta+3\lambda\delta))B_1]+(1-k)(2-m-m^2)(1+2(\lambda-\delta+3\lambda\delta))B_1^2}{2(2-m-m^2)(1+2(\lambda-\delta+3\lambda\delta))B_1^2}$$

$$\leq \mu$$

$$\leq \frac{2(1-m)(1+\lambda-\delta+2\lambda\delta)^2 k[((1-m)B_2+(1+m)B_1^2)+(1-m)(1+2(\lambda-\delta+3\lambda\delta))B_1]+(1-k)(2-m-m^2)(1+2(\lambda-\delta+3\lambda\delta))B_1^2}{2(2-m-m^2)(1+2(\lambda-\delta+3\lambda\delta))B_1^2}$$

which implies, $-1 \leq t \leq 1$ and $|W_2^2 - tw_1^2| \leq 1$, thus

$$|\rho_{2k+1}-\mu\rho_{k+1}^2| \leq \frac{(2-m-m^2)B_1}{(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2 k} \tag{2.23}$$

Case (iii): If $\mu \geq \eta_2$, then

$$\mu \geq \frac{2(1-m)(1+\lambda-\delta+2\lambda\delta)^2 k[((1-m)B_2+(1+m)B_1^2)+(1-m)(1+2(\lambda-\delta+3\lambda\delta))B_1]+(1-k)(2-m-m^2)(1+2(\lambda-\delta+3\lambda\delta))B_1^2}{2(2-m-m^2)(1+2(\lambda-\delta+3\lambda\delta))B_1^2}$$

which implies $t \geq 1$ and $|W_2^2 - tw_1^2| \leq t$, it is clear that

$$|\rho_{2k+1}-\mu\rho_{k+1}^2| \leq \frac{(2-m-m^2)B_1}{(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2 k}\Bigg|\frac{\mu(2-m-m^2)B_1}{(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2 k} - \frac{(1-m)B_2+(1+m)B_1^2}{(1-m)(1+2(\lambda-\delta+3\lambda\delta))B_1} - \frac{(1-k)(2-m-m^2)B_1}{2(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2 k}\Bigg| \tag{2.24}$$

The result in (2.20) follows from (2.22), (2.23) and (2.24).

It is also note that when
(i) $\mu \leq \eta_1$, the equality holds if and only if $w(z) = z$ or one of its rotation.
(ii) $\eta_1 \leq \mu \leq \eta_2$, then the equality holds if and only if $w(z) = z^2$ or one of its rotation.
(iii) $\mu \geq \eta_2$, then the equality holds when $w(z) = \frac{z(\epsilon+z)}{1+\epsilon z}(0 \leq \epsilon \leq 1)$ or one of its rotation.
This completes the proof of Theorem 2.11. $\square$

Theorem 2.12. *For $0 \leq \lambda < 1$, $0 \leq \delta < 1$, $|m| \leq 1$ but $m \neq 1$, if $f \in C_s(\lambda, \delta, m, \phi)$ and F is the k^{th} root transform of the function f defined by* (1.6), *then for any real number μ and for*

$$\gamma_1 = \frac{\begin{aligned}8(1-m)(1+\lambda-\delta+2\lambda\delta)^2 k[((1-b)B_2+(1+b)B_1^2)-(1-b)B_1(1+2(\lambda-\delta \\ +3\lambda\delta))]+3(1-k)(2-m-m^2)(1+2(\lambda-\delta+3\lambda\delta))B_1^2\end{aligned}}{6\,(2-m-m^2)\,(1+2(\lambda-\delta+3\lambda\delta))B_1^2} \tag{2.25}$$

$$\gamma_2 = \frac{\begin{aligned}8(1-m)(1+\lambda-\delta+2\lambda\delta)^2 k[((1-b)B_2+(1+b)B_1^2)+(1-b)B_1(1+2(\lambda-\delta \\ +3\lambda\delta))]+3(1-k)(2-m-m^2)(1+2(\lambda-\delta+3\lambda\delta))B_1^2\end{aligned}}{6\,(2-m-m^2)\,(1+2(\lambda-\delta+3\lambda\delta))B_1^2} \tag{2.26}$$

We have

$$|\rho_{2k+1}-\mu\rho_{k+1}^2| = \begin{cases} \frac{B_1}{3(2-m-m^2)k}\left[\frac{(1-m)B_2+(1+m)B_1^2}{(1-m)(1+2(\lambda-\delta+3\lambda\delta))} - \frac{3(k-1)(2-m-m^2)B_1}{8(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2k} \right. \\ \left. -\frac{3\mu(2-m-m^2)B_1}{4(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2k}\right], & \mu \leq \gamma_1 \\ \frac{B_1}{3(2-m-m^2)k}, & \gamma_1 \leq \mu \leq \gamma_2 \\ \frac{B_1}{3(2-m-m^2)k}\left[\frac{3(k-1)(2-m-m^2)B_1}{8(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2k} - \frac{(1-m)B_2+(1+m)B_1^2}{(1-m)(1+2(\lambda-\delta+3\lambda\delta))} \right. \\ \left. +\frac{3\mu(2-m-m^2)B_1}{4(1-m)^2(1+\lambda-\delta+2\lambda\delta)^2k}\right], & \mu \geq \gamma_2 \end{cases} \tag{2.27}$$

Each of the estimate (2.27) *is sharp.*

Proof. The proof of Theorem is much akin to proof of Theorem 2.11 and hence omit it. $\square$

Remark 2.13. The result is sharp for the extremal functions

$$\frac{(1-m)(\lambda\delta z^3 f'''(z)+(2\lambda\delta+\lambda-\delta)z^2 f''(z)+zf'(z))}{\lambda\delta z^2(f''(z)-m^2 f''(mz))+(\lambda-\delta)z(f'(z)-mf'(mz))+(1-\lambda+\delta)(f(z)-f(mz))} = p(z) = \frac{z(z-t)}{1-zt}, \quad |t|<1$$

and the result is sharp for the functions given by

$$\frac{(1-m)(\lambda\delta z^3 f'''(z)+(2\lambda\delta+\lambda-\delta)z^2 f''(z)+zf'(z))'}{(\lambda\delta z^2(f''(z)-m^2 f''(mz))+(\lambda-\delta)z(f'(z)-mf'(mz))+(1-\lambda+\delta)(f(z)-f(mz)))'} = p(z) = \frac{z}{1+z^2}.$$

By choosing $\lambda = 0, \delta = 0$ and $m = -1$, obtained by Panigrahi et al. [13]. Further, by taking $k = 1$ in Theorem 2.7 and Theorem 2.12, the result was studied by Shanmugam et al. ([18], Theorem 2.1) and ([18], Corollary 2.4) respectively.

References

[1] O. P. Ahuja and M. Jahangiri, Fekete-Szegö problem for a unified class of analytic functions, *Panamer. Math. J.* 7(2) (1997), 67-78.

[2] R. M. Ali, S. K. Lee, V. Ravichandran and S. Supramaniam, The Fekete-Szegö coefficient functional for the transforms of analytic functions, *Bull. Iranian Math. Soc.* 35(2) (2009), 119-142.

[3] R. M. Ali, V. Ravichandran and N. Seerivasagan, Coefficient bounds for p-valent functions, *Appl. Math. Comput.* 187(1) (2007), 35-46.

[4] L. de Branges de Bourcia, A proof of Bieberbach conjecture, *Acta Mathematica* (1985), 137-152. https://doi.org/10.1007/bf02392821

[5] N. E. Cho and S. Owa, On the Fekete-Szegö problem for strongly α logarithmic quasi convex functions, *Southeast Asian Bull. Math.* 28(3) (2004), 421-430. https://doi.org/10.5556/j.tkjm.34.2003.269

[6] R. N. Das and P. Singh, On subclasses of schlicht mapping, *Indian J. Pure Appl. Math.* 8 (1977), 864-872.

[7] R. M. El-Ashwah, M. K. Aouf and F. M. Abdulkaren, Fekete-Szegö inequality for certain class of analytic function of complex order, *Int. J. Open Problem Complex Anal.* 6(1) (2014). https://doi.org/10.12816/0006022

[8] F. R. Keogh and E. P. Merkes, A coefficient inequality for certain classes of analytic functions, *Proc. Amer. Math. Soc.* 20 (1969), 8-12. https://doi.org/10.1090/s0002-9939-1969-0232926-9

[9] O. S. Kwon and N. E. Cho, On the Fekete-Szegö problem for certain analytic functions, *J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math.* 10(4) (2003), 265-271.

[10] S. S. Miller and P. T. Mocanu, Differential Subordination: Theory and Applications, *Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker, New York*, 2000.

[11] J. W. Noonan and D. K. Thomas, On the second Hankel Determinant of a really mean p-valent functions, *Trans. Amer. Math. Soc.* 223 (1976), 337-346. https://doi.org/10.1090/s0002-9947-1976-0422607-9

[12] T. Panigrahi and S. K. Mohapatra, Coefficient functional for the kth root transform of analytic function and applications to fractional derivatives, *Fract. Diff. Cal.* 8(1) (2018), 191-203. https://doi.org/10.7153/fdc-2018-08-12

[13] T. Panigrahi and G. Murugusundaramoorthy, The Fekete-Szegö inequality for subclass of analytic function of complex order, *Adv. Stud. Contemp. Math.* 24(1) (2014), 67-75.

[14] T. Panigrahi, G. Murugusundaramoorthy and S. K. Mohapatra, Fekete-Szegö problems for the kth root transform of subclasses of starlike convex functions with respect to symmetric points, *Monties Taurus J. Pure Appl. Math.* 3(3) (2021), 172-181. https://doi.org/10.21136/mb.2014.143938

[15] T. RamReddy, D. Shalini, D. Vamshee Krishna and B. Venkateswarlu, Coefficient inequality for transforms of certain subclass of analytic functions, *Acta Et Commentationes Universitati Startuensis De Mathematica* 21(2) (2017), 185-193. https://doi.org/10.12697/acutm.2017.21.12

[16] V. Ravichandran, Starlike and convex functions with respect to conjugate points, *Acta Math. Acad. Paedagog. Nyhazi (N.S.)* 20(1) (2004), 31-37.

[17] K. Sakaguchi, On a certain univalent mapping, *J. Math. Soc. Japan* 11 (1959), 72-75.

[18] T. N. Shanmugam, C. Ramachandran and V. Ravichandran, Fekete-Szegö problem for subclasses of starlike functions with respect to symmetric points, *Bull. Korean Math. Soc.* 43, 589-598. https://doi.org/10.4134/bkms.2006.43.3.589