Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 13, Number 1, 2023, Pages 121-131
https://doi.org/10.34198/ejms.13123.121131

On Inequalities for the Ratio of v-Gamma and v-Polygamma Functions

İnci Ege

Department of Mathematics, Aydın Adnan Menderes University, Aydın, Turkey
e-mail: iege@adu.edu.tr

Abstract

In this paper, the author presents some double inequalities involving a ratio of v-Gamma and v-polygamma functions. The approach makes use of the log-convexity property of v-Gamma function and the monotonicity property of v-polygamma function. Some of the results also give generalizations and extensions of some previous results.

1 Introduction and Preliminaries

Logarithmically convex (log-convex) functions are of interest in many areas of mathematics. They have been found to play an important role and have many applications, especially in the theory of special functions, [3, 6, 9, 15].

Let $C \in \mathbb{R}$ be a convex set. The function $f : C \to [0,\infty)$ is said to be convex on C if it satisfies the inequality

$$f(cr + (1-c)t) \leq c\, f(r) + (1-c)\, f(t), \quad r, t \in C,\ 0 \leq c \leq 1.$$

Also, a function $f : C \to [0,\infty)$ is said to be logarithmically convex (log-convex) if $\log f$ is convex or equivalently it satisfy the inequality

$$f(cr + (1-c)t) \leq (f(r))^c (f(t))^{1-c}, \quad r, t \in C,\ 0 \leq c \leq 1.$$

Note that, a log-convex function is convex, and a family of log-convex functions is closed under both addition and multiplication.

Received: April 27, 2023; Revised & Accepted: May 16, 2023; Published: May 20, 2023
2020 Mathematics Subject Classification: 33B15, 26D07, 26A48.
Keywords and phrases: Gamma function, v-Gamma function, v-polygamma function, convexity, inequality, monotonicity.

Perhaps, the most known and used of the special functions is Euler's Gamma function. It is defined by

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx, \quad t > 0.$$

The Psi (or digamma) function is defined as the logarithmic derivative of the Gamma function, and the polygamma function is defined as the rth order derivative of the digamma function.

The subject of present new inequalities including the Gamma function has attracted the attention of many mathematicians. For example, Shabani, in [13] proved the following inequalities:

$$\frac{\Gamma(a+b)^c}{\Gamma(d+e)^f} \leq \frac{\Gamma(a+bt)^c}{\Gamma(d+et)^f} \leq \frac{\Gamma(a)^c}{\Gamma(d)^f}, \quad t \in [0,1] \tag{1}$$

where a, b, c, d, e and f are real numbers such that $a+bt > 0$, $d+et > 0$, $a+bt \leq d+et$, $ef \geq bc > 0$ and $\psi(a+bt) > 0$ or $\psi(d+et) > 0$.

Also, Vinh and Ngoc, in [14] proved the following inequalities:

$$\frac{\prod_{i=1}^{n} \Gamma(1+\alpha_i)}{\Gamma(\beta + \sum_{i=1}^{n} \alpha_i)} \leq \frac{\prod_{i=1}^{n} \Gamma(1+\alpha_i t)}{\Gamma(\beta + \sum_{i=1}^{n} \alpha_i t)} \leq \frac{1}{\Gamma(\beta)}, \quad t \in [0,1].$$

where $\beta \geq 1$, $\alpha_i > 0$, $n \in \mathbb{N}$.

Some new extensions of the Gamma function and including inequalities of them have been given by many researchers, [1, 5, 7, 8, 11, 12, 13]. Recently, a new one-parameter deformation of the classical Gamma function is introduced as a v-analogue (v-deformation or v-generalization) of the Gamma function, [4]. It is defined as

$$\Gamma_v(t) = \int_0^\infty \left(\frac{x}{v}\right)^{\frac{t}{v}-1} e^{-x}\, dx, \quad t, v > 0.$$

Note that when $v = 1$, we have $\Gamma_v(t) = \Gamma(t)$. The logarithmic derivative of Γ_v is called v-digamma or v-psi function and denoted by ψ_v and the rth order derivative of ψ_v is called v-polygamma function. The series representations are given in [4] as

$$\psi_v(t) = -\frac{\ln v + \gamma}{v} - \frac{1}{t} + \sum_{n=1}^{\infty}\left[\frac{1}{nv} - \frac{1}{t+nv}\right], \tag{2}$$

and

$$\psi_v^{(r)}(t) = (-1)^{r+1} r! + \sum_{n=0}^{\infty} \frac{1}{(t+nv)^{r+1}}. \tag{3}$$

The main aim of the present study is to give some new generalized inequalities including the functions Γ_v and $\psi_v^{(r)}$ by using similar methods, used in [10, 13]. This method is based on some monotonicity properties of certain functions associated with v-Gamma and v-polyamma functions.

2 Main Results

Theorem 1. *Let* $0 < a_i + \sum_{j=1}^{m} b_j t \le d_i + \sum_{k=1}^{l} e_k t$, $\left(\sum_{k=1}^{l} e_k\right) f_i \ge \left(\sum_{j=1}^{m} b_j\right) c_i$, $a_i, c_i, d_i, f_i > 0$, *and* $\left(\sum_{j=1}^{m} b_j\right) c_i > 0$ *for* $i = 1, 2, \ldots n$. *If*

$$\psi_v\left(a_i + \sum_{j=1}^{m} b_j t\right) > 0 \quad or \quad \psi_v\left(d_i + \sum_{k=1}^{l} e_k t\right) > 0 \tag{4}$$

then the function

$$F(t) = \prod_{i=1}^{n} \frac{\Gamma_v\left(a_i + \sum_{j=1}^{m} b_j t\right)^{c_i}}{\Gamma_v\left(d_i + \sum_{k=1}^{l} e_k t\right)^{f_i}}$$

is decreasing on $[0, \infty)$ *and the following inequalities bounding a ratio of the* v*-Gamma function hold for* $v > 0$*:*

$$\prod_{i=1}^{n} \frac{\Gamma_v\left(a_i+\sum_{j=1}^{m} b_j\right)^{c_i}}{\Gamma_v\left(d_i+\sum_{k=1}^{l} e_k\right)^{f_i}} \leq F(t) \leq \prod_{i=1}^{n} \frac{\Gamma_v\left(a_i\right)^{c_i}}{\Gamma_v\left(d_i\right)^{f_i}}, \quad t \in [0,1], \tag{5}$$

and

$$F(t) \leq \prod_{i=1}^{n} \frac{\Gamma_v\left(a_i+\sum_{j=1}^{m} b_j\right)^{c_i}}{\Gamma_v\left(d_i+\sum_{k=1}^{l} e_k\right)^{f_i}}, \quad t \in (1,\infty). \tag{6}$$

Proof. Let $G(t)=\ln F(t)$. Then

$$G(t)=\sum_{i=1}^{n} c_i \ln \Gamma_v\left(a_i+\sum_{j=1}^{m} b_j t\right)-\sum_{i=1}^{n} f_i \ln \Gamma_v\left(d_i+\sum_{k=1}^{l} e_k t\right),$$

and

$$\begin{aligned} G^{'}(t) &= \left(\sum_{j=1}^{m} b_j\right) \sum_{i=1}^{n}\left[c_i \psi_v\left(a_i+\sum_{j=1}^{m} b_j t\right)\right]-\left(\sum_{k=1}^{l} e_k\right) \sum_{i=1}^{n}\left[f_i \psi_v\left(d_i+\sum_{k=1}^{l} e_k t\right)\right] \\ &= \sum_{i=1}^{n}\left[\left(\sum_{j=1}^{m} b_j\right) c_i \psi_v\left(a_i+\sum_{j=1}^{m} b_j t\right)-\left(\sum_{k=1}^{l} e_k\right) f_i \psi_v\left(d_i+\sum_{k=1}^{l} e_k t\right)\right]. \end{aligned}$$

Let $\psi_v\left(a_i+\sum_{j=1}^{m} b_j t\right)>0$ for $i=1,2,\ldots n$. Since from the equation (2) we have that ψ_v is an increasing function on $[0,\infty)$ we get

$$\psi_v\left(a_i+\sum_{j=1}^{m} b_j t\right) \leq \psi_v\left(d_i+\sum_{k=1}^{l} e_k t\right),$$

so we have $\psi_v\left(d_i+\sum_{k=1}^{l} e_k t\right)>0$. Then

$$\begin{aligned}\left(\sum_{j=1}^{m} b_j\right) c_i \psi_v\left(a_i+\sum_{j=1}^{m} b_j t\right) &\leq \left(\sum_{j=1}^{m} b_j\right) c_i \psi_v\left(d_i+\sum_{k=1}^{l} e_k t\right) \\ &\leq \left(\sum_{k=1}^{l} e_k\right) f_i \psi_v\left(d_i+\sum_{k=1}^{l} e_k t\right).\end{aligned}$$

This proves the inequality

$$\left(\sum_{j=1}^{m} b_j\right) c_i \psi_v\left(a_i+\sum_{j=1}^{m} b_j t\right)-\left(\sum_{k=1}^{l} e_k\right) f_i \psi_v\left(d_i+\sum_{k=1}^{l} e_k t\right) \leq 0 \tag{7}$$

for $i=1,2,\ldots n$. Now, let $\psi_v\left(d_i+\sum_{k=1}^{l} e_k t\right)>0$. Then if $\psi_v\left(a_i+\sum_{j=1}^{m} b_j t\right)>0$, we have the inequality (7) with the above discussion.

If $\psi_v\left(a_i+\sum_{j=1}^{m} b_j t\right) \leq 0$ with $\psi_v\left(d_i+\sum_{k=1}^{l} e_k t\right)>0$, we can write

$$\begin{aligned}\left(\sum_{k=1}^{l} e_k\right) f_i \psi_v\left(d_i+\sum_{k=1}^{l} e_k t\right) &\geq \left(\sum_{j=1}^{m} b_j\right) c_i \psi_v\left(d_i+\sum_{k=1}^{l} e_k t\right) \\ &\geq \left(\sum_{j=1}^{m} b_j\right) c_i \psi_v\left(a_i+\sum_{j=1}^{m} b_j t\right),\end{aligned}$$

and the inequality (7) again follows for $i=1,2,\ldots n$. Then we have $G^{'}(t) \leq 0$. It implies that G and so F is decreasing on $[0,\infty)$. Then for every $t\in[0,1]$, we have

$$F(1) \leq F(t) \leq F(0),$$

and for $t\in(1,\infty)$ we have

$$F(t) \leq F(1),$$

and the inequalities (5) and (6) are valid. □

Theorem 2. *Let F be a function given in the Theorem 1, where* $0 < a_i + \sum_{j=1}^{m} b_j t \leq d_i + \sum_{k=1}^{l} e_k t$, $\left(\sum_{j=1}^{m} b_j\right) c_i \geq \left(\sum_{k=1}^{l} e_k\right) f_i$, $\quad a_i, c_i, d_i, f_i > 0$, *and* $\sum_{k=1}^{l} e_k f_i > 0$ *for* $i = 1, 2, \dots n$. *If* $\psi_v\left(a_i + \sum_{j=1}^{m} b_j t\right) < 0$ *or* $\psi_v\left(d_i + \sum_{k=1}^{l} e_k t\right) < 0$, *then the function F is decresing for $t \in [0, \infty)$ and the inequalities (5) and (6) hold.*

Proof. It can be proved by using the similar way in the Theorem 1. □

Now, we give the following remarks by using the Theorems 1 and 2.

Remark 3. *If in the Theorem 1 we take $n = m = l = v = 1$, we obtain the inequality (1).*

Remark 4. *If we take $F : [0, \infty) \to \mathbb{R}$ as any differentiable log-convex function instead of Γ_v satisfying the conditions in the Theorems 1 and 2, the results of the Theorems still hold, since the logarithmic convexity of F on $[0, \infty)$ implies that its logarithmic derivative is an increasing function on $[0, \infty)$. For example, one can take the log-convex function Γ or any log-convex generalizations such as Γ_k, $\Gamma_{q,k}$, $\Gamma_{p,q,k}$ functions instead of Γ_v, [1, 2, 5].*

Now, we give some inequalities including the v-polygamms functions.

Theorem 5. *Let $0 < a + bt \leq c + dt$, $a, c, \alpha, \beta > 0$, $\alpha b > 0$ and*

$$H(t) = \frac{\left[\psi_v^{(r)}(a + bt)\right]^{\alpha}}{\left[\psi_v^{(r)}(c + dt)\right]^{\beta}}.$$

If $\alpha b \leq \beta d$ and $r = 2k + 1$, $k \in \mathbb{N} \cup \{0\}$, then H is decreasing function on $[0, \infty)$ and the following inequalities hold for $t \in [0, 1]$:

$$\frac{\left[\psi_v^{(r)}(a + b)\right]^{\alpha}}{\left[\psi_v^{(r)}(c + d)\right]^{\beta}} \leq H(t) \leq \frac{\left[\psi_v^{(r)}(a)\right]^{\alpha}}{\left[\psi_v^{(r)}(c)\right]^{\beta}}, \tag{8}$$

and if $\alpha b \geq \beta d$ *and* $r = 2k,\ k \in \mathbb{N} \cup \{0\}$, *then* H *is increasing function on* $[0, \infty)$ *and the inequalities (8) are reversed.*

Proof. Let $I(t) = \ln H(t)$. Then

$$\begin{aligned} I^{'}(t) &= \alpha b\,\frac{\psi_v^{(r+1)}(a+bt)}{\psi_v^{(r)}(a+bt)} - \beta d\,\frac{\psi_v^{(r+1)}(c+dt)}{\psi_v^{(r)}(c+dt)} \\ &= \frac{\alpha b\,\psi_v^{(r+1)}(a+bt)\psi_v^{(r)}(c+dt) - \beta d\,\psi_v^{(r+1)}(c+dt)\psi_v^{(r)}(a+bt)}{\psi_v^{(r)}(a+bt)\psi_v^{(r)}(c+dt)}. \end{aligned}$$

Now, by using equation (3) observe that we have $\psi_v^{(r)} > 0$ for $r = 2k+1$, $k \in \mathbb{N} \cup \{0\}$ and $\psi_v^{(r)} < 0$ for $r = 2k$, $k \in \mathbb{N} \cup \{0\}$. Firstly, let $\alpha b \leq \beta d$ and $r = 2k+1$, $k \in \mathbb{N} \cup \{0\}$. Then $\psi_v^{(r)}(a+bt) > 0$, $\psi_v^{(r)}(c+dt) > 0$, $\psi_v^{(r+1)}(a+bt) < 0$ and $\psi_v^{(r+1)}(c+dt) < 0$, $\psi_v^{(r)}$ is decreasing and $\psi_v^{(r+1)}$ is increasing. Then we have $\psi_v^{(r)}(a+bt) \geq \psi_v^{(r)}(c+dt)$ and $\psi_v^{(r+1)}(a+bt) \leq \psi_v^{(r+1)}(c+dt)$. Then we have

$$\begin{aligned} \psi_v^{(r)}(c+dt)\psi_v^{(r+1)}(a+bt) &\leq \psi_v^{(r)}(c+dt)\psi_v^{(r+1)}(c+dt) \\ &\leq \psi_v^{(r)}(a+bt)\psi_v^{(r+1)}(c+dt). \end{aligned}$$

Now, using the condition $\alpha b \leq \beta d$ we get

$$\begin{aligned} \alpha b\,\psi_v^{(r)}(c+dt)\psi_v^{(r+1)}(a+bt) &\leq \alpha b\,\psi_v^{(r)}(c+dt)\psi_v^{(r+1)}(c+dt) \\ &\leq \alpha b\,\psi_v^{(r)}(a+bt)\psi_v^{(r+1)}(c+dt) \\ &\leq \beta d\,\psi_v^{(r)}(a+bt)\psi_v^{(r+1)}(c+dt), \end{aligned}$$

that is

$$\alpha b\,\psi_v^{(r)}(c+dt)\psi_v^{(r+1)}(a+bt) - \beta d\,\psi_v^{(r)}(a+bt)\psi_v^{(r+1)}(c+dt) \leq 0,$$

so we have $I^{'}(t) \leq 0$. That implies that I is decreasing on $[0, \infty)$. Hence H is decreasing on $[0, \infty)$. Then for $t \in [0, 1]$ we have

$$H(1) \leq H(t) \leq H(0),$$

and the inequalities (8) follow.

Let $\alpha b \geq \beta d$ and $r = 2k$, $k \in \mathbb{N} \cup \{0\}$. Then we have $\psi_v^{(r)}(a+bt) < 0$, $\psi_v^{(r)}(c+dt) < 0$, $\psi_v^{(r+1)}(a+bt) > 0$ and $\psi_v^{(r+1)}(c+dt) > 0$, $\psi_v^{(r)}$ is increasing and $\psi_v^{(r+1)}$ is decreasing. Then we have

$$\begin{aligned}\psi_v^{(r+1)}(c+dt)\psi_v^{(r)}(a+bt) &\leq \psi_v^{(r+1)}(c+dt)\psi_v^{(r)}(c+dt)\\ &\leq \psi_v^{(r+1)}(a+bt)\psi_v^{(r)}(c+dt),\end{aligned}$$

and using the condition $\alpha b \geq \beta d$ we get

$$\begin{aligned}\alpha b\, \psi_v^{(r+1)}(a+bt)\psi_v^{(r)}(c+dt) &\geq \alpha b\, \psi_v^{(r+1)}(c+dt)\psi_v^{(r)}(c+dt)\\ &\geq \alpha b\, \psi_v^{(r+1)}(c+dt)\psi_v^{(r)}(a+bt)\\ &\geq \beta d\, \psi_v^{(r+1)}(c+dt)\psi_v^{(r)}(a+bt),\end{aligned}$$

so we have $I^{'}(t) \geq 0$, means that I is increasing for $t \in [0,\infty)$. Hence H is increasing on $[0,\infty)$. Then the reverse of the inequality (8) is valid. □

Theorem 6. *Let* $0 < a_i + \sum_{j=1}^{m} b_j t \leq c_i + \sum_{k=1}^{l} d_k t$, $a_i, c_i, \alpha_i, \beta_i > 0$ *for* $i = 1, 2, \ldots n$, $r \in \mathbb{N} \cup \{0\}$ *and*

$$J(t) = \prod_{i=1}^{n} \frac{\left[\psi_v^{(r)}\left(a_i + \sum_{j=1}^{m} b_j t\right)\right]^{\alpha_i}}{\left[\psi_v^{(r)}\left(c_i + \sum_{k=1}^{l} d_k t\right)\right]^{\beta_i}}.$$

If $\left(\sum_{j=1}^{m} b_j\right)\alpha_i < 0$ *and* $\left(\sum_{k=1}^{l} d_k\right)\beta_i > 0$ *for* $i = 1, 2, \ldots n$. *Then* J *is increasing on* $[0,\infty)$ *and the following inequalities hold on* $[0,1]$*:*

$$\prod_{i=1}^{n} \frac{\left[\psi_v^{(r)}(a_i)\right]^{\alpha_i}}{\left[\psi_v^{(r)}(c_i)\right]^{\beta_i}} \leq J(t) \leq \prod_{i=1}^{n} \frac{\left[\psi_v^{(r)}\left(a_i + \sum_{j=1}^{m} b_j\right)\right]^{\alpha_i}}{\left[\psi_v^{(r)}\left(c_i + \sum_{k=1}^{l} d_k\right)\right]^{\beta_i}}, \tag{9}$$

and if $\left(\sum_{j=1}^{m} b_j\right)\alpha_i > 0$ *and* $\left(\sum_{k=1}^{l} d_k\right)\beta_i < 0$ *for* $i = 1, 2, \ldots n$. *Then* J *is increasing function and the inequalities (9) are reversed.*

Proof. Let $K(t) = \ln J(t)$. Then

$$K^{'}(t) = \sum_{i=1}^{n}\left[\left(\sum_{j=1}^{m} b_j\right)\alpha_i \frac{\psi_v^{(r+1)}\left(a_i + \sum_{j=1}^{m} b_j t\right)}{\psi_v^{(r)}\left(a_i + \sum_{j=1}^{m} b_j t\right)} - \left(\sum_{k=1}^{t} d_k\right)\beta_i \frac{\psi_v^{(r+1)}\left(c_i + \sum_{k=1}^{l} d_k t\right)}{\psi_v^{(r)}\left(c_i + \sum_{k=1}^{l} d_k t\right)}\right].$$

Now, observe that $\dfrac{\psi_v^{(r+1)}\left(a_i + \sum_{j=1}^{m} b_j t\right)}{\psi_v^{(r)}\left(a_i + \sum_{j=1}^{m} b_j t\right)} < 0$ and $\dfrac{\psi_v^{(r+1)}\left(c_i + \sum_{k=1}^{l} d_k t\right)}{\psi_v^{(r)}\left(c_i + \sum_{k=1}^{l} d_k t\right)} < 0$.

For the case $\left(\sum_{j=1}^{m} b_j\right)\alpha_i < 0$ with $\left(\sum_{k=1}^{l} d_k\right)\beta_i > 0$ for $i = 1, 2, \ldots n$, we get $K^{'}(t) > 0$. Then we have J is increasing on $[0, \infty)$ and the inequalities (9) follow on $[0, 1]$. Now, for the second case $\left(\sum_{j=1}^{m} b_j\right)\alpha_i > 0$ with $\left(\sum_{k=1}^{l} d_k\right)\beta_i < 0$ for $i = 1, 2, \ldots n$, we get J is decreasing and then the inequalities (9) are reversed on $[0, 1]$. □

References

[1] R. Diaz and E. Pariguan, On hypergeometric functions and Pochhammer k-symbol, *Divulgaciones Matematicas* 15 (2007), 179-192.

[2] R. Diaz and C. Teruel, q, k-Generalized Gamma and beta functions, *Journal of Nonlinear Mathematical Physics* 12(1) (2005), 118-134. https://doi.org/10.2991/jnmp.2005.12.1.10

[3] S. S. Dragomir and C. Pearce, Selected topics on Hermite-Hadamard inequalities and applications, *Science direct working paper* (S1574-0358), 04 (2003).

[4] E. Djabang, K. Nantomah and M. M. Iddrisu, On a v-analogue of the Gamma function and some associated inequalities, *J. Math. Comput. Sci.* 11(1) (2020), 74-86. https://doi.org/10.28919/jmcs/5047

[5] İ. Ege, On defining the generalized Gamma function, *Note di Matematica* 39(1) (2019), 107-116. https://doi.org/10.1285/i15900932v39n1p107

[6] D. Karp and S. M. Sitnik, Log-convexity and log-concavity of hypergeometric-like functions, *Journal of Mathematical Analysis and Applications* 364(2) (2010), 384-394. https://doi.org/10.1016/j.jmaa.2009.10.057

[7] V. Krasniqi and F. Merovci, Generalization of some inequalities for the special functions, *J. Inequal. Spec. Funct.* 3 (2012), 34-40.

[8] V. Krasniqi, H. M. Srivastava and S. S. Dragomir, Some complete monotonicity properties for the (p, q)-gamma function, *Applied Mathematics and Computation* 219(21) (2013), 10538-10547. https://doi.org/10.1016/j.amc.2013.04.034

[9] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis. Mathematics and its Applications* (East European Series), Vol. 61, Springer, 2010.

[10] K. Nantomah and M. M. Iddrisu, The k-analogue of some inequalities for the gamma function, *Electronic Journal of Mathematical Analysis and Applications* 2(2) (2014), 172-177.

[11] K. Nantomah and İ Ege, A lambda analogue of the Gamma function and its properties, *Researches in Mathematics* 30(2) (2022), 18-29. https://doi.org/10.15421/242209

[12] K. Nantomah, E. Prempeh and S. B. Twum, On a (p, k)-analogue of the Gamma function and some associated Inequalities, *Moroccan Journal of Pure and Applied Analysis* 2(2) (2016), 79-90. https://doi.org/10.7603/s40956-016-0006-0

[13] A. S. Shabani, Generalization of some inequalities for the Gamma function, *Mathematical Communications* 13(2) (2008), 271-275.

[14] N. V. Vinh and N. P. N. Ngoc, An inequality for the Gamma function, *International Mathematical Forum* 4(28) (2009), 1379-1382.

[15] Z.-H. Yang, On the log-convexity of two-parameter homogeneous functions, *Mathematical Inequalities and Applications* 10(3) (2007), 499-506. https://doi.org/10.7153/mia-10-47