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The Inverse Burr-Generalized Family of Distributions: Theory and Applications

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Abstract

This paper presents a new class of generalized distributions based on the inverse Burr (Burr III) distribution. Statistical properties of the proposed family of distributions such as the density and cumulative distribution functions, survival and hazard rate functions, quantile, moments, moment generating function, probability weighted moments, Renyi entropy and distribution of order statistics are derived. The maximum likelihood estimation method is employed to obtain the parameter estimates of the family of distributions. A Monte Carlo simulation study is conducted in order to investigate the asymptotic behaviour of the parameter estimates of a sub-model from the proposed family of distributions. Finally, the utility of proposed family of distributions in lifetime data fittings is illustrated using two real data sets and the results obtained were compared to some existing non-nested models. Based on some model selection criteria and goodness of fit test statistics, it was evident that the sub-model from the proposed family of distributions performed reasonably better than the competitor distributions in fitting the two data sets.

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1 Introduction

Burr (1942) has introduced various forms of probability distributions (Burr II, III, X, XII) which have been widely studied and applied to fit lifetime data. Many researchers have shown interest in the Burr XII distribution since it could be used to fit almost any form of unimodal (left-skewed, right-skewed or symmetric) data. Silva et al. (2008) developed the log Burr XII regression model, Silva and Cordeiro (2015) introduced the Burr XII power series distribution, Lanjoni et al. (2015) introduced the extended Burr XII regression models, Osatohanmwen et al. (2017) proposed the Gumbel Burr XII distribution, Bhati et al. (2018) introduced the generalized log Burr XII distribution, Korkmaz and Chesneau (2021) studied the unit-Burr XII distribution.

Tadikamalla (1980) noted that while the Burr XII distribution yields a wide range of values of skewness $\left(\sqrt{\beta_1}\right)$ and kurtosis (β_2), the inverse of the Burr XII distribution referred to as Burr III distribution covers a wider region in the $\left(\sqrt{\beta_1}, \beta_2\right)$ plane than the Burr XII distribution, the Weibull family, the gamma family, the log-normal family, the normal distribution, the logistic distribution, etc. On this note, Osemwenkhae and Iyenoma (2018) gave a comprehensive study on the mathematical properties of the inverse Burr (Burr III) distribution and illustrated the applicability of the distribution in lifetime data analysis. More study on the Burr III distribution can be found in the works of Burr and Cislak (1968), Johnson et al. (1995), Modi and Gill (2019).

Let T be a Burr III random variable, then the cumulative distribution function of T is defined as

$$G(t) = \left(1 + t^{-a}\right)^{-b} \tag{1}$$

with the corresponding density function given as

$$g(t) = abt^{-(a+1)} \left(1 + t^{-a}\right)^{-(b+1)}, \quad t > 0, a, b > 0 \tag{2}$$

where a and b are shape parameters.

To enhance flexibility, several methods of adding extra parameter(s) to existing distributions have been introduced and widely studied in the literature. It has become unarguably that the flexibility of classical distributions in fitting lifetime (survival) data can be enhanced through the addition of extra parameter(s). Eugene et al. (2002) introduced the beta-generated family of distributions, Marshall and Olkin (2007) studied the Marshall-Olkin extended family of distributions, Shaw and Buckley (2009) proposed the transmuted-G family of distributions, Alzaatreh et al. (2013) introduced and extension of the beta-G family which they called the transformed-transformer $(T - X)$ family of distributions, Cordeiro and de Castro (2011) studied the Kumaraswamy-G family.

The cdf of the $T - X$ family of distributions suggested by Alzaatreh et al. (2013) is given by

$$F(t) = \int_0^{-\log[1-G(t)]} r(x)dx, \tag{3}$$

where $r(x)$ is the density function of a known probability distribution.

Using the idea in equation (3), Bourguignon et al. (2014) proposed the Weibull-G family of distributions by allowing the density function in Equation (3) to follow the Weibull distribution. Nadarajah et al. (2015) considered the density function of the gamma distribution to introduce the Zografos-Balakrishnan-G family of distributions.

In this paper, we use the inverse Burr III distribution (Burr distribution) as the generator to introduce the inverse Burr generated family of distributions. The remaining sections of this paper are organized as follows: the inverse Burr generated family of distributions is defined in Section 2. Section 3 present sub-models from the inverse Burr-G family. In Section 4, some general mathematical properties and useful expansions relating to the inverse Burr-G family of distributions. Section 5 presents the real life data fittings, while Section 6 provides the concluding remark.

2 The Inverse Burr Generated Family of Distributions

Considering the framework in Alzaatreh et al. (2013) defined in equation (3), and employing the inverse Burr distribution as the generator, we define the cumulative distribution function of the inverse Burr generated family of distributions as

$$\begin{aligned} F(t,a,b,\xi) &= ab\int_0^{-\log[1-G(t,\xi)]} x^{-(a+1)}\left(1+x^{-a}\right)^{-(b+1)}dx, \\ &= \left[1+\{-\log[1-G(t,\xi)]\}^{-a}\right]^{-b}, \quad t>0, a, b>0, \end{aligned} \tag{4}$$

where a and b are the shape parameters and $G(t,\xi)$ is the baseline distribution which depends on a parameter vector ξ. The pdf associated with equation (4) is given by

$$\begin{aligned} f(t,a,b,\xi) = abg(t,\xi)[1-G(t,\xi)]^{-1}\{-\log[1-G(t,\xi)]\}^{-(a+1)} \\ \times\left[1+\{-\log[1-G(t,\xi)]\}^{-a}\right]^{-(b+1)}. \end{aligned} \tag{5}$$

Subsequently, we shall denote a random variable T having the pdf in equation (5) by $T \sim$ IB-G.

The survival and hazard rate functions of the IB-G family of distributions are defined, respectively, as

$$S(t,a,b,\xi) = 1-\left[1+\{-\log[1-G(t,\xi)]\}^{-a}\right]^{-b}, \tag{6}$$

and

$$h(t,a,b,\xi) = \frac{abg(t,\xi)\{-\log[1-G(t,\xi)]\}^{-(a+1)}\left[1+\{-\log[1-G(t,\xi)]\}^{-a}\right]^{-(b+1)}}{[1-G(t,\xi)]\left[1-\left[1+\{-\log[1-G(t,\xi)]\}^{-a}\right]^{-b}\right]}. \tag{7}$$

3 Sub-models of the IB-G family of Distributions

In this section, we introduce five sub-models from IB-G family by letting the baseline distribution in equation (5) to follow Kumaraswamy, Lomax, Normal, logistic, and Topp-Leone distributions.

3.1 Inverse Burr-Kumaraswamy (IB-K) distribution

Kumaraswamy (1980) introduced a unit interval probability distribution and called it Kumaraswamy distribution. The cumulative distribution function (cdf) and the probability density function (pdf) of the distribution are, respectively, defined by

$$G(t,\beta,\alpha) = 1 - \left(1 - t^{\beta}\right)^{\alpha}, \quad \beta,\alpha > 0, 0 < t < 1 \tag{8}$$

and

$$g(t,\beta,\alpha) = \alpha\beta t^{\beta-1}\left(1 - t^{\beta}\right)^{\alpha-1}, \quad \beta,\alpha > 0, 0 < t < 1. \tag{9}$$

Applying equation (8) in equation (4), we define the cumulative distribution function of the inverse Burr-Kumaraswamy (IB-K) distribution as

$$F(t,a,b,\alpha,\beta) = \left[1 + \left\{-\log\left(1 - t^{\beta}\right)^{\alpha}\right\}^{-a}\right]^{-b}, \quad 0 < t < 1, a, b, \alpha, \beta > 0. \tag{10}$$

The pdf associated with (10) is defined by

$$\begin{aligned} f(t,a,b,\alpha,\beta) = ab\alpha\beta t^{\beta-1}\left(1 - t^{\beta}\right)^{-1}\left\{-\log\left(1 - t^{\beta}\right)^{\alpha}\right\}^{-(a+1)} \\ \times\left[1 + \left\{-\log\left(1 - t^{\beta}\right)^{\alpha}\right\}^{-a}\right]^{-(b+1)}. \end{aligned} \tag{11}$$

From equations (10) and (11), we obtained the survival and hazard rate functions of the inverse Burr Kumaraswamy (IB-K) distribution, respectively, as follows

$$\begin{aligned} S(t,a,b,\alpha,\beta) &= 1 - F(t,a,b,\alpha,\beta), \\ &= 1 - \left[1 + \left\{-\log\left(1 - t^{\beta}\right)^{\alpha}\right\}^{-a}\right]^{-b}, \end{aligned} \tag{12}$$

and

$$\begin{aligned} h(t,a,b,\alpha,\beta) = \frac{f(t,a,b,\alpha,\beta)}{S(t,a,b,\alpha,\beta)} = \frac{ab\alpha\beta t^{\beta-1}\left(1 - t^{\beta}\right)^{-1}\left\{-\log\left(1 - t^{\beta}\right)^{\alpha}\right\}^{-(a+1)}}{1 + \left\{-\log\left(1 - t^{\beta}\right)^{\alpha}\right\}^{-a}} \\ \times\left[1 + \left\{-\log\left(1 - t^{\beta}\right)^{\alpha}\right\}^{-a}\right]^{-(b+1)}. \end{aligned} \tag{13}$$

3.2 Inverse Burr-Lomax (IB-L) distribution

Suppose a random variable T follow the Lomax distribution, then the cdf and pdf of t is given by

$$G(t, \beta, \alpha) = 1 - \left(1 + \frac{t}{\beta}\right)^{-\alpha}, \quad \beta, \alpha > 0, t > 0 \tag{14}$$

and

$$g(t, \beta, \alpha) = \frac{\alpha}{\beta}\left(1 + \frac{t}{\beta}\right)^{-(\alpha+1)}, \quad \beta, \alpha > 0, t > 0. \tag{15}$$

By inserting equations (14) and (15) into equations (4) and (5), we define the cdf and pdf of the inverse Burr-Lomax (IB-L) distribution, respectively, as

$$F(t, a, b, \alpha, \beta) = \left[1 + \left\{-\alpha \log\left(1 + \frac{t}{\beta}\right)\right\}^{-a}\right]^{-b}, \quad t > 0, a, b, \alpha, \beta > 0, \tag{16}$$

and

$$\begin{aligned} f(t, a, b, \alpha, \beta) =& \frac{ab\alpha}{\beta}\left(1 + \frac{t}{\beta}\right)^{-1}\left\{-\alpha \log\left(1 + \frac{t}{\beta}\right)\right\}^{-(a+1)} \\ & \times \left[1 + \left\{-\alpha \log\left(1 + \frac{t}{\beta}\right)\right\}^{-a}\right]^{-(b+1)}. \end{aligned} \tag{17}$$

The survival and hazard rate functions of the inverse Burr-Lomax (IB-L) distribution are, respectively, defined using equations (16) and (17) as follows

$$\begin{aligned} S(t, a, b, \alpha, \beta) &= 1 - F(t, a, b, \alpha, \beta) \\ &= 1 - \left[1 + \left\{-\alpha \log\left(1 + \frac{t}{\beta}\right)\right\}^{-a}\right]^{-b}, \end{aligned} \tag{18}$$

and

$$\begin{aligned} h(t,a,b,\alpha,\beta) &= \frac{f(t,a,b,\alpha,\beta)}{S(t,a,b,\alpha,\beta)}, \\ &= \frac{ab\alpha\beta^{-1}\left(1+\frac{t}{\beta}\right)^{-1}\left\{-\alpha\log\left(1+\frac{t}{\beta}\right)\right\}^{-(a+1)}}{1-\left[1+\left\{-\alpha\log\left(1+\frac{t}{\beta}\right)\right\}^{-a}\right]^{-b}} \\ &\quad \times\left[1+\left\{-\alpha\log\left(1+\frac{t}{\beta}\right)\right\}^{-a}\right]^{-(b+1)}. \end{aligned} \tag{19}$$

3.3 Inverse Burr-Logistic (IB-Logistic) distribution

The cdf and pdf of a random variable T following the logistic distribution are, respectively, defined by

$$G(t,\alpha) = \left(1+e^{-\alpha t}\right)^{-1}, \quad \alpha > 0, t > 0, \tag{20}$$

and

$$g(t,\alpha) = \alpha e^{-\alpha t}\left(1+e^{-\alpha t}\right)^{-2}, \quad \alpha > 0, t > 0. \tag{21}$$

By inserting equations (20) and (21) into equations (4) and (5), we define the cdf and pdf of the inverse Burr logistic (IB-Logistic) distribution, respectively, as

$$F(t,a,b,\alpha) = \left[1+\left\{-\log\left[1-\left(1+e^{-\alpha t}\right)^{-1}\right]\right\}^{-a}\right]^{-b}, \quad t > 0, a, b, \alpha > 0 \tag{22}$$

and

$$\begin{aligned} f(t,a,b,\alpha) &= \frac{ab\alpha e^{-\alpha t}\left\{-\log\left[1-\left(1+e^{-\alpha t}\right)^{-1}\right]\right\}^{-(a+1)}}{(1+e^{-\alpha t})^{2}\left[1-(1+e^{-\alpha t})^{-1}\right]} \\ &\quad \times\left[1+\left\{-\log\left[1-\left(1+e^{-\alpha t}\right)^{-1}\right]\right\}^{-a}\right]^{-(b+1)}. \end{aligned} \tag{23}$$

The survival and hazard rate functions of the inverse Burr-logistic (IB-Logistic) distribution are, respectively, defined using equations (22) and (23) as follows

$$\begin{aligned} S(t,a,b,\alpha) =& 1 - F(t,a,b,\alpha), \\ &= 1 - \left[1 + \left\{-\log\left[1 - \left(1+e^{-\alpha t}\right)^{-1}\right]\right\}^{-a}\right]^{-b}, \end{aligned} \tag{24}$$

and

$$\begin{aligned} h(t,a,b,\alpha) &= \frac{f(t,a,b,\alpha)}{S(t,a,b,\alpha)}, \\ =& \frac{ab\alpha e^{-\alpha t}\left\{-\log\left[1-\left(1+e^{-\alpha t}\right)^{-1}\right]\right\}^{-(a+1)}}{(1+e^{-\alpha t})^2\left[1-(1+e^{-\alpha t})^{-1}\right]\left[1-\left[1+\left\{-\log\left[1-(1+e^{-\alpha t})^{-1}\right]\right\}^{-a}\right]^{-b}\right]} \\ &\times \left[1+\left\{-\log\left[1-\left(1+e^{-\alpha t}\right)^{-1}\right]\right\}^{-a}\right]^{-(b+1)}. \end{aligned} \tag{25}$$

3.4 Inverse Burr Topp-Leone (IBTL) distribution

Suppose a random variable T follow the one-parameter Topp-Leone distribution reported in Opone et al. (2022), with $G(t,\lambda) = [t(2-t)]^{\lambda}$, and $g(t,\lambda) = 2\lambda(1-t)[t(2-t)]^{\lambda-1}, \lambda > 0, 0 < t < 1$. We defined the cumulative distribution function (cdf) and probability distribution function (pdf) of the inverse Burr-Topp-Leone (IB-TL) distribution, respectively, as

$$F(t,a,b,\lambda) = \left[1+\left\{-\log\left[1-[t(2-t)]^{\lambda}\right]\right\}^{-a}\right]^{-b}, \quad t>0, a,b,\lambda>0 \tag{26}$$

and

$$\begin{aligned} f(t,a,b,\lambda) = & \frac{2ab\lambda(1-t)[t(2-t)]^{\lambda-1}}{[1-[t(2-t)]^{\lambda}]}\left\{-\log\left[1-[t(2-t)]^{\lambda}\right]\right\}^{-(a+1)} \\ &\times\left[1+\left\{-\log\left[1-[t(2-t)]^{\lambda}\right]\right\}^{-a}\right]^{-(b+1)}. \end{aligned} \tag{27}$$

The survival and hazard rate functions of the inverse Burr Topp-Leone (IB-TL) distribution are, respectively, defined using equations (26) and (27) as follows

$$S(t,a,b,\lambda) = 1 - F(t,a,b,\lambda) = 1 - \left[1 + \left\{-\log\left[1 - [t(2-t)]^{\lambda}\right]\right\}^{-a}\right]^{-b} \tag{28}$$

and

$$\begin{aligned} h(t,a,b,\alpha) &= \frac{f(t,a,b,\alpha)}{S(t,a,b,\alpha)} \\ &= \frac{2ab\lambda(1-t)[t(2-t)]^{\lambda-1}\left\{-\log\left[1-[t(2-t)]^{\lambda}\right]\right\}^{-(a+1)}}{[1-[t(2-t)]^{\lambda}]\left[1-\left[1+\{-\log[1-[t(2-t)]^{\lambda}]\}^{-a}\right]^{-b}\right]} \\ &\quad \times \left[1 + \left\{-\log\left[1 - [t(2-t)]^{\lambda}\right]\right\}^{-a}\right]^{-(b+1)}. \end{aligned} \tag{29}$$

3.5 Inverse Burr-Normal (IB-N) distribution

Suppose a random variable T is normally distributed with $G(t,\mu,\sigma) = \Phi\left(\frac{t-\mu}{\sigma}\right)$, and $g(t,\mu,\sigma) = \phi\left(\frac{t-\mu}{\sigma}\right), \mu, \sigma > 0, -\infty < t < \infty$. The cdf and pdf of the inverse Burr-Normal (IB-N) distribution are, respectively, defined as

$$F(t,a,b,\mu,\sigma) = \left[1 + \left\{-\log\left[1 - \Phi\left(\frac{t-\mu}{\sigma}\right)\right]\right\}^{-a}\right]^{-b}, \quad -\infty < t < \infty, a, b, \mu, \sigma > 0 \tag{30}$$

and

$$f(t,a,b,\mu,\sigma) = \frac{ab\phi\left(\frac{t-\mu}{\sigma}\right)\left\{-\log\left[1-\Phi\left(\frac{t-\mu}{\sigma}\right)\right]\right\}^{-(a+1)}\left[1+\left\{-\log\left[1-\Phi\left(\frac{t-\mu}{\sigma}\right)\right]\right\}^{-a}\right]^{-(b+1)}}{\left[1-\Phi\left(\frac{t-\mu}{\sigma}\right)\right]}. \tag{31}$$

Using equations (30) and (31), we defined the survival and hazard rate functions of the inverse Burr Normal (IB-N) distribution, respectively, as follows

$$\begin{aligned} S(t,a,b,\mu,\sigma) &= 1 - F(t,a,b,\mu,\sigma) \\ &= 1 - \left[1 + \left\{-\log\left[1 - \Phi\left(\frac{t-\mu}{\sigma}\right)\right]\right\}^{-a}\right]^{-b}, \end{aligned} \tag{32}$$

and

$$h(t,a,b,\mu,\sigma)=\frac{f(t,a,b,\mu,\sigma)}{S(t,a,b,\mu,\sigma)}$$
$$=\frac{ab\phi\left(\frac{t-\mu}{\sigma}\right)\left\{-\log\left[1-\Phi\left(\frac{t-\mu}{\sigma}\right)\right]\right\}^{-(a+1)}\left[1+\left\{-\log\left[1-\Phi\left(\frac{t-\mu}{\sigma}\right)\right]\right\}^{-a}\right]^{-(b+1)}}{\left[1-\Phi\left(\frac{t-\mu}{\sigma}\right)\right]\left[1-\left[1+\left\{-\log\left[1-\Phi\left(\frac{t-\mu}{\sigma}\right)\right]\right\}^{-a}\right]^{-b}\right]}. \tag{33}$$

4 Some Mathematical Properties of the IB-G family of Distributions

This section presents some mathematical properties of the IB-G family of distributions such as the linear representation of the cumulative distribution function and probability density function, quantile function, r^{th} ordinary moments, moment generating function (mgf), probability weighted moments (PWMs), Renyi entropy and distribution of order statistics.

4.1 Linear Representation

The usefulness of obtaining a linear representation for the density function and the cumulative distribution function of a new model is to allow easy derivation of some other mathematical properties of the model such as the moments, probability weighted moments (PWMs), moment generating function, the distribution of order statistics, etc. The following lemmas will guide us in the derivation of the linear representation of the IB-G family of distributions.

Lemma I. For any real number $s>0$, consider the generalized binomial series expansion

$$(1+y)^{-s}=\sum_{k=0}^{\infty}\binom{s+k-1}{k}(-1)^k y^k.$$

[See Prudnikov et al., 1986, page 712].

Lemma II. For any real parameter $\alpha > 0$ the convergent series holds

$$[-\log(1-x)]^{\alpha-1} = x^{\alpha-1}\left[\sum_{m=0}^{\infty}\binom{\alpha-1}{m}x^m\left(\sum_{s=0}^{\infty}\frac{x^s}{s+2}\right)^m\right], \quad 0 < x < 1.$$

Applying the result on power series raised to a positive integer, with $a_s = (s+2)$, that is,

$$\left(\sum_{s=0}^{\infty}a_s x^s\right)^m = \sum_{s=0}^{\infty}b_{s,m}x^s$$

so that

$$[-\log(1-x)]^{\alpha-1} = \sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{\alpha-1}{m}b_{s,m}x^{\alpha+m+s-1}$$

where

$$b_{s,m} = (sa_0)^{-1}\sum_{q=0}^{s}(m(q+1)-s)a_q b_{s-q,m}, \qquad \text{and} \qquad b_{0,m} = a_0^m$$

[See Gradsteyn and Ryzhik, 2000].

Now applying the above lemmas to the last expression in equation (5), we have

$$\left[1+\{-\log[1-G(t,\xi)]\}^{-a}\right]^{-(b+1)} = \sum_{j=0}^{\infty}\binom{b+j}{j}(-1)^j\{-\log[1-G(t,\xi)]\}^{-aj}$$

$$[-\log[1-G(t,\xi)]]^{-a((j+1)+1)} = \sum_{m=0}^{\infty}\sum_{s=0}^{\infty}b_{s,m}\binom{-a((j+1)+1)}{m}[G(t,\xi)]^{m+s-a(j+1)-1}$$

$$[1-(1-G(t,\xi))]^{m+s-a(j+1)-1} = \sum_{k=0}^{\infty}\binom{m+s-a(j+1)-1}{k}(-1)^k(1-G(t,\xi))^k$$

$$(1-G(t,\xi))^{k-1} = \sum_{n=0}^{\infty}\binom{k-1}{n}(-1)^n[G(t,\xi)]^n,$$

Substituting the expansions into equation (5),

$$\begin{aligned} f(t,a,b,\xi) = ab \sum_{j,m,s=0}^{\infty} \sum_{k=0}^{m+s-a(j+1)-1} \sum_{n=0}^{k-1} \binom{b+j}{j} \binom{-a((j+1)+1)}{m} \\ \times \binom{m+s-a(j+1)-1}{k} \\ \times \binom{k-1}{n} (-1)^{j+k+n} b_{s,m} g(t,\xi)[G(t,\xi)]^{n}, \\ = \sum_{j,m,s=0}^{\infty} \psi_{k,n} \pi_{(n+1)}(t,a,b,\xi) \end{aligned} \tag{34}$$

where

$$\psi_{k,n} = \frac{ab}{(n+1)} \sum_{k=0}^{m+s-a(j+1)-1} \sum_{n=0}^{k-1} \binom{b+j}{j} \binom{-a((j+1)+1)}{m} \times \binom{m+s-a(j+1)-1}{k} \binom{k-1}{n} (-1)^{j+k+n} b_{s,m}$$

and $\pi_{(n+1)}(t,a,b,\xi) = (n+1)g(t,\xi)[G(t,\xi)]^{(n+1)-1}$.

The density function of IB-G family of distributions defined in equation (34) is expressed as an infinite linear combination of exp-G densities with power parameter $(n+1)$.

Consequently, the cumulative distribution function of the IB-G family of distributions is expressed as a linear combination of exp-G cdfs with power parameter $(n+1)$.

$$F(t,a,b,\xi) = \sum_{j,m,s=0}^{\infty} \psi_{k,n} \Pi_{(n+1)}(t,a,b,\xi), \tag{35}$$

where $\Pi_{(n+1)}(t,a,b,\xi)$ is the exp-G cdf with $(n+1)$ as the power parameter.

4.2 Quantile Function of IB-G family

The quantile function of IB-G family of distributions is obtained as

$$Q_T(u) = G^{-1}\left[1 - e^{-\left(u^{-1/b}-1\right)^{-1/a}}\right], \quad u \in (0,1) \tag{36}$$

where u is a uniformly generated random variable $(0 < u < 1)$ and $G^{-1}(.)$ is the inverse function of $G(.)$. In particular, the median of IB-G family of distributions is obtained by setting $u = 0.5$ as shown in equation (36a).

$$Q_T(0.5) = G^{-1}\left[1 - e^{-\left((0.5)^{-1/b}-1\right)^{-1/a}}\right] \tag{36a}$$

Equation (36) is useful in generating random samples from the IB-G family of distributions for simulation purposes. Let $G^{-1}(.)$ be the inverse function of the Topp-Leone distribution. From equation (36), we generate some quantiles from the inverse IB-TL distribution for some selected parameter values as shown in Table 1.

Table 1: Some quantiles of the IB-TL distribution (a, b, λ).

u	$(4.0, 3.0, 0.3)$	$(2.0, 1.5, 0.5)$	$(3.0, 2.5, 0.7)$	$(5.0, 2.0, 4.0)$
0.05	0.0863	0.0550	0.2281	0.6229
0.25	0.1501	0.1686	0.3389	0.6709
0.29	0.1606	0.1900	0.3552	0.6770
0.34	0.1737	0.2175	0.3752	0.6841
0.39	0.1870	0.2461	0.3950	0.6911
0.44	0.2009	0.2763	0.4151	0.6979
0.49	0.2155	0.3085	0.4357	0.7047
0.54	0.2312	0.3433	0.4573	0.7116
0.59	0.2485	0.3813	0.4803	0.7188
0.64	0.2676	0.4235	0.5051	0.7265

Table 1 shows some quantiles from the IB-TL distribution at different choice of parameter values. We observe that the values are bounded within the unit interval, which agrees with the support of a random variable T following the IB-TL distribution.

4.3 The r^{th} moments and related measures of IB-G family

Let T be a random variable having the density function of the IB-G family, then from equation (34), the r^{th} moments of T is defined by

$$\begin{aligned} E\left(T^r\right) = \mathrm{M}'_r &= \sum_{j,m,s=0}^{\infty} \psi_{k,n} \int_{-\infty}^{\infty} t^r \pi_{(n+1)}(t,a,b,\xi)dt \\ &= \sum_{j,m,s=0}^{\infty} \psi_{k,n} E\left[Y^r_{(n+1)}\right], \quad r = 1,2,3,4,\ldots \end{aligned} \tag{37}$$

where $E\left[Y^r_{(n+1)}\right]$ is the r^{th} moments of the exp-G family with power parameter $(n+1)$.

The mean (M'_1) of the IB-G family is obtained from equation (37) when $r = 1$. The variance $\left(\sigma^2\right)$, skewness (S) and kurtosis (K) are obtained as

$$\text{variance}\left(\sigma^2\right) = \mathrm{M}'_2 - \left(\mathrm{M}'_1\right)^2, \quad \text{skewness } (s) = \frac{\mathrm{M}'_3 - 3\mathrm{M}'_2\mathrm{M}'_1 + 2\left(\mathrm{M}'_1\right)^3}{\left(\mathrm{M}'_2 - \left(\mathrm{M}'_1\right)^2\right)^{\frac{3}{2}}}$$

$$\text{kurtosis}(K) = \frac{\mathrm{M}'_4 - 4\mathrm{M}'_3\mathrm{M}'_1 + 6\mathrm{M}'_3\left(\mathrm{M}'_1\right)^2 - 3\left(\mathrm{M}'_1\right)^4}{\left(\mathrm{M}'_2 - \left(\mathrm{M}'_1\right)^2\right)^2}.$$

Again, allowing the baseline distribution follow the one-parameter Topp-Leone distribution, the numerical values of the mean (M'_1), variance $\left(\sigma^2\right)$, measures of skewness (S) and kurtosis (K) of the IB-TL distribution for some selected parameter values are computed in Table 2.

Table 2: Theoretical moments of the IB-TL distribution for ($\lambda = 2$).

a	b	M'_1	σ^2	S	K
2.0	0.3	0.3465	0.0468	0.4283	2.5822
	0.5	0.4394	0.0502	0.6469	2.8611
	0.8	0.5234	0.0413	0.2789	2.5474
4.0	0.3	0.4131	0.0235	0.0334	2.8390
	0.5	0.4788	0.0177	0.0773	2.9291
	0.8	0.5309	0.0137	0.1692	2.9257
6.0	0.3	0.4473	0.0136	-0.2994	3.0557
	0.5	0.4974	0.0090	-0.0935	2.4544
	0.8	0.5347	0.0066	-0.0996	8.2257
8.0	0.3	0.4678	0.0090	-0.6451	4.5954
	0.5	0.5082	0.0054	0.3382	5.2197
	0.8	0.5372	0.0037	0.3723	0.3477

Observations from the table reveal that IB-TL distribution exhibits a left-skewed, right-skewed, approximately symmetric, platykurtic, leptokurtic as well as mesokurtic properties.

4.4 Moment generating function of IB-G family

The moment generating function of IB-G family is obtained by

$$\begin{aligned} M_T(q) = E\left[e^{qT}\right] &= \sum_{j,m,s=0}^{\infty} \psi_{k,n} \int_{-\infty}^{\infty} e^{qt} \pi_{(n+1)}(t,a,b,\xi)dt \\ &= \sum_{j,m,s,w=0}^{\infty} \psi^*_{k,n} E\left[Y^w_{(n+1)}\right], \quad w = 2,3,4,\ldots \end{aligned} \tag{38}$$

where,

$$\psi^*_{k,n} = \frac{abq^w}{(n+1)w!} \sum_{k=0}^{m+s-a(j+1)-1} \sum_{n=0}^{k-1} \binom{b+j}{j} \binom{-a((j+1)+1)}{m} \times \binom{m+s-a(j+1)-1}{k} \binom{k-1}{n} (-1)^{j+k+n} b_{s,m}$$

and $E\left[Y^w_{(n+1)}\right]$ is the w^{th} moment of the exp-G family with power parameter $(n+1)$.

4.5 Probability Weighted Moments (PWMs) of IB-G family

Let T be a random variable following the pdf and cdf of a known probability distribution. Greenwood et al. (1979) defined the probability weighted moments (PWMs) of random variable T as

$$\rho_{q,w} = E\left[T^w F^q(t)\right] = \int_{-\infty}^{\infty} t^w f(t) F^q(t) dt. \quad (39)$$

By inserting equation (4) and (5) into equation (39), the $(q,w)^{th}$ PWMs of IB-G family is obtained as follows

$$f(t,a,b,\xi)F^q(t,a,b,\xi) = abg(t,\xi)[1-G(t,\xi)]^{-1}\{-\log[1-G(t,\xi)]\}^{-(a+1)} \times \left[1+\{-\log[1-G(t,\xi)]\}^{-a}\right]^{-(b(q+1)+1)} \quad (40)$$

Applying Lemma I and II into equation (40),

$$\left[1+\{-\log[1-G(t,\xi)]\}^{-a}\right]^{-(b(q+1)+1)} = \sum_{j=0}^{\infty} \binom{b(q+1)+j}{j} (-1)^j \{-\log[1-G(t,\xi)]\}^{-aj}$$

$$[-\log[1-G(t,\xi)]]^{-a((j+1)+1)} = \sum_{m=0}^{\infty}\sum_{s=0}^{\infty} b_{s,m} \binom{-a((j+1)+1)}{m} [G(t,\xi)]^{m+s-a(j+1)-1}$$

$$[1-(1-G(t,\xi))]^{m+s-a(j+1)-1} = \sum_{k=0}^{m+s-a(j+1)-1} \binom{m+s-a(j+1)-1}{k} (-1)^k (1-G(t,\xi))^k$$

$$(1-G(t,\xi))^{k-1} = \sum_{n=0}^{k-1} \binom{k-1}{n} (-1)^n [G(t,\xi)]^n$$

Substituting these expansions into equation (40),

$$\begin{aligned}
f(t,a,b,\xi)F^{q}(t,a,b,\xi) &= ab\sum_{j,m,s=0}^{\infty}\sum_{k=0}^{m+s-a(j+1)-1}\sum_{n=0}^{k-1}\binom{b(q+1)+j}{j} \\
&\times\binom{-a((j+1)+1)}{m} \\
&\times\binom{m+s-a(j+1)-1}{k}\binom{k-1}{n}(-1)^{j+k+n}b_{s,m}g(t,\xi)[G(t,\xi)]^{n}, \\
&= \sum_{j,m,s=0}^{\infty}\psi^{**}_{k,n}\pi_{(n+1)}(t,a,b,\xi).
\end{aligned} \tag{41}$$

where

$$\begin{aligned}
\psi^{**}_{k,n} =& \frac{ab}{(n+1)}\sum_{k=0}^{m+s-a(j+1)-1}\sum_{n=0}^{k-1}\binom{b(q+1)+j}{j}\binom{-a((j+1)+1)}{m} \\
&\times\binom{m+s-a(j+1)-1}{k}\binom{k-1}{n}(-1)^{j+k+n}b_{s,m}.
\end{aligned}$$

Hence, the $(q,w)^{th}$ PWMs of IB-G family is obtained as

$$\begin{aligned}
\rho_{q,w} = E\left[T^{w}F^{q}(t)\right] &= \sum_{j,m,s=0}^{\infty}\psi^{*}_{k,n}\int_{-\infty}^{\infty}t^{w}\pi_{(n+1)}(t,a,b,\xi)dt, \\
&= \sum_{j,m,s=0}^{\infty}\psi^{*}_{k,n}E\left[Y^{w}_{(n+1)}\right].
\end{aligned} \tag{42}$$

4.6 Renyi entropy of IB-G family

Renyi (1961) defined the entropy of a random variable T following a known probability distribution with pdf, $f(t)$ as

$$\tau_R(\omega) = \frac{1}{1-\omega}\log\int_{-\infty}^{\infty}f^{\omega}(t)dt, \quad \omega>0, \omega\neq 1. \tag{43}$$

By inserting equation (5) into equation (43), the Renyi entropy of IB-G family is defined as follows:

$$\begin{aligned}\tau_R(\omega) =& \frac{1}{1-\omega}\log\left[\left[\frac{\log\alpha}{\alpha-1}\right]^{\omega}(ab)^{\omega}\int_{-\infty}^{\infty} g^{\omega}(t,\xi)[1-G(t,\xi)]^{-\omega}\right.\\ &\left.\times\{-\log[1-G(t,\xi)]\}^{-\omega(a+1)}\left[1+\{-\log[1-G(t,\xi)]\}^{-\alpha}\right]^{-\omega(b+1)}dt\right].\end{aligned} \tag{44}$$

Applying Lemma I and II in equation (44), yields the following expansions

$$\left[1+\{-\log[1-G(t,\xi)]\}^{-a}\right]^{-\omega(b+1)} = \sum_{j=0}^{\infty}\begin{pmatrix}\omega(b+1)+j-1\\ j\end{pmatrix}(-1)^j\{-\log[1-G(t,\xi)]\}^{-aj}$$

$$[-\log[1-G(t,\xi)]]^{-a(\omega+j)-\omega} = \sum_{m=0}^{\infty}\sum_{s=0}^{\infty} b_{s,m}\begin{pmatrix}-a(\omega+j)-\omega\\ m\end{pmatrix}[G(t,\xi)]^{m+s-a(\omega+j)-\omega}$$

$$[1-(1-G(t,\xi))]^{m+s-a(\omega+j)-\omega} = \sum_{k=0}^{m+s-a(\omega+j)-\omega}\begin{pmatrix}m+s-a(\omega+j)-\omega\\ k\end{pmatrix}(-1)^k(1-G(t,\xi))^k$$

$$(1-G(t,\xi))^{k-\omega} = \sum_{n=0}^{k-\omega}\begin{pmatrix}k-\omega\\ n\end{pmatrix}(-1)^n[G(t,\xi)]^n.$$

Substituting these expansions into equation (44), the Renyi entropy of IB-G family is obtained as

$$\tau_R(\omega) = \frac{1}{1-\omega}\log\left[\left[\frac{\log\alpha}{\alpha-1}\right]^{\omega}(ab)^{\omega}\sum_{j,m,s=0}^{\infty}\psi_{k,n}^{*m}\int_{-\infty}^{\infty} g^{\omega}(t,\xi)G^n(t,\xi)dt\right] \tag{45}$$

where,

$$\begin{aligned}\psi_{k,n}^{***k} = \textstyle\sum_{k=0}^{m+s-a(\omega+j)-\omega_k-\omega}\sum_{n=0}&\begin{pmatrix}\boldsymbol{\omega}(b+1)+j-1\\ j\end{pmatrix}\begin{pmatrix}-a(\omega+j)-\boldsymbol{\omega}\\ m\end{pmatrix}\\ &\begin{pmatrix}m+s-a(\omega+j)-\boldsymbol{\omega}\\ k\end{pmatrix}\begin{pmatrix}k-\omega\\ n\end{pmatrix}(-1)^{j+k+n}b_{s,m}.\end{aligned}$$

4.7 Distribution of order Statistics of IB-G family

Suppose that $T_1, T_2, \ldots, T_l$ is a random sample from IB-G family of distributions. Let $T_{r:l}$ denote the r^{th} order statistic, then the density function of $T_{r:n}$ is defined as

$$f_{r:l}(t,a,b,\xi)=\frac{1}{B(r,l-r+1)}\sum_{p=0}^{l-r}\binom{l-r}{p}(-1)^p f(t,a,b,\xi)F^{r+p-1}(t,a,b,\xi). \tag{46}$$

By inserting equations (4) and (5) into equation (46), the density function of IB-G r^{th} order statistics is defined as follows.

$$f(t,a,b,\xi)F^{r+p-1}(t,a,b,\xi)=abg(t,\xi)[1-G(t,\xi)]^{-1}\{-\log[1-G(t,\xi)]\}^{-(a+1)} \times\left[1+\{-\log[1-G(t,\xi)]\}^{-a}\right]^{-(b(r+p)+1)}. \tag{47}$$

Applying Lemma I and II into equation (47),

$$\left[1+\{-\log[1-G(t,\xi)]\}^{-a}\right]^{-(b(r+p)+1)}=\sum_{j=0}^{\infty}\binom{b(r+p)+j}{j}(-1)^j\{-\log[1-G(t,\xi)]\}^{-aj},$$

$$[-\log[1-G(t,\xi)]]^{-a((j+1)+1)}=\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}b_{s,m}\binom{-a((j+1)+1)}{m}[G(t,\xi)]^{m+s-a(j+1)-1}$$

$$[1-(1-G(t,\xi))]^{m+s-a(j+1)-1}=\sum_{k=0}^{m+s-a(j+1)-1}\binom{m+s-a(j+1)-1}{k}(-1)^k(1-G(t,\xi))^k$$

$$(1-G(t,\xi))^{k-1}=\sum_{n=0}^{k-1}\binom{k-1}{n}(-1)^n[G(t,\xi)]^n.$$

Substituting these expansions into equation (47),

$$\begin{aligned}&f(t,a,b,\xi)F^{r+p-1}(t,a,b,\xi)\\&=ab\sum_{j,m,s=0}^{\infty}\sum_{k=0}^{m+s-a(j+1)-1}\sum_{l=0}^{k-1}\binom{b(r+p)+j}{j}\binom{-a((j+1)+1)}{m}\\&\times\binom{m+s-a(j+1)-1}{k}\binom{k-1}{n}(-1)^{j+k+n}b_{s,m}g(t,\xi)[G(t,\xi)]^n.\end{aligned} \tag{48}$$

Applying equation (48) in equation (46), we have

$$f_{r:l}(t,a,b,\xi)=\frac{1}{B(r,l-r+1)}\sum_{j,m,s=0}^{\infty}\psi^{****}_{l,k,n}\pi_{(n+1)}(t,a,b,\xi) \tag{49}$$

where,

$$\Psi^{*+*k}_{l,k,n} = \frac{ab}{(n+1)} \sum_{l=0}^{l-1} \sum_{k=0}^{m+-a(j+1)-1} \sum_{n=0}^{k-1} \binom{l-r}{l} \binom{b(r+p)+j}{j}$$

$$\times \binom{-a((j+1)+1)}{m} \binom{m+s-a(j+1)-1}{k} \binom{k-1}{n} (-1)^{p+j+k+n} b_{s,m}.$$

The w^{th} moment of IB-G r^{th} order statistic can be expressed from equation (49) as

$$E\left(T_r^w\right) = \frac{1}{B(r,l-r+1)} \sum_{j,m,s=0}^{\infty} \psi^{****}_{l,k,n} E\left[Y^w_{(n+1)}\right] \tag{50}$$

where $E\left[Y^w_{(n+1)}\right]$ is the w^{th} moment of exp-G family with power parameter $(n+1)$.

4.8 Parameter estimation of IB-G family

4.8.1 Maximum likelihood estimation

Suppose $(t_1, t_2, \ldots, t_n)$ are random samples generated from the IB-G family of distributions. The likelihood function of T is given as

$$\begin{aligned} L(t,\varpi) = \prod_{i=1}^{n} \Big[& abg\left(t_i,\xi\right)\left[1-G\left(t_i,\xi\right)\right]^{-1}\left\{-\log\left[1-G\left(t_i,\xi\right)\right]\right\}^{-(a+1)} \\ & \times \left[1+\left\{-\log\left[1-G\left(t_i,\xi\right)\right]\right\}^{-a}\right]^{-(b+1)}\Big], \varpi = (a,b,\xi)^T. \end{aligned} \tag{51}$$

Taking the natural logarithm of equation (51), yields

$$\begin{aligned} \ell(t,\varpi) = & n\ln(ab) + \sum_{i=1}^{n}\ln(g(t,\xi)) - \sum_{i=1}^{n}\ln(1-G(t,\xi)) \\ & -(a+1)\sum_{i=1}^{n}\ln(-\ln(1-G(t,\xi))) \\ & -(b+1)\sum_{i=1}^{n}\ln\left[1+(-\ln(1-G(t,\xi)))^{-a}\right]. \end{aligned} \tag{52}$$

The associated gradients are obtained by differentiating the log-likelihood function in equation (52) with respect to the parameters.

$$\begin{aligned}
\frac{\ell(t,\varpi)}{\partial a} &= \frac{n}{a} - \sum_{i=1}^{n} \ln\left(-\ln\left(1-G\left(t_i,\xi\right)\right)\right) \\
&+ (b+1)\sum_{i=1}^{n} \frac{\left(-\ln\left(1-G\left(t_i,\xi\right)\right)\right)^{-a} \ln\left[-\ln\left(1-G\left(t_i,\xi\right)\right)\right]}{\left[1+\left(-\ln\left(1-G\left(t_i,\xi\right)\right)\right)^{-a}\right]}, \\
\frac{\ell(t,\varpi)}{\partial b} &= \frac{n}{b} - \sum_{i=1}^{n} \ln\left[1+\left(-\ln\left(1-G\left(t_i,\xi\right)\right)\right)^{-a}\right], \\
\frac{\ell(t,\varpi)}{\partial \xi} &= \sum_{i=1}^{n} \frac{g'(t,\xi)}{g(t,\xi)} + \sum_{i=1}^{n} \frac{g'\left(t_i,\xi\right)}{\left(1-G\left(t_i,\xi\right)\right)} \\
&- (a+1)\sum_{i=1}^{n} \frac{g(b+1)\sum_{i=1}^{n} \frac{g(t_i,\xi)}{(1-G(t_i,\xi))(-\ln(1-G(t_i,\xi)))}\left[-\ln\left(1-G\left(t_i,\xi\right)\right)\right)^{-(a+1)}}{\left[1-G\left(t_i,\xi\right)\right)\right)^{-a}\right]\left(1-G\left(t_i,\xi\right)\right)}
\end{aligned} \tag{53}$$

where $g'\left(t_i,\xi\right) = \frac{\partial g(t_i,\xi)}{\partial \xi_i}$ and $\partial\xi_i$ is the i^{th} element of the vector of parameter ξ.

The maximum likelihood estimates (MLEs) of ϖ say $\hat{\varpi} = (\hat{a}, \hat{b}, \hat{\xi})^T$, is obtained from the solution of the score function $U\left(t_i,\varpi\right) = \left[\frac{\ell(t,\varpi)}{\partial a}, \frac{\ell(t,\varpi)}{\partial b}, \frac{\ell(t,\varpi)}{\partial \xi}\right]^T = 0$. These numerical solutions can be obtained using Statistical packages such as fitdisrtplus and optim in R program.

4.8.2 Simulation study

Again, taking the Topp-Leone distribution as the generator, the study investigates the performance of the parameter estimates of the IBTL distribution via Monte Carlo simulation study. Random samples of size $n = (25, 50, 100, 200, 500)$ are generated from the IBTL distribution at three distinct sets of parameter values $(a = 2.0, b = 0.4, \lambda = 0.5)$, $(a = 2.5, b = 0.3, \lambda = 0.7)$ and $(a = 3.0, b = 0.2, \lambda = 2.0)$. At each case, the simulation is repeated 1000 times and the following quantities are computed:

i) mean estimate $(\bar{\psi}) = \frac{1}{N}\sum_{i=1}^{N} \hat{\psi}_i$,

ii) average bias $= \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\psi}_i - \bar{\psi}\right)$,

iii) root mean square error (RMSE) $= \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\psi}_i - \bar{\psi}\right)^2}$,

iv) Coverage Probability of the 95% confidence interval of the estimates $\hat{\psi}_i$ given by

$$CP(\hat{\psi}) = \frac{1}{N}\sum_{i=1}^{N} I\left(\hat{\psi}_i - Z_{\delta/2}\sqrt{\text{var}(\hat{\varphi})} < \psi_0 < \hat{\psi}_i + Z_{\delta/2}\sqrt{\text{var}(\hat{\varphi})}\right),$$

where $I(.)$ is an indicator function and $(\hat{\psi})$ is the standard error of the estimate ψ_i.

Tables 3 - 6 display the mean estimate, average bias, root mean square error and coverage probability of the 95% confidence interval of the parameter estimates of the IBTL distribution.

Table 3: Simulation results for the mean estimates of the parameters of the IBTL distribution.

Parameters	n	Mean(a)	Mean(b)	Mean(λ)
	25	2.6236	1.2405	0.4995
$a = 2.0$	50	2.3530	0.6628	0.5271
$b = 0.4$	100	2.1743	0.5744	0.5386
$\lambda = 0.5$	200	2.0890	0.4898	0.5317
	500	2.0350	0.4114	0.5279
	25	3.4615	0.6818	0.7209
$a = 2.5$	50	2.9798	0.4156	0.7458
$b = 0.3$	100	2.7051	0.3524	0.7459
$\lambda = 0.7$	200	2.5948	0.3112	0.7384
	500	2.5227	0.3091	0.7082
	25	3.9173	0.3282	1.8407
$a = 3.0$	50	3.7643	0.2670	2.0193
$b = 0.2$	100	3.3766	0.2135	2.0686
$\lambda = 2.0$	200	3.1830	0.2075	2.0547
	500	3.0535	0.2025	2.0185

Table 4: Simulation results for the Bias of the parameters of the IBTL distribution.

Parameters	n	Bias(a)	Bias(b)	Bias(λ)
	25	0.6236	0.8405	-0.0005
$a = 2.0$	50	0.3530	0.2628	0.0271
$b = 0.4$	100	0.1743	0.1744	0.0386
$\lambda = 0.5$	200	0.0890	0.0898	0.0316
	500	0.0350	0.0114	0.0279
	25	0.9615	0.3815	0.0209
$a = 2.5$	50	0.4798	0.1156	0.0458
$b = 0.3$	100	0.2051	0.0524	0.0459
$\lambda = 0.7$	200	0.0948	0.0112	0.0384
	500	0.0227	0.0091	0.0082
	25	0.9173	0.1282	-0.1592
$a = 3.0$	50	0.7643	0.0670	0.0193
$b = 0.2$	100	0.3766	0.0135	0.0686
$\lambda = 2.0$	200	0.1830	0.0075	0.0548
	500	0.0535	0.0025	0.0185

Table 5: Simulation results for the RMSE of the parameters of the IBTL distribution.

Parameters	n	RMSE(a)	RMSE(b)	RMSE(λ)
	25	1.1339	2.6137	0.3869
$a = 2.0$	50	0.7507	0.8872	0.3596
$b = 0.4$	100	0.3720	0.7533	0.3422
$\lambda = 0.5$	200	0.2447	0.3679	0.2626
	500	0.1437	0.1285	0.1689
	25	2.0986	1.9290	0.4556
$a = 2.5$	50	1.2833	0.4496	0.3948
$b = 0.3$	100	0.6162	0.2365	0.3367
$\lambda = 0.7$	200	0.3252	0.1188	0.2355
	500	0.1701	0.0718	0.1375
	25	2.0161	0.4124	0.9454
$a = 3.0$	50	1.1198	0.3365	0.8416
$b = 0.2$	100	0.9885	0.1135	0.6522
$\lambda = 2.0$	200	0.5888	0.0833	0.5267
	500	0.2830	0.0419	0.3007

Table 6: Simulation results of the CP of 95% CI of the parameters of the IBTL distribution.

Parameters	n	CP(a)	CP(b)	CP(λ)
	25	1.1339	0.912	0.826
$a = 2.0$	50	0.7507	0.904	0.842
$b = 0.4$	100	0.3720	0.920	0.866
$\lambda = 0.5$	200	0.2447	0.938	0.888
	500	0.1437	0.920	0.912
	25	2.0986	0.866	0.862
$a = 2.5$	50	1.2833	0.876	0.872
$b = 0.3$	100	0.6162	0.896	0.906
$\lambda = 0.7$	200	0.3252	0.924	0.926
	500	0.1701	0.940	0.934
	25	2.0161	0.934	0.828
$a = 3.0$	50	1.1198	0.888	0.888
$b = 0.2$	100	0.9885	0.886	0.926
$\lambda = 2.0$	200	0.5888	0.906	0.926
	500	0.2830	0.946	0.958

Simulation results from Tables 3 − 6 are discussed as follows:

i) The mean estimates in Table 3 approaches the true parameter value as the sample size n increases;

ii) Table 4 shows that the parameter estimate $\hat{a}$ and $\hat{b}$ are positively biased while $\hat{\lambda}$ is both negatively and positively biased. Furthermore, the bias of $\hat{a}$ and $\hat{b}$ decrease as the sample size n increases;

iii) From Table 5, the root mean square error of the parameter estimates $\hat{a}, \hat{b}$ and $\hat{\lambda}$ decrease as the sample size n increases;

iv) Finally, Table 6 shows that the coverage probability of the 95% confidence interval of the estimates are very close to the nominal level of 95%.

These properties are what one should expect from a good estimator.

5 Data Analysis

This section is devoted to illustrate the applicability of the proposed family of distributions in lifetime data fittings. In order to achieve this, two real data sets are employed and the fits of the sub-model from the proposed family of distributions together with the fits attained by some non-nested distributions are compared. The non-nested distributions with their density function are defined as follows:

1. Marshall-Olkin extended Kumaraswamy distribution (MOEKD) introduced by George and Thobias (2017);

$$f(t,a,b,\alpha)=\frac{\alpha abt^{a-1}\left(1-t^{a}\right)^{b-1}}{\left[1-\alpha\left(1-t^{a}\right)^{b}\right]^{2}}.$$

2. Unit-Burr XII distribution (UBXIID) developed by Korkmaz and Chesneau (2021);

$$f(t,\alpha,\beta)=\alpha\beta t^{-1}(-\log t)^{\beta-1}\left(1+(-\log t)^{\beta}\right)^{-(\alpha+1)}.$$

3. Unit-Burr III distribution (UBIIID) proposed by Modi and Gill (2020);

$$f(t,\lambda,\beta)=\lambda\beta t^{-2}\left(t^{-1}-1\right)^{\beta-1}\left(1+\left(t^{-1}-1\right)^{\beta}\right)^{-(\lambda+1)}.$$

4. Unit-Weibull distribution (UWD) proposed by Mazucheli et al. (2019);

$$f(t,\alpha,\beta) = \frac{1}{t}\alpha\beta(-\log t)^{\beta-1}\exp\left[-\alpha(-\log t)^{\beta}\right].$$

5. Unit-Gompertz distribution (UGD) proposed by Mazucheli et al. (2019);

$$f(t,a,b) = abt^{-(a+1)}e^{-b\left(t^{-a}-1\right)}.$$

6. Log-weighted exponential distribution (LWED) proposed by Altun (2019);

$$f(t,\alpha,\beta) = \frac{\alpha+1}{\alpha}\beta\exp(-\beta t)\left(1-e^{-\alpha\beta t}\right).$$

7. Alpha power Topp-Leone distribution (APTLD) proposed by Ehiwario et al. (2023);

$$f(t,\alpha,\lambda) = \frac{\log\alpha}{\alpha-1}2\lambda(1-t)\left[1-(1-t)^2\right]^{\lambda-1}\alpha^{\left[1-(1-)^2\right]^{\lambda}}.$$

8. Marshall-Olkin extended Topp-Leone distribution (MOETLD) due to Opone and Iwerumor (2021);

$$f(t,\alpha,\lambda) = \frac{2\alpha\lambda(1-t)\left[1-(1-t)^2\right]^{\lambda-1}}{\left[1-\bar{\alpha}\left\{1-(1-(1-t)^2)^{\lambda}\right\}\right]^2}.$$

9. Power continuous Bernoulli distribution proposed by Chesneau and Opone (2022);

$$f(t,\alpha,\lambda) = \frac{\lambda^{\alpha^{\alpha}}(1-\lambda)^{1t^{\alpha}}+\lambda-1}{2\lambda-1}.$$

10. Beta distribution reported in Opone and Ekhosuehi (2017);

$$f(t,a,b) = \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}, \quad B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

11. Kumaraswamy distribution (KwD) developed by Kumaraswamy (1980);

$$f(t, a, b) = abt^{a-1} (1 - t^{a})^{b-1}.$$

Data set 1: The first data set consists of trade share data from Bantan et al. (2021). The trade share data are as follows:
0.140501976, 0.156622976, 0.157703221, 0.160405084, 0.160815045, 0.22145839, 0.299405932, 0.31307286, 0.324612707, 0.324745566, 0.329479247, 0.330021679, 0.337879002, 0.339706242, 0.352317631, 0.358856708, 0.393250912, 0.41760394, 0.425837249, 0.43557933, 0.442142904, 0.444374621, 0.450546652, 0.4557693, 0.46834656, 0.473254889, 0.484600782, 0.488949597, 0.509590268, 0.517664552, 0.527773321, 0.534684658, 0.543337107, 0.544243515, 0.550812602, 0.552722335, 0.56064254, 0.56074965, 0.567130983, 0.575274825, 0.582814276, 0.603035331, 0.605031252, 0.613616884, 0.626079738, 0.639484167, 0.646913528, 0.651203632, 0.681555152, 0.699432909, 0.704819918, 0.729232311, 0.742971599, 0.745497823, 0.779847085, 0.798375845, 0.814710021, 0.822956383, 0.830238342, 0.834204197, and 0.979355395.

Details of this data set can be accessed in Stock and Watson (2007).

Data set 2: The second data set reported in Nigm et al. (2003) is concerned with ordered failure of components. The data are presented as follows:
0.0009, 0.004, 0.0142, 0.0221, 0.0261, 0.0418, 0.0473, 0.0834, 0.1091, 0.1252, 0.1404, 0.1498, 0.175, 0.2031, 0.2099, 0.2168, 0.2918, 0.3465, 0.4035, 0.6143.

Figures 1 and 2 present the boxplot and histogram of the two data sets, respectively. From the boxplot in Figure 1, we observe that there is no presence of outliers, whereas, Figure 2 shows the presence of outlier in the data set.

To obtain the appropriate model for analyzing the two data sets, we considered some model selection criteria such as the maximized log-likelihood (LogL), Akaike Information Criteria (AIC), and some goodness of fit test statistics such as the Komolgorov-Smirnov (K-S), Crammer von Mises (W^*) and Anderson Darling (A^*) test statistics.

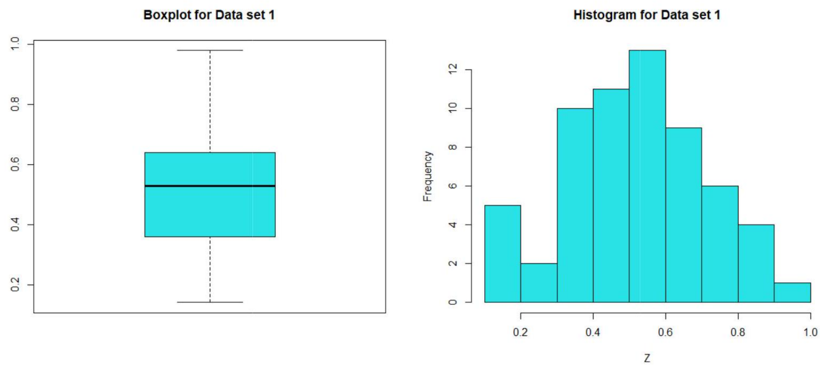

Figure 1: The Boxplot and Histogram of Data set 1.

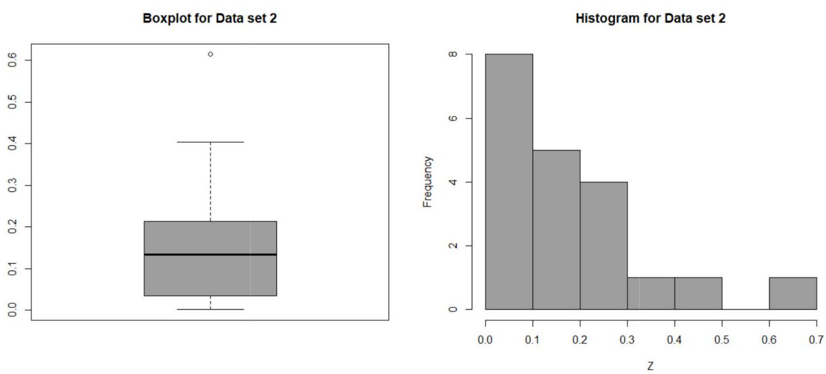

Figure 2: The Boxplot and Histogram of Data set 2.

Tables 7 and 8 present the summary results of the fit of the distributions for the two data sets, respectively.

Table 7: Summary Statistics for Data set 1.

Models	Parameter estimates	log L	AIC	$K-S$ (*p-value*)	W^* (*p-value*)	A^* (*p-value*)
IB-TL	$a = 2.1881$	15.2310	-25.4621	0.0528	0.0257	0.2545
	$b = 0.3003$			(**0.9924**)	(**0.9888**)	(**0.9679**)
	$\lambda = 4.4174$					
MOEK	$a = 3.0578$	14.3183	-22.6366	0.0582	0.0439	0.4014
	$b = 1.9514$			(**0.9784**)	(**0.9137**)	(**0.8568**)
	$\alpha = 0.3015$					
UBXII	$\alpha = 2.1247$	14.1186	-24.2371	0.0538	0.0314	0.2858
	$\beta = 2.2237$			(**0.9904**)	(**0.9724**)	(**0.9482**)
UBIII	$\lambda = 1.0984$	14.6571	-25.3142	0.0500	0.0291	0.2742
	$\beta = 1.8704$			(**0.9961**)	(**0.9799**)	(**0.9560**)
UW	$\alpha = 1.3396$	14.2436	-24.4872	0.0615	0.0617	0.5034
	$\beta = 1.7346$			(**0.9210**)	(**0.8049**)	(**0.7427**)
UG	$a = 0.6162$	10.8759	-17.7518	0.1098	0.0276	1.4468
	$b = 1.0921$			(**0.4235**)	(**0.2535**)	(**0.1897**)
LWE	$\alpha = -0.00003$	13.0830	-22.1660	0.1025	0.1356	0.8576
	$\lambda = 2.6578$			(**0.5108**)	(**0.4376**)	(**0.4408**)
APTL	$\alpha = 0.4158$	14.4209	-24.8417	0.0575	0.0416	0.3765
	$\lambda = 3.3733$			(**0.9808**)	(**0.9260**)	(**0.8713**)
MOETL	$\alpha = 0.6630$	14.3606	-24.7211	0.0568	0.0447	0.3925
	$\lambda = 3.3521$			(**0.9831**)	(**0.9092**)	(**0.8557**)
PCB	$\alpha = 2.8491$	15.1002	-25.2005	0.0565	0.0305	0.2964
	$\lambda = 0.0094$			(**0.9837**)	(**0.9754**)	(**0.9406**)
Beta	$a = 2.7940$	13.9561	-23.9121	0.0618	0.0470	0.3760
	$b = 2.6038$			(**0.9629**)	(**0.8958**)	(**0.8717**)
Kum	$a = 2.3301$	13.6251	-23.2503	0.0690	0.0558	0.4115
	$b = 2.7638$			(**0.9143**)	(**0.8423**)	(**0.8367**)

Table 8: Summary Statistics for Data set 2.

Models	Parameter estimates	log L	AIC	$K-S$ (*p-value*)	W^* (*p-value*)	A^* (*p-value*)
IB-TL	$a = 3.3104$	17.3080	-30.6159	0.0846	0.0175	0.1139
	$b = 0.2589$			(**0.9963**)	(**0.9992**)	(**0.9999**)
	$\lambda = 0.6771$					
MOEK	$a = 0.6225$	17.3089	-28.6179	0.0878	0.0206	0.1320
	$b = 3.7649$			(**0.9941**)	(**0.9972**)	(**0.9996**)
	$\alpha = 2.1331$					
UBXII	$\alpha = 0.2783$	14.3451	-24.6902	0.2274	0.1781	0.9250
	$\beta = 4.3784$			(**0.2163**)	(**0.3159**)	(**0.3978**)
UBIII	$\lambda = 0.1309$	15.4957	-26.9913	0.2159	0.1486	0.7819
	$\beta = 3.1866$			(**0.2678**)	(**0.3965**)	(**0.4925**)
UW	$\alpha = 0.1598$	16.4575	-28.9150	0.1319	0.0531	0.3117
	$\beta = 1.7269$			(**0.8335**)	(**0.8625**)	(**0.9284**)
UG	$a = 0.7741$	14.7625	-25.5251	0.1494	0.0996	0.6509
	$b = 0.2782$			(**0.7093**)	(**0.5911**)	(**0.5991**)
LWE	$\alpha = 0.0003$	16.4330	-28.8659	0.1351	0.0521	0.3371
	$\lambda = 0.7807$			(**0.8120**)	(**0.8689**)	(**0.9069**)
APTL	$\alpha = 0.0957$	16.8009	-29.6017	0.1158	0.0382	0.2260
	$\lambda = 0.8350$			(**0.9236**)	(**0.9463**)	(**0.9817**)
MOETL	$\alpha = 0.3520$	16.5625	-29.1249	0.1096	0.0407	0.2605
	$\lambda = 0.8346$			(**0.9485**)	(**0.9342**)	(**0.9643**)
PCB	$\alpha = 0.8734$	16.7779	-29.5558	0.1177	0.0408	0.2403
	$\lambda = 0.0067$			(**0.9145**)	(**0.9336**)	(**0.9752**)
Beta	$a = 0.7134$	17.2532	-30.5067	0.0981	0.0261	0.1542
	$b = 3.7453$			(**0.9803**)	(**0.9891**)	(**0.9984**)
Kum	$a = 0.7640$	17.2047	-30.4095	0.1027	0.0286	0.1678
	$b = 3.4347$			(**0.9699**)	(**0.9829**)	(**0.9970**)

An appropriate model suitable for analyzing a real data set corresponds to the model with the maximized log-likelihood value, least value in terms of the AIC, K-S, W^* and A^* test statistics with the corresponding highest p-value. Clearly, from Tables 7 and 8, we observe that the inverse Burr Topp Leone (IBTL) distribution belonging to the proposed family of distributions satisfies the conditions and thus, outperforms the competitor distributions in analyzing the two data sets under study. The density and cumulative distribution fits as well as the probability-probability (p-p) plots of the distributions for the two data sets are examined in Figures 3 − 6, to further support the flexibility of the ITBL distribution.

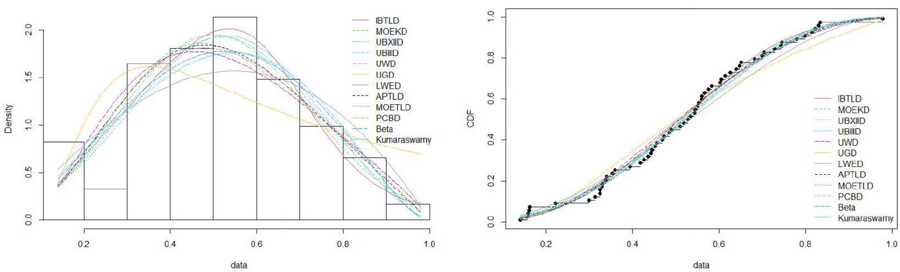

Figure 3: The fitted pdf and cdf of the distributions for Data set 1.

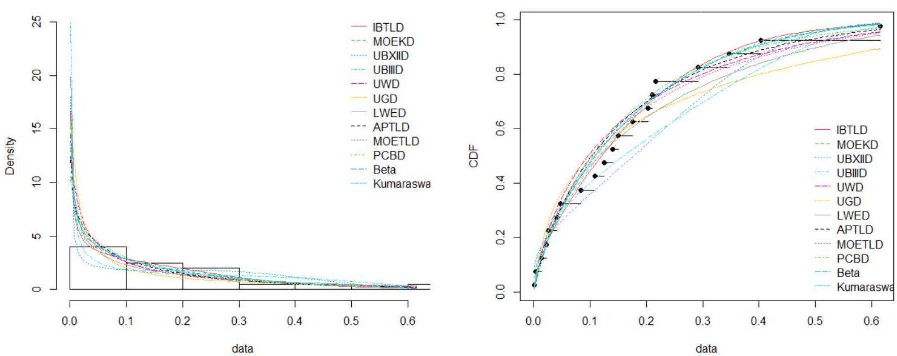

Figure 4: The fitted pdf and cdf of the distributions for Data set 2.

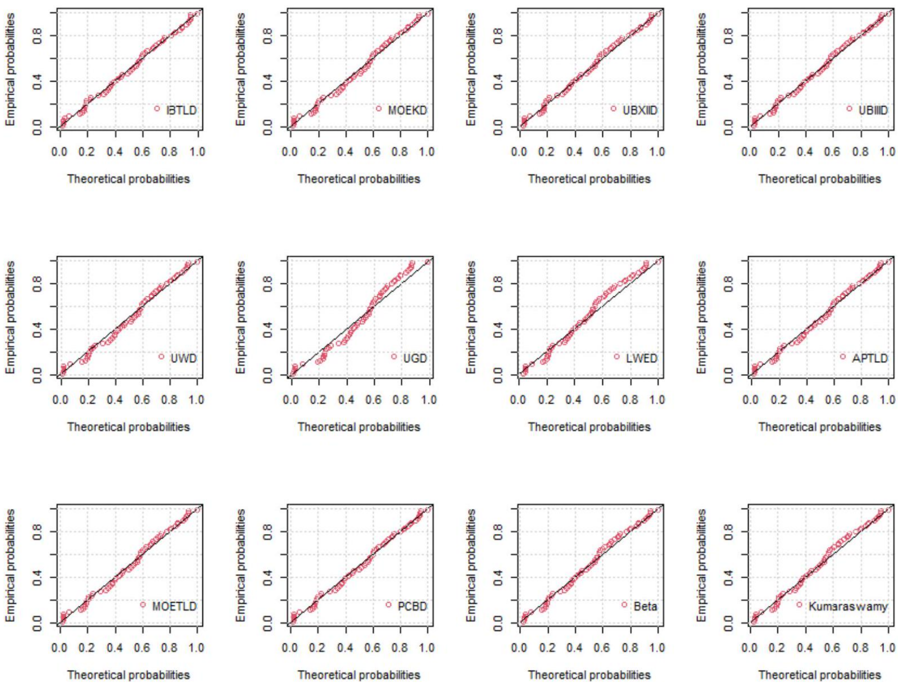

Figure 5: The probability-probability (p-p) plots of the distributions for Data set 1.

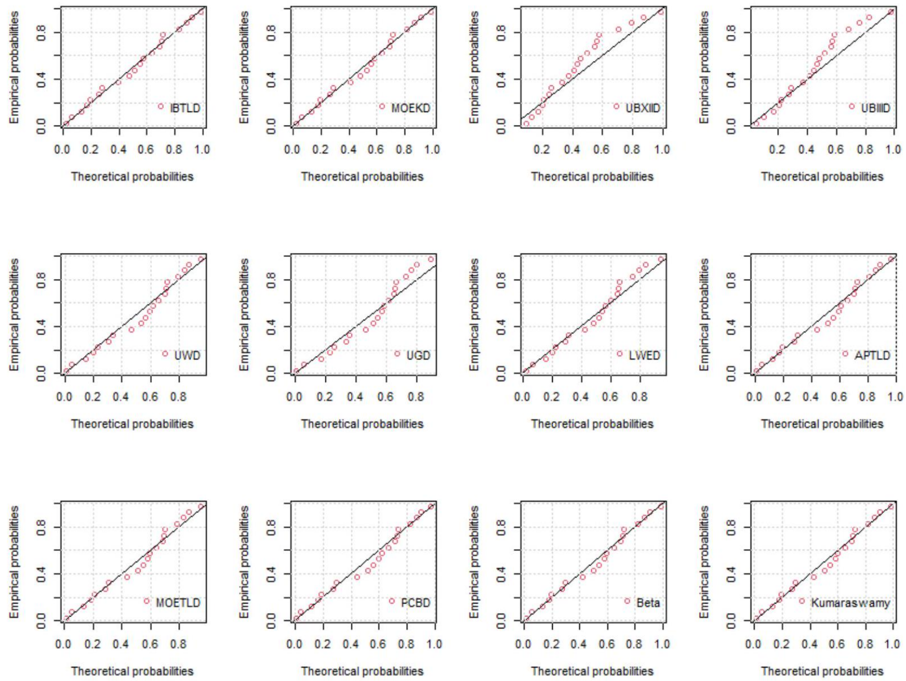

Figure 6: The probability-probability (p-p) plots of the distributions for Data set 2.

6 Conclusion

In this paper, we have introduced a new class of generalized distributions based on the inverse Burr distribution. Basic Statistical properties of the proposed family of distributions such as the density and cumulative distribution functions, survival and hazard rate functions, quantile, moments, moment generating function, probability weighted moments, Renyi entropy and distribution of order statistics were derived. The parameter estimates of the family of distributions were derived via the maximum likelihood estimation method. The performance of the parameter estimates of sub-model from the proposed family of distributions were examined through a Monte Carlo simulation study. Finally, we illustrate the utility of the proposed family of distributions in lifetime data fittings using two real data sets and the results obtained were compared with some existing non-nested models. Based on some model selection criteria and goodness of fit test statistics, it was evident that the IBTL distribution belonging to the proposed family of distributions performed reasonably better than the competitor distributions in fitting the two data sets under study.

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