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Differential Sandwich Theorems for Mittag-Leffler Function Associated with Fractional Integral Defined by Convolution Structure

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Abstract

In this work, we use fractional integral and Mittag-Leffler function to obtain some results related to differential subordination and superordination defined by Hadamard product for univalent analytic functions defined in the open unit disk. These results are applied to obtain differential sandwich results. Our results extend corresponding previously known results.

1. Introduction and Preliminaries

Denote by $\mathcal{H}$ the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For a positive integer n and $a \in \mathbb{C}$, assume that $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions that have the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots . \tag{1.1}$$

Also, let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.2}$$

For the functions $f \in \mathcal{A}$ given by (1.2) and $g \in \mathcal{A}$ defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z).$$

Let $f, g \in \mathcal{A}$. The function f is said to be subordinate to g, or g is said to be superordinate to f, if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$. In such a case we write $f \prec g$ or $f(z) \prec g(z) (z \in U)$. Furthermore, if g is univalent in U, then we have the following equivalent (see [8]), $f \prec g \Leftrightarrow f(0) = g(0)$ and $f(U) \subset g(U)$.

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Let $p, h \in \mathcal{H}$ and $\psi(r, s, t; z): \mathbb{C}^3 \times U \longrightarrow \mathbb{C}$. If p and $\psi(p(z), zp'(z), z^2p''(z); z)$ are univalent functions in U and if p satisfies the second-order differential superordination:

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \tag{1.3}$$

then p is called a solution of the differential superordination (1.3). (If f is subordinate to g, then g is superordinate to f). An analytic function q is called a subordinant of (1.3), if $q \prec p$ for all the functions p satisfying (1.3). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all the subordinants q of (1.3) is called the best subordinant.

The Hadamard product $f * g$ of f and g is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_n b_n z^n = (g * f)(z).$$

The Mittag-Leffler function $E_\alpha(z), (z \in \mathbb{C})$ (see [10,11]) is defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (\alpha \in \mathbb{C}, Re(\alpha) > 0).$$

Several researchers have investigated properties of Mittag-Leffer function and generalized Mittag-Leffler function, see for example [1,6,7,9,15]. Moreover, Srivastava and Tomovski [11] introduced the function $E_{\alpha,\beta}^{\xi,\omega}(z), (z \in \mathbb{C})$ in the form:

$$E_{\alpha,\beta}^{\xi,\omega}(z) = \sum_{n=0}^{\infty} \frac{(\xi)_{n\omega} z^n}{\Gamma(\alpha n + \beta) n!},$$

where $\alpha, \beta, \xi \in \mathbb{C}, Re(\alpha) > \max\{0, Re(\omega) - 1\}, Re(\omega) > 0$ and $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n = \frac{\Gamma(x + n)}{\Gamma(x)} = \begin{cases} 1 & (n = 0), \\ x(x+1)\ldots(x+n-1) & (n \in \mathbb{N}). \end{cases}$$

Definition 1.1 [3]**.** For $f \in \mathcal{A}$ the operator $\mathcal{H}_{\alpha,\beta}^{\xi,\omega} : \mathcal{A} \longrightarrow \mathcal{A}$ is defined by

$$\mathcal{H}_{\alpha,\beta}^{\xi,\omega} f(z) = Q_{\alpha,\beta}^{\xi,\omega}(z) * f(z) (z \in U),$$

where

$$Q_{\alpha,\beta}^{\xi,\omega}(z) = \frac{\Gamma(\alpha + \beta)}{(\gamma)_k}\left(E_{\alpha,\beta}^{\xi,\omega}(z) - \frac{1}{\Gamma(\beta)}\right),$$

$\beta, \gamma \in \mathbb{C}, Re(\alpha) > \max\{0, Re(k) - 1\}, Re(k) > 0.$

By some easy calculations, we have

$$\mathcal{H}_{\alpha,\beta}^{\xi,\omega} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\gamma + nk)\Gamma(\alpha + \beta)}{\Gamma(\gamma + k)\Gamma(\beta + \alpha n) n!} a_n z^n.$$

Definition 1.2 [17]**.** The fractional integral of order $\lambda, (\lambda > 0)$ is defined for a function f by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z - \zeta)^{1-\lambda}} d\zeta,$$

where f is analytic function in a simply-connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real, when $(z-\zeta)>0$.

We now, by making use of Definition 1.1 and Definition 1.2, we have

$$D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}f(z)=\frac{1}{\Gamma(2+\lambda)}z^{1+\lambda}+\sum_{n=2}^{\infty}\frac{\Gamma(\omega+n\xi)\Gamma(\alpha+\beta)}{\Gamma(n+1+\lambda)\Gamma(\omega+\xi)\Gamma(\beta+\alpha n)}a_n z^{n+\lambda}. \tag{1.4}$$

It is easily verified from (1.4) that

$$z\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}f(z)\right)'=\left(\frac{\omega+\xi}{\xi}\right)D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}f(z)-\left(\frac{\omega-\lambda\xi}{\xi}\right)D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}f(z),\quad Re(\omega-\lambda\xi)\neq 0. \tag{1.5}$$

Very recently, Xu et al. [21], Tang and Deniz [18], Rahrovi [12], Attiya and Yassen [4], Seoudy [14], Wanas and Alina [19], Aydogan et al. [2], Sakar and Canbulat [13] and Wanas and Khudher [20] have studied differential subordinations and superordinations for different conditions of analytic functions.

The main object of the present paper is to find sufficient condition for certain normalized analytic functions f in U such that $(f*\Psi)(z)\neq 0$ and f to satisfy

$$q_1(z)\prec\left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)}\right)^{\delta}\prec q_2(z),$$

and

$$q_1(z)\prec\left(\frac{\Gamma(2+\lambda)\left[t_1D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z)+t_2D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)\right]}{(t_1+t_2)z^{1+\lambda}}\right)^{\delta}\prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U and $\Phi(z)=z+\sum_{n=2}^{\infty}t_nz^n$, $\Psi(z)=z+\sum_{n=2}^{\infty}\varphi_nz^n$ are analytic functions in U with $t_n\geq 0,\varphi_n\geq 0$ and $t_n\geq\varphi_n$. Also, we obtain the number of results as their special cases.

To establish our main results, we need the following definition and lemmas:

Definition 1.3 [8]**.** Denote by Q the set of all functions f that are analytic and injective on $\overline{U}\setminus E(f)$, where

$$E(f)=\left\{\zeta\in\partial U:\lim_{z\to\zeta}f(z)=\infty\right\}$$

and are such that $f'(\zeta)\neq 0$ for $\zeta\in\partial U\setminus E(f)$.

Lemma 1.1 [8]**.** *Let q be univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w)\neq 0$ when $w\in q(U)$. Set $Q(z)=zq'(z)\phi(q(z))$ and $h(z)=\theta(q(z))+Q(z)$. Suppose that*

(1) $Q(z)$ *is starlike univalent in* U,

(2) $Re\left\{\frac{zh'(z)}{Q(z)}\right\}>0$ *for* $z\in U$.

If

$$\theta(p(z))+zp'(z)\phi(p(z))\prec\theta(q(z))+zq'(z)\phi(q(z)), \tag{1.6}$$

then $p\prec q$ and q is the best dominant of (1.6).

Lemma 1.2 [5]. *Let q be convex univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that*

(1) $Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0$ *for* $z \in U$,

(2) $Q(z) = zq'(z)\phi(q(z))$ *is starlike univalent in* U.

If $p \in \mathcal{H}\,[q(0),1] \cap Q$, *with* $p(U) \subset D, \theta(p(z)) + zp'(z)\phi(p(z))$ *is univalent in U and*

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(p(z)) + zp'(z)\phi(p(z)), \tag{1.7}$$

then $q \prec p$ *and* q *is the best subordinant of* (1.7).

2. Main Results

Theorem 2.1. *Let* $\Phi, \Psi \in \mathcal{A}$, $\rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta \in \mathbb{C}$ *such that* $\delta \neq 0$ *and* σ, τ *are not simultaneously zero,* q *be convex univalent in U with* $q(0) = 1$ *and assume that*

$$Re\left\{\frac{1}{\sigma q(z) + \tau}\left(-\varepsilon + \gamma q^2(z) + 2\mu q^3(z) - \sigma zq'(z) - \frac{2\tau zq'(z)}{q(z)}\right) + \frac{zq''(z)}{q'(z)} + 1\right\} > 0. \tag{2.1}$$

If $f \in \mathcal{A}$ *satisfies the differential subordination*

$$N_1(f, \Phi, \Psi, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, \xi, \omega, \lambda; z) \prec \rho + \gamma q(z) + \mu q^2(z) + \frac{\varepsilon}{q(z)} + \left(\frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)}\right) zq'(z), \tag{2.2}$$

where

$$N_1(f, \Phi, \Psi, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, \xi, \omega, \lambda; z)$$

$$= \rho + \left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f * \Phi)(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f * \Psi)(z)}\right)^{\delta}\left(\gamma + \mu\left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f * \Phi)(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f * \Psi)(z)}\right)^{\delta}\right)$$

$$+\varepsilon\left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f * \Psi)(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f * \Phi)(z)}\right)^{\delta} + \delta\left(\sigma + \tau\left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f * \Psi)(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f * \Phi)(z)}\right)^{\delta}\right) \times$$

$$\left(\left(\frac{\omega + 1 + \xi}{\xi}\right)\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+2}(f * \Phi)(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f * \Phi)(z)} - \left(\frac{\omega + \xi}{\xi}\right)\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f * \Psi)(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f * \Psi)(z)}\right), \tag{2.3}$$

then

$$\left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f * \Phi)(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f * \Psi)(z)}\right)^{\delta} \prec q(z)$$

and q *is the best dominant of* (2.2).

Proof. Let the function p be defined by

$$p(z) = \left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)}\right)^{\delta}, \quad (z \in U). \tag{2.4}$$

Then the function p is analytic in U and $p(0) = 1$.

A simple computation using (2.4) gives

$$\frac{zp'(z)}{p(z)} = \delta\left(\frac{z\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z)\right)'}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z)} - \frac{z\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)\right)'}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)}\right).$$

In view of (1.5), we obtain

$$\frac{zp'(z)}{p(z)} = \delta\left(\left(\frac{\omega+1+\xi}{\xi}\right)\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+2}(f*\Phi)(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z)} - \left(\frac{\omega+\xi}{\xi}\right)\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Psi)(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)}\right).$$

Also, we find that

$$\rho + \gamma p(z) + \mu p^2(z) + \frac{\varepsilon}{p(z)} + \left(\frac{\sigma}{p(z)} + \frac{\tau}{p^2(z)}\right)zp'(z) = N_1(f,\Phi,\Psi,\rho,\gamma,\mu,\varepsilon,\sigma,\tau,\delta,\alpha,\beta,\xi,\omega,\lambda;z), \tag{2.5}$$

where $N_1(f,\Phi,\Psi,\rho,\gamma,\mu,\varepsilon,\sigma,\tau,\delta,\alpha,\beta,\xi,\omega,\lambda;z)$ is given by (2.3).

By using (2.5) in (2.2), we have

$$\rho + \gamma p(z) + \mu p^2(z) + \frac{\varepsilon}{p(z)} + \left(\frac{\sigma}{p(z)} + \frac{\tau}{p^2(z)}\right)zp'(z)$$
$$\prec \rho + \gamma q(z) + \mu q^2(z) + \frac{\varepsilon}{q(z)} + \left(\frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)}\right)zq'(z).$$

By setting

$$\theta(w) = \rho + \gamma w + \mu w^2 + \frac{\varepsilon}{w} \quad \text{and} \quad \phi(w) = \frac{\sigma}{w} + \frac{\tau}{w^2},$$

it can be easily observed that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0, w \in \mathbb{C} \setminus \{0\}$. Also, we get

$$Q(z) = zq'(z)\phi\big(q(z)\big) = \left(\frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)}\right)zq'(z)$$

and

$$h(z) = \theta\big(q(z)\big) + Q(z) = \rho + \gamma q(z) + \mu q^2(z) + \frac{\varepsilon}{q(z)} + \left(\frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)}\right)zq'(z).$$

In light of the hypothesis of Theorem 2.1, we see that $Q(z)$ is starlike univalent in U and

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{\frac{1}{\sigma q(z)+\tau}\left(-\varepsilon + \gamma q^2(z) + 2\mu q^3(z) - \sigma zq'(z) - \frac{2\tau zq'(z)}{q(z)}\right) + \frac{zq''(z)}{q'(z)} + 1\right\} > 0.$$

Hence the result is now followed by an application of Lemma 1.1.

By fixing $\Phi(z) = \Psi(z) = \frac{z}{1-z}$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.1. *Let $\rho,\gamma,\mu,\varepsilon,\sigma,\tau,\delta\in\mathbb{C}$ such that $\delta\neq 0$ and σ,τ are not simultaneously zero, q be convex univalent in U with $q(0)=1$ and assume that* (2.1) *holds true. If $f\in\mathcal{A}$ satisfies the differential subordination*

$$N_2(f,\rho,\gamma,\mu,\varepsilon,\sigma,\tau,\delta,\alpha,\beta,\xi,\omega,\lambda;z)\prec\rho+\gamma q(z)+\mu q^2(z)+\frac{\varepsilon}{q(z)}+\left(\frac{\sigma}{q(z)}+\frac{\tau}{q^2(z)}\right)zq'(z), \quad (2.6)$$

where

$$N_2(f,\rho,\gamma,\mu,\varepsilon,\sigma,\tau,\delta,\alpha,\beta,\xi,\omega,\lambda;z)$$

$$=\rho+\left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}f(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}f(z)}\right)^{\delta}\left(\gamma+\mu\left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}f(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}f(z)}\right)^{\delta}\right)$$

$$+\varepsilon\left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}f(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}f(z)}\right)^{\delta}+\delta\left(\sigma+\tau\left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}f(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}f(z)}\right)^{\delta}\right)\times$$

$$\left(\left(\frac{\omega+1+\xi}{\xi}\right)\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+2}f(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}f(z)}-\left(\frac{\omega+\xi}{\xi}\right)\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}f(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}f(z)}\right), \quad (2.7)$$

then

$$\left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}f(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}f(z)}\right)^{\delta}\prec q(z)$$

and q is the best dominant of (2.6).

Theorem 2.2. *Let $\Phi,\Psi\in\mathcal{A}$, $\rho,\gamma,\mu,\varepsilon,\sigma,\tau,\delta\in\mathbb{C}$ such that $\delta\neq 0$ and σ,τ are not simultaneously zero, q be convex univalent in U with $q(0)=1$ and assume that*

$$Re\left\{\frac{q'(z)}{\sigma q(z)+\tau}\left(\gamma q^2(z)+2\mu q^3(z)-\varepsilon\right)\right\}>0. \quad (2.8)$$

Let $f\in\mathcal{A}$,

$$\left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q$$

and $N_1(f,\Phi,\Psi,\rho,\gamma,\mu,\varepsilon,\sigma,\tau,\delta,\alpha,\beta,\xi,\omega,\lambda;z)$ as defined by (2.3) *be univalent in U. If*

$$\rho+\gamma q(z)+\mu q^2(z)+\frac{\varepsilon}{q(z)}+\left(\frac{\sigma}{q(z)}+\frac{\tau}{q^2(z)}\right)zq'(z)\prec N_1(f,\Phi,\Psi,\rho,\gamma,\mu,\varepsilon,\sigma,\tau,\delta,\alpha,\beta,\xi,\omega,\lambda;z), \quad (2.9)$$

then

$$q(z)\prec\left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)}\right)^{\delta}$$

and q is the best subordinant of (2.9).

Proof. Let the function p be defined by (2.4).

In view of (1.5), the superordination (2.9) becomes

$$\rho+\gamma q(z)+\mu q^2(z)+\frac{\varepsilon}{q(z)}+\left(\frac{\sigma}{q(z)}+\frac{\tau}{q^2(z)}\right)zq'(z) \prec \rho+\gamma p(z)+\mu p^2(z)+\frac{\varepsilon}{p(z)}+\left(\frac{\sigma}{p(z)}+\frac{\tau}{p^2(z)}\right)zp'(z).$$

By setting $\theta(w)=\rho+\gamma w+\mu w^2+\frac{\varepsilon}{w}$ and $\phi(w)=\frac{\sigma}{w}+\frac{\tau}{w^2}$, it is easily observed that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}\setminus\{0\}$ and that $\phi(w)\neq 0, w\in\mathbb{C}\setminus\{0\}$. Also, we get

$$Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\}=Re\left\{\frac{q'(z)}{\sigma q(z)+\tau}(\gamma q^2(z)+2\mu q^3(z)-\varepsilon)\right\}>0.$$

Now Theorem 2.2 follows by applying Lemma 1.2.

By fixing $\Phi(z)=\Psi(z)=\frac{z}{1-z}$ in Theorem 2.2, we obtain the following corollary:

Corollary 2.2. *Let $\rho,\gamma,\mu,\varepsilon,\sigma,\tau,\delta\in\mathbb{C}$ such that $\delta\neq 0$ and σ,τ are not simultaneously zero, q be convex univalent in U with $q(0)=1$ and assume that* (2.2) *holds true. Let $f\in\mathcal{A}$,*

$$\left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}f(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}f(z)}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q$$

and $N_2(f,\rho,\gamma,\mu,\varepsilon,\sigma,\tau,\delta,\alpha,\beta,\xi,\omega,\lambda;z)$ as defined by (2.7) *be univalent in U. If*

$$\rho+\gamma q(z)+\mu q^2(z)+\frac{\varepsilon}{q(z)}+\left(\frac{\sigma}{q(z)}+\frac{\tau}{q^2(z)}\right)zq'(z)\prec N_2(f,\rho,\gamma,\mu,\varepsilon,\sigma,\tau,\delta,\alpha,\beta,\xi,\omega,\lambda;z), \tag{2.10}$$

then

$$q(z)\prec\left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}f(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}f(z)}\right)^{\delta}$$

and q is the best subordinant of (2.10).

Concluding the results of differential subordination and superordination, we state at the following sandwich result.

Theorem 2.3. *Let q_1 and q_2 be convex univalent in U with $q_1(0)=q_2(0)=1$, $\rho,\gamma,\mu,\varepsilon,\sigma,\tau,\delta\in\mathbb{C}$ such that $\delta\neq 0$ and σ,τ are not simultaneously zero. Suppose q_2 satisfies* (2.1) *and q_1 satisfies* (2.8). *For $f,\Phi,\Psi\in\mathcal{A}$, let*

$$\left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)}\right)^{\delta}\in\mathcal{H}[1,1]\cap Q$$

and $N_1(f,\Phi,\Psi,\rho,\gamma,\mu,\varepsilon,\sigma,\tau,\delta,\alpha,\beta,\xi,\omega,\lambda;z)$ as defined by (2.3) *be univalent in U. If*

$$\rho+\gamma q_1(z)+\mu q_1^2(z)+\frac{\varepsilon}{q_1(z)}+\left(\frac{\sigma}{q_1(z)}+\frac{\tau}{q_1^2(z)}\right)zq_1'(z)\prec N_1(f,\Phi,\Psi,\rho,\gamma,\mu,\varepsilon,\sigma,\tau,\delta,\alpha,\beta,\xi,\omega,\lambda;z)$$

$$\prec \rho + \gamma q_2(z) + \mu q_2^2(z) + \frac{\varepsilon}{q_2(z)} + \left(\frac{\sigma}{q_2(z)} + \frac{\tau}{q_2^2(z)}\right) z q_2'(z),$$

then

$$q_1(z) \prec \left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)}\right)^{\delta} \prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and the best dominant.*

By making use of Corollaries 2.1 and 2.2, we obtain the following corollary:

Corollary 2.3. *Let* q_1 *and* q_2 *be convex univalent in* U *with* $q_1(0) = q_2(0) = 1$, $\rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta \in \mathbb{C}$ *such that* $\delta \neq 0$ *and* σ, τ *are not simultaneously zero. Suppose* q_2 *satisfies* (2.1) *and* q_1 *satisfies* (2.8). *For* $f \in \mathcal{A}$, *let*

$$\left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}f(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}f(z)}\right)^{\delta} \in \mathcal{H}[1,1] \cap Q$$

and $N_2(f, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, \xi, \omega, \lambda; z)$ *as defined by* (2.7) *be univalent in* U. *If*

$$\rho + \gamma q_1(z) + \mu q_1^2(z) + \frac{\varepsilon}{q_1(z)} + \left(\frac{\sigma}{q_1(z)} + \frac{\tau}{q_1^2(z)}\right) z q_1'(z) \prec N_2(f, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, \xi, \omega, \lambda; z)$$
$$\prec \rho + \gamma q_2(z) + \mu q_2^2(z) + \frac{\varepsilon}{q_2(z)} + \left(\frac{\sigma}{q_2(z)} + \frac{\tau}{q_2^2(z)}\right) z q_2'(z),$$

then

$$q_1(z) \prec \left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}f(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}f(z)}\right)^{\delta} \prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and the best dominant.*

Theorem 2.4. *Let* $\Phi, \Psi \in \mathcal{A}$, $\rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, t_1, t_2 \in \mathbb{C}$ *such that* $\delta \neq 0, t_1 + t_2 \neq 0$, *and* σ, τ *are not simultaneously zero,* q *be convex univalent in* U *with* $q(0) = 1$ *and assume that* (2.1) *holds true. If* $f \in \mathcal{A}$ *satisfies the differential subordination*

$$N_3(f, \Phi, \Psi, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, \xi, \omega, \lambda, t_1, t_2; z) \prec \rho + \gamma q(z) + \mu q^2(z) + \frac{\varepsilon}{q(z)} + \left(\frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)}\right) z q'(z), \quad (2.11)$$

where

$$N_3(f, \Phi, \Psi, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, \omega, \xi, \lambda, t_1, t_2; z)$$
$$= \rho + \left(\frac{\Gamma(2+\lambda)\left[t_1 D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z) + t_2 D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)\right]}{(t_1+t_2)z^{1+\lambda}}\right)^{\delta}$$
$$\times \left(\gamma + \mu\left(\frac{\Gamma(2+\lambda)\left[t_1 D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z) + t_2 D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)\right]}{(t_1+t_2)z^{1+\lambda}}\right)^{\delta}\right)$$

$$+\varepsilon\left(\frac{(t_1+t_2)z^{1+\lambda}}{\Gamma(2+\lambda)\left[t_1D_Z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z)+t_2D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)\right]}\right)^{\delta}$$

$$+\delta\left(\sigma+\tau\left(\frac{(t_1+t_2)z^{1+\lambda}}{\Gamma(2+\lambda)\left[t_1D_Z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z)+t_2D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)\right]}\right)^{\delta}\right)$$

$$\times\left(\frac{t_1\left[\left(\frac{\omega+1+\xi}{\xi}\right)D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+2}(f*\Phi)(z)+\left(\frac{\omega+1-\xi(2\lambda-1)}{\xi}\right)D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z)\right]}{t_1D_Z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\mathfrak{z},\omega+1}(f*\Phi)(z)+t_2D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)}\right.$$

$$\left.+\frac{t_2\left[\left(\frac{\omega+\xi}{\xi}\right)D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Psi)(z)+\left(\frac{\omega-\xi(2\lambda-1)}{\xi}\right)D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)\right]}{t_1D_Z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z)+t_2D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)}\right), \quad (2.12)$$

then

$$\left(\frac{\Gamma(2+\lambda)\left[t_1D_Z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z)+t_2D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)\right]}{(t_1+t_2)z^{1+\lambda}}\right)^{\delta}\prec q(z)$$

and q is the best dominant of (2.11).

Proof. Let the function p be defined by

$$p(z)=\left(\frac{\Gamma(2+\lambda)\left[t_1D_Z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z)+t_2D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)\right]}{(t_1+t_2)z^{1+\lambda}}\right)^{\delta}, \quad (z \in U). \quad (2.13)$$

Then the function p is analytic in U and $p(0)=1$.

A simple computation using (2.13) gives

$$\frac{zp'(z)}{p(z)}=\delta\left(\frac{t_1z\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z)\right)'+t_2z\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)\right)'}{t_1D_Z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z)+t_2D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)}-(1+\lambda)\right).$$

In view of (1.5), we obtain

$$\frac{zp'(z)}{p(z)}=\delta\left(\frac{t_1\left[\left(\frac{\omega+1+\xi}{\xi}\right)D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+2}(f*\Phi)(z)+\left(\frac{\omega+1-\xi(2\lambda-1)}{\xi}\right)D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z)\right]}{t_1D_Z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z)+t_2D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)}\right.$$

$$\left.+\frac{t_2\left[\left(\frac{\omega+\xi}{\xi}\right)D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Psi)(z)+\left(\frac{\omega-\xi(2\lambda-1)}{\xi}\right)D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)\right]}{t_1D_Z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z)+t_2D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)}\right). \quad (2.14)$$

Also, we find that

$$\rho+\gamma p(z)+\mu p^2(z)+\frac{\varepsilon}{p(z)}+\left(\frac{\sigma}{p(z)}+\frac{\tau}{p^2(z)}\right)zp'(z)=N_3(f,\Phi,\Psi,\rho,\gamma,\mu,\varepsilon,\sigma,\tau,\delta,\alpha,\beta,\xi,\omega,\lambda,t_1,t_2;z), \quad (2.15)$$

where $N_3(f,\Phi,\Psi,\rho,\gamma,\mu,\varepsilon,\sigma,\tau,\delta,\alpha,\beta,\xi,\omega,\lambda,t_1,t_2;z)$ is given by (2.12).

By using (2.15) in (2.11), we have

$$\rho+\gamma p(z)+\mu p^2(z)+\frac{\varepsilon}{p(z)}+\left(\frac{\sigma}{p(z)}+\frac{\tau}{p^2(z)}\right)zp'(z)$$
$$\prec \rho+\gamma q(z)+\mu q^2(z)+\frac{\varepsilon}{q(z)}+\left(\frac{\sigma}{q(z)}+\frac{\tau}{q^2(z)}\right)zq'(z).$$

By setting

$$\theta(w)=\rho+\gamma w+\mu w^2+\frac{\varepsilon}{w} \quad \text{and} \quad \phi(w)=\frac{\sigma}{w}+\frac{\tau}{w^2},$$

it can be easily observed that $\theta(w)$ and $\phi(w)$ are analytic in $\mathrm{C}\setminus\{0\}$ and that $\phi(w)\neq 0, w\in\mathbb{C}\setminus\{0\}$. Also, we get

$$Q(z)=zq'(z)\phi\big(q(z)\big)=\left(\frac{\sigma}{q(z)}+\frac{\tau}{q^2(z)}\right)zq'(z)$$

and

$$h(z)=\theta\big(q(z)\big)+Q(z)=\rho+\gamma q(z)+\mu q^2(z)+\frac{\varepsilon}{q(z)}+\left(\frac{\sigma}{q(z)}+\frac{\tau}{q^2(z)}\right)zq'(z).$$

In light of the hypothesis of Theorem 2.4, we see that $Q(z)$ is starlike univalent in U and

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\}=Re\left\{\frac{1}{\sigma q(z)+\tau}\left(-\varepsilon+\gamma q^2(z)+2\mu q^3(z)-\sigma zq'(z)-\frac{2\tau zq'(z)}{q(z)}\right)+\frac{zq''(z)}{q'(z)}+1\right\}>0.$$

Hence the result is now followed by an application of Lemma 1.1.

By fixing $\Phi(z)=\Psi(z)=\frac{z}{1-z}$ in Theorem 2.4, we obtain the following corollary:

Corollary 2.4. *Let $\rho,\gamma,\mu,\varepsilon,\sigma,\tau,\delta,t_1,t_2\in\mathbb{C}$ such that $\delta\neq 0, t_1+t_2\neq 0$ and σ,τ are not simultaneously zero, q be convex univalent in U with $q(0)=1$ and assume that* (2.1) *holds true. If $f\in\mathcal{A}$ satisfies the differential subordination*

$$N_4(f,\rho,\gamma,\mu,\varepsilon,\sigma,\tau,\delta,\alpha,\beta,\xi,\omega,\lambda,t_1,t_2;z)\prec\rho+\gamma q(z)+\mu q^2(z)+\frac{\varepsilon}{q(z)}+\left(\frac{\sigma}{q(z)}+\frac{\tau}{q^2(z)}\right)zq'(z), \quad (2.16)$$

where

$N_4(f,\rho,\gamma,\mu,\varepsilon,\sigma,\tau,\delta,\alpha,\beta,\xi,\omega,\lambda,t_1,t_2;z)$

$$=\rho+\left(\frac{\Gamma(2+\lambda)[t_1D_Z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}f(z)+t_2D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}f(z)]}{(t_1+t_2)z^{1+\lambda}}\right)^{\delta}$$
$$\times\left(\gamma+\mu\left(\frac{\Gamma(2+\lambda)[t_1D_Z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}f(z)+t_2D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}f(z)]}{(t_1+t_2)z^{1+\lambda}}\right)^{\delta}\right)$$
$$+\varepsilon\left(\frac{\Gamma(2+\lambda)[t_1D_Z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}f(z)+t_2D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}f(z)]}{(t_1+t_2)z^{1+\lambda}}\right)^{\delta}$$

$$+\delta\left(\sigma+\tau\left(\frac{\Gamma(2+\lambda)[t_1 D_Z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}f(z)+t_2 D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}f(z)]}{(t_1+t_2)z^{1+\lambda}}\right)^{\delta}\right)$$

$$\times\left(\frac{t_1\left[\left(\frac{\omega+1+\xi}{\xi}\right)D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+2}f(z)+\left(\frac{\omega+1-\xi(2\lambda-1)}{\xi}\right)D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}f(z)\right]}{t_1 D_Z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}f(z)+t_2 D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}f(z)}\right.$$

$$\left.+\frac{t_2\left[\left(\frac{\omega+\xi}{\xi}\right)D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}f(z)+\left(\frac{\omega-\xi(2\lambda-1)}{\xi}\right)D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}f(z)\right]}{t_1 D_Z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}f(z)+t_2 D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}f(z)}\right), \quad (2.17)$$

then

$$\left(\frac{\Gamma(2+\lambda)[t_1 D_Z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}f(z)+t_2 D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}f(z)]}{(t_1+t_2)z^{1+\lambda}}\right)^{\delta}\prec q(z)$$

and q is the best dominant of (2.16).

Theorem 2.5. *Let $\Phi,\Psi\in\mathcal{A},\rho,\gamma,\mu,\varepsilon,\sigma,\tau,\delta,t_1,t_2\in\mathbb{C}$ such that $\delta\neq 0, t_1+t_2\neq 0$ and σ,τ are not simultaneously zero, q be convex univalent in U with $q(0)=1$ and assume that* (2.8) *holds true. Let $f\in\mathcal{A}$,*

$$\left(\frac{\Gamma(2+\lambda)[t_1 D_Z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z)+t_2 D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)]}{(t_1+t_2)z^{1+\lambda}}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q$$

and $N_3(f,\Phi,\Psi,\rho,\gamma,\mu,\varepsilon,\sigma,\tau,\delta,\alpha,\beta,\xi,\omega,\lambda,t_1,t_2;z)$ as defined by (2.12) *be univalent in U. If*

$$\rho+\gamma q(z)+\mu q^2(z)+\frac{\varepsilon}{q(z)}+\left(\frac{\sigma}{q(z)}+\frac{\tau}{q^2(z)}\right)zq'(z)$$
$$\prec N_3(f,\Phi,\Psi,\rho,\gamma,\mu,\varepsilon,\sigma,\tau,\delta,\alpha,\beta,\xi,\omega,\lambda,t_1,t_2;z), \quad (2.18)$$

then

$$q(z)\prec\left(\frac{\Gamma(2+\lambda)[t_1 D_Z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z)+t_2 D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)]}{(t_1+t_2)z^{1+\lambda}}\right)^{\delta}$$

and q is the best subordinant of (2.18).

Proof. Let the function p be defined by (2.13).

In view of (1.5), the superordination (2.18) becomes

$$\rho+\gamma q(z)+\mu q^2(z)+\frac{\varepsilon}{q(z)}+\left(\frac{\sigma}{q(z)}+\frac{\tau}{q^2(z)}\right)zq'(z)$$
$$\prec\rho+\gamma p(z)+\mu p^2(z)+\frac{\varepsilon}{p(z)}+\left(\frac{\sigma}{p(z)}+\frac{\tau}{p^2(z)}\right)zp'(z).$$

By setting $\theta(w)=\rho+\gamma w+\mu w^2+\frac{\varepsilon}{w}$ and $\phi(w)=\frac{\sigma}{w}+\frac{\tau}{w^2}$, it is easily observed that $\theta(w)$ and $\phi(w)$ are analytic in C \ {0} and that $\phi(w)\neq 0, w\in\mathbb{C}\setminus\{0\}$. Also, we get

$$Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} = Re\left\{\frac{q'(z)}{\sigma q(z)+\tau}(\gamma q^2(z)+2\mu q^3(z)-\varepsilon)\right\} > 0.$$

Now Theorem 2.5 follows by applying Lemma 1.2.

By fixing $\Phi(z) = \Psi(z) = \frac{z}{1-z}$ in Theorem 2.5, we obtain the following corollary:

Corollary 2.5. *Let $\rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, t_1, t_2 \in \mathbb{C}$ such that $\delta \neq 0, t_1 + t_2 \neq 0$ and σ, τ are not simultaneously zero, q be convex univalent in U with $q(0) = 1$ and assume that* (2.8) *holds true. Let $f \in \mathcal{A}$,*

$$\left(\frac{\Gamma(2+\lambda)[t_1 D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1} f(z) + t_2 D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega} f(z)]}{(t_1+t_2)z^{1+\lambda}}\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q$$

and $N_4(f, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, \xi, \omega, \lambda, t_1, t_2; z)$ as defined by (2.17) *be univalent in U. If*

$$\rho + \gamma q(z) + \mu q^2(z) + \frac{\varepsilon}{q(z)} + \left(\frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)}\right) z q'(z) \prec N_4(f, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, \xi, \omega, \lambda, t_1, t_2; z), \quad (2.19)$$

then

$$q(z) \prec \left(\frac{\Gamma(2+\lambda)[t_1 D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1} f(z) + t_2 D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega} f(z)]}{(t_1+t_2)z^{1+\lambda}}\right)^{\delta}$$

and q is the best subordinant of (2.19).

Concluding the results of differential subordination and superordination, we state at the following sandwich result.

Theorem 2.6. *Let q_1 and q_2 be convex univalent in U with $q_1(0) = q_2(0) = 1$, $\rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, t_1, t_2 \in \mathbb{C}$ such that $\delta \neq 0, t_1 + t_2 \neq 0$, and σ, τ are not simultaneously zero. Suppose q_2 satisfies* (2.1) *and q_1 satisfies* (2.8). *For $f, \Phi, \Psi \in \mathcal{A}$, let*

$$\left(\frac{\Gamma(2+\lambda)[t_1 D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z) + t_2 D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)]}{(t_1+t_2)z^{1+\lambda}}\right)^{\delta} \in \mathcal{H}[1,1] \cap Q$$

and $N_3(f, \Phi, \Psi, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, \xi, \omega, \lambda, t_1, t_2; z)$ as defined by (2.12) *be univalent in U. If*

$$\rho + \gamma q_1(z) + \mu q_1^2(z) + \frac{\varepsilon}{q_1(z)} + \left(\frac{\sigma}{q_1(z)} + \frac{\tau}{q_1^2(z)}\right) z q_1'(z) \prec N_3(f, \Phi, \Psi, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, \xi, \omega, \lambda, t_1, t_2; z)$$
$$\prec \rho + \gamma q_2(z) + \mu q_2^2(z) + \frac{\varepsilon}{q_2(z)} + \left(\frac{\sigma}{q_2(z)} + \frac{\tau}{q_2^2(z)}\right) z q_2'(z),$$

then

$$q_1(z) \prec \left(\frac{\Gamma(2+\lambda)[t_1 D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega+1}(f*\Phi)(z) + t_2 D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\xi,\omega}(f*\Psi)(z)]}{(t_1+t_2)z^{1+\lambda}}\right)^{\delta} \prec q_2(z)$$

and q_1, q_2 are respectively the best subordinant and the best dominant.

By making use of Corollaries 2.4 and 2.5, we obtain the following corollary:

Corollary 2.6. *Let q_1 and q_2 be convex univalent in U with $q_1(0) = q_2(0) = 1, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, t_1, t_2 \in \mathbb{C}$ such that $\delta \neq 0, t_1 + t_2 \neq 0$, and σ, τ are not simultaneously zero. Suppose q_2 satisfies* (2.8) *and q_1 satisfies* (2.1). *For $f \in \mathcal{A}$, let*

$$\left(\frac{\Gamma(2+\lambda)[t_1 D_z^{-\lambda} \mathcal{H}_{\alpha,\beta}^{\xi,\omega+1} f(z) + t_2 D_z^{-\lambda} \mathcal{H}_{\alpha,\beta}^{\xi,\omega} f(z)]}{(t_1+t_2) z^{1+\lambda}}\right)^{\delta} \in \mathcal{H}[1,1] \cap Q$$

and $N_4(f, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, \xi, \omega, \lambda, t_1, t_2; z)$ as defined by (2.17) *be univalent in U. If*

$$\rho + \gamma q_1(z) + \mu q_1^2(z) + \frac{\varepsilon}{q_1(z)} + \left(\frac{\sigma}{q_1(z)} + \frac{\tau}{q_1^2(z)}\right) z q_1'(z) \prec N_4(f, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, \xi, \omega, \lambda, t_1, t_2; z)$$
$$\prec \rho + \gamma q_2(z) + \mu q_2^2(z) + \frac{\varepsilon}{q_2(z)} + \left(\frac{\sigma}{q_2(z)} + \frac{\tau}{q_2^2(z)}\right) z q_2'(z),$$

then

$$q_1(z) \prec \left(\frac{\Gamma(2+\lambda)[t_1 D_z^{-\lambda} \mathcal{H}_{\alpha,\beta}^{\xi,\omega+1} f(z) + t_2 D_z^{-\lambda} \mathcal{H}_{\alpha,\beta}^{\xi,\omega} f(z)]}{(t_1+t_2) z^{1+\lambda}}\right)^{\delta} \prec q_2(z)$$

and q_1, q_2 are respectively the best subordinant and the best dominant.

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